केन्द्रीय विद्यालय संगठन

आंचलिक शिक्षा एवं प्रशिक्षण संस्थान, भुवनेश्वर

KENDRIYA VIDYALAYA SANGATHAN

ZONAL INSTITUTE OF EDUCATION & TRAINING, BHUBANESWAR

MATH OLYMPIAD

THE JOY OF MATHEMATICAL PROBLEM SOLVING

(A JOURNEY TO OLYMPIAD EXCELLENCE)

प्रस्तावना / FOREWORD

It gives me immense pleasure to present the Mathematics Olympiad Study Material prepared as a part of the workshop conducted from 7th to 11th July 2025 at ZIET, Bhubaneswar, on ***"Capacity Building of Teachers in Preparing Students for Mathematics Olympiad"*** exclusively for PGTs (Mathematics) from six feeder regions—Bhubaneswar, Kolkata, Ranchi, Guwahati, Silchar, and Tinsukia. A total of 39 participants enthusiastically engaged in this enriching academic exercise aimed at deepening their understanding of core mathematical concepts aligned with Olympiad-level problem solving.

The workshop focused on advanced mathematical domains such as Number Theory, Combinatorics, Inequalities, Algebra, and Geometry—not only through theoretical exploration but also through intensive hands-on sessions dedicated to rigorous problem solving. These topics, central to the spirit of mathematical inquiry, play a crucial role in fostering logical reasoning, creativity, and analytical thinking among students.

In alignment with the vision of the National Education Policy 2020 (NEP 2020) and the guiding principles of the National Curriculum Framework for School Education 2023 (NCF SE 2023), this initiative reflects a shift towards competency-based learning, critical thinking, and nurturing excellence through enrichment programmes such as Olympiads. These national frameworks emphasize the importance of identifying and nurturing talent at an early stage, making such capacity-building initiatives both timely and essential.

Recognizing the pivotal role of Mathematics Olympiads in stimulating young minds and encouraging deeper exploration beyond conventional classroom learning, we are proud to present the specially curated study material titled:
"The Joy of Mathematical Problem Solving: A Journey to Olympiad Excellence."

This resource is the outcome of a collaborative effort enriched by the insightful deliberations of the participants, expert contributions of the resource persons, the thoughtful planning and academic support of the Training Associate (Mathematics), and the visionary leadership and academic guidance of the Course Director during the course of the workshop. It is designed to serve as a valuable reference for both students and teachers of Kendriya Vidyalaya Sangathan, fostering curiosity, precision, and joy in mathematical thinking.

As the Director of ZIET, Bhubaneswar, I extend my heartfelt appreciation to all the participants for their active involvement and meaningful contributions. I also acknowledge the dedicated efforts of the entire team; whose commitment and collaboration made this workshop a resounding success.

May this initiative inspire many more such academic endeavours in the future, driving forward our shared vision of holistic, future-ready, and excellence-oriented education.

(Dr. Anurag Yadav)
Director
ZIET Bhubaneswar

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(Participants of the Workshop conducted from 07th to 11th July 2025 on "Capacity Building of Teachers in preparing students for Mathematics Olympiad")

Compiled & Edited By:

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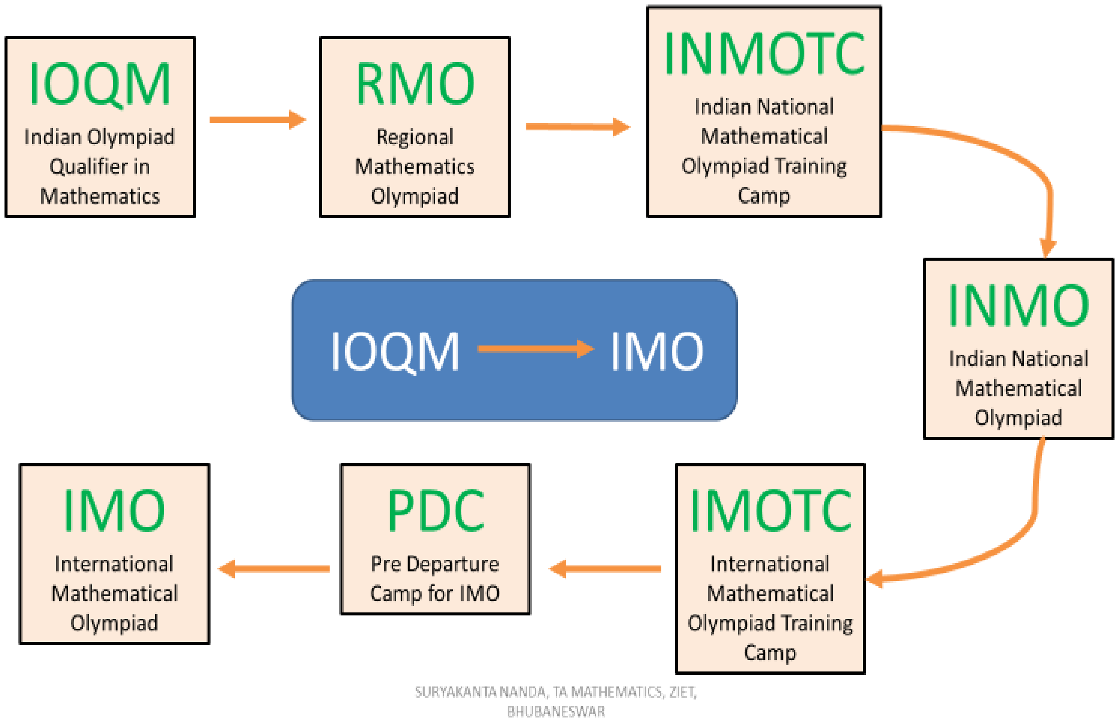
ABOUT MATHEMATICS OLYMPIAD PROGRAMME

The Mathematical Olympiad Programme in India, which leads to the participation of Indian students in the International Mathematical Olympiad (IMO) is organized **by the Homi Bhabha Centre for Science Education (HBCSE)** on behalf of the National Board for Higher Mathematics (NBHM) of the Department of Atomic Energy (DAE), Government of India. This programme is one of the major initiatives undertaken by the NBHM. Its main purpose is to spot mathematical talent among pre-university students in the country. For the purpose of training and selection of students for the Olympiad contest several regions all over the country have been designated and each assigned a Regional Coordinator. Additionally, two groups of schools: Jawahar Navodaya Vidyalayas (JNV) and Kendriya Vidyalayas (KV) are also treated as separate regions and have a 'Regional Coordinator' each.



Stages of Mathematics Olympiad

Stage 1 : IOQM

- The first stage examination, the Indian Olympiad Qualifier in Mathematics (IOQM) (Previously PRMO) is a three-hour examination with 30 questions. The answer to each question is either a single-digit number or a two-digit number and will need to be marked on a machine-readable OMR response sheet.
- The IOQM question paper will be in English and Hindi.
- The IOQM is conducted by the Mathematics Teachers' Association (India).
- Duration: 3 hours
- Total marks: 100
- The answer to each question is an integer in the range 00-99.
- No negative marking.
- OMR-based exam.
- Composition of the paper: 10 questions of 2 marks each; 10 questions of 3 marks each; 10 questions of 5 marks each.

Stage 2: RMO

- The second stage examination, the Regional Mathematical Olympiad (RMO) is a three-hour examination with six problems.
- The RMOs are offered in English, Hindi and other regional languages as deemed appropriate by the respective Regional Coordinators.
- The problems under each topic involve a high level of difficulty and sophistication.

Stage 3: INMO

- The best-performing students from the RMO qualify for the third stage – the Indian National Mathematical Olympiad (INMO).
- The INMO is held on the third Sunday of January across the country.
- Number of questions: 06
- Duration: 4.5 hours
- Each question requires writing detailed proof.

Stage 4: IMOTC

- The top students from the INMO (approximately 65) are invited for the fourth stage, the International Mathematical Olympiad Training Camp (IMOTC) held at HBCSE (or any other institute in India) from April to May.
- At this camp, orientation is provided to students for the International Mathematical Olympiad (IMO).
- Emphasis is laid on developing conceptual foundations and problem-solving skills.
- Several selection tests are held during this camp.
- On the basis of performance in these tests, six students are selected to represent India at the IMO.
- Resource persons from different institutions across the country are invited to the training camps.

PDC (Pre-Departure Camp for IMO):

- The selected team of 6 students undergoes a rigorous training programme for about 8-10 days at HBCSE prior to its departure for the IMO.

Stage 5: IMO

- This is the final stage where the Olympiad program concludes with the participation of the Indian students in the IMO.
- The six students are accompanied by four teachers or mentors.
- The competition consists of 6 problems.
- The competition is held over two consecutive days with 3 problems each.
- Each day the contestants have four-and-a-half hours to solve three problems.
- Each problem is worth 7 points for a maximum total score of 42 points.

ELIGIBILITY CRITERIA FOR INDIAN OLYMPIAD QUALIFIER IN MATHEMATICS (IOQM), 2025

- The student must NOT have qualified (or scheduled to appear) class 12 board examination earlier than 30 October, 2025.
- The student must NOT have commenced (or planning to commence) studies in a university or equivalent institution by 1 June, 2026.
- The INMO 2025 Awardees are eligible to write INMO 2026 directly WITHOUT qualifying through IOQM 2025 and RMO 2025 provided they fulfil the age criteria mentioned in point no.1, they haven't commenced (or planning to commence by June 1, 2026) studies in a university or equivalent institution and they fulfil the eligibility criteria for IMO 2026
- A student has to score at least 10% of the total marks of the IOQM 2025 paper in order to be eligible to appear for RMO 2025 but this is NOT the sole qualifying criterion.
- Any student who has scored at least 10% of the total marks of the IOQM 2025 paper and enrolled for IOQM 2025 as a student of one of the classes 8,9,10,11 will be classified as a Category A student.
- Any student who has scored at least 10% of the total marks of the IOQM 2025 paper and enrolled for IOQM 2025 as a student of class 12 will be classified as a Category B student.
- The list of students who qualify for the RMO 2025 will be prepared according to the following rule:

(a) From each region

i. the top 200 students from Category A will qualify for RMO 2025 along with those tied in the 200th position;

ii. the top 40 students from Category B will qualify for RMO 2025 along with those tied in the 40th position;

iii. 5 additional girl students from Category A irrespective of the number of girl students qualifying in the top 200 students from Category A will qualify for RMO 2025 under Girls' quota.

(b) There is no separate Girls' quota for Category B. A girl student of Category B can qualify for RMO 2025 from IOQM 2025 if and only if she is selected among the top 40 students in Category B as described in the previous section (point 4(a)ii. above)

IOQM SYLLABUS

IOQM SYLLABUS

- must be familiar with all the topics covered in NCERT Mathematics books of Class VIII, IX and X.
- In addition to the topics covered in point no. 1 above the following topics are to be given importance while preparing for the Olympiad examinations.
- The major areas from which problems are posed are algebra, combinatorics, geometry and number theory.
- The difficulty level increases from IOQM to RMO to INMO to IMO.

ALGEBRA

Inequalities, Progressions (A.P, G.P, H.P), Theory of indices, System of linear equations, Theory of equations, Binomial theorem and properties of binomial coefficients, Complex Numbers, Polynomials in one and two variables, Functional equations, Sequences.

Recommended Books:

- Higher Algebra; H.S.Hall & S.R.Knight
- Higher Algebra; Barnard & Child
- Polynomials; Ed Barbeau
- Functional Equations: A Problem Solving Approach; B.J.Venkatachala (Prism Books Pvt. Ltd., Bangalore)
- Inequalities: An Approach Through Problems (texts & readings in mathematics); B.J.Venkatachala, (Hindustan Book Agency)

PLANE GEOMETRY

Triangles, quadrilaterals, circles and their properties; standard Euclidean constructions; concurrency and collinearity (Theorems of Ceva and Menelaus); basic trigonometric identities, compound angles, multiple and submultiple angles, general solutions, sine rule, cosine rule, properties of triangles and polygons, Coordinate Geometry (straight line, circle, conics,3-D geometry), vectors.

Recommended Books:

- Geometry Revisited; H.S.M Coxeter & S.L.Greitzer
- Problems in Plane Geometry; I.F.Sharygin
- Plane Trigonometry; S.L.Loney

- The Elements of Coordinate Geometry; S.L.Loney

COMBINATORICS

Basic enumeration, pigeonhole principle and its applications, recursion, elementary graph theory.

Recommended Books:

- Introductory Combinatorics; Richard A. Brualdi
- Discrete Mathematics: Elementary and Beyond; László Lovász, József Pelikán, Katalin Vesztergombi
- Combinatorial Techniques; S. S. Sane
- Combinatorics For Mathematical Olympiad; S. Muralidharan

NUMBER THEORY

Divisibility theory in the Integers (The Division Algorithm, the Greatest Common Divisor, The Euclidean Algorithm, The Diophantine Equation ax + by = c) , Fundamental Theorem of Arithmetic, Basic properties of congruence, Linear congruences, Chinese Remainder Theorem, Fermat's Little Theorem, Wilson's Theorem, Euler's Phi function and Euler's generalisation of Fermat's Theorem, Pythagorean triples (definition and properties), Diophantine equations.

Recommended Books:

- Elementary Number Theory; David M. Burton
- An Introduction to the Theory of Numbers; Niven, Zuckerman, Montgomery

<u>FEW MORE REFERENCES:</u>

- Problem Primer for Olympiads
  C. R. Pranesachar, B. J. Venkatachala and C. S. Yogananda (Prism Books Pvt. Ltd., Bangalore).
- Challenge and Thrill of Pre-College Mathematics
  V. Krishnamurthy, C. R. Pranesachar, K. N. Ranganathan and B. J. Venkatachala (New Age International Publishers, New Delhi).
- An Excursion in Mathematics
  Editors: M. R. Modak, S. A. Katre and V. V. Acharya and V. M. Sholapurkar (Bhaskaracharya Pratishthana, Pune).
- Problem Solving Strategies
  A Engel (Springer-Verlag, Germany).
- Mathematical Circles
  Fomin and others (University Press, Hyderabad).

NUMBER THEORY

IMPORTANT CONCEPT ON NUMBER THEORY

Prime numbers: If the numbers of divisors of any natural number is 2 (including 1) then it is a Prime number.

Composite numbers: If the number of divisors of any natural number is more than 2 (including 1) then it is a composite number.

The Fundamental Theorem of Arithmetic: Every natural number, other than 1, can be factored into a product of primes in only one way, apart from the order of the factors.

Total number of Divisors:

Suppose the natural number n has the prime decomposition

$\mathbf{n = p_1^{n_1}.p_2^{n_2}p_3^{n_3} \dots p_k^{n_k}}$, where $p_1, p_2.., p_k$ is a collection of distinct primes.

Then the number of distinct divisors of n is

$$\boldsymbol{\tau(n) = (n_1 + 1)(n_2 + 1) \cdots (n_k + 1)}$$

Sum of all Divisors:

Let $\mathbf{n = p_1^{n_1}.p_2^{n_2}p_3^{n_3} \dots p_k^{n_k}}$

Sum of all divisors $\boldsymbol{\sigma(n) = \left(\frac{p_1^{n_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{n_2+1}-1}{p_2-1}\right) \dots \left(\frac{p_k^{n_k+1}-1}{p_k-1}\right)}$

Euclidean and Division Algorithm

(Division Algorithm). For every integer pair a, b, there exists distinct integer quotient and remainders, q and r, that satisfy $a = bq + r, 0 \leq r < b$
(Euclid). For natural numbers ab, we use the division algorithm to determine a quotient and remainder, q, r, such that $a = bq + r$. Then gcd(a,b) = gcd(b, r).

Lemma On Remainders. The Product of any two natural numbers has the same remainder, when divided by p, as the Product of their remainders.

The Units Digit of Powers of Positive Integers $\mathbf{a^n}$

Let P be the unit's digit of a positive integer a, and n be the positive integer power of a. then the last(unit) digit of a^n is last(unit) digit of of P.

The sequence takes constant values for P = 0,1,5, 6, i.e. U(P) does not change as n changes.

The sequence is periodic with a period 2 for P = 4 or 9. The sequence is periodic with a period 4 for P = 2,3,7,8.

Congruence of Integers

When an integer n is divided by a non-zero integer m, there must be an integral quotient q and a remainder r, where $\mathbf{0} < |\mathbf{r}| < \mathbf{m}$. This relation is denoted by

$$\mathbf{n} = \mathbf{mq} + \mathbf{r}$$

and the process for getting this relation is called division lemma.

Two integers a and b are said to be congruent modulo m, denoted by

$$\mathbf{a} \equiv \mathbf{b}\ (\mathbf{mod}\ \mathbf{m}),$$

If a and b have the same remainder when they are divided by a non-zero integer m.

If the remainders are different, then a and b are said to be not congruent modulo m, denoted by $\mathbf{a} \not\equiv \mathbf{b}\ (\mathbf{mod}\ \mathbf{m})$.

By the definition following is obvious

$$a \equiv b\ (mod\ m) \iff a - b = km \iff a - b \equiv 0 (mod\ m) \quad \iff m/(a - b).$$

BASIC PROPERTIES OF CONGRUENCES:

The letters a, b, c, d, k represents integers. The letters m, n represent positive integers. The notation a ≡ b (mod m) means that m divides a − b. We then say that a is congruent to b modulo m.

1. (Reflexive Property): a ≡ a (mod m)

2. (Symmetric Property): If a ≡ b (mod m), then b ≡ a (mod m).

3. (Transitive Property): If a ≡ b (mod m) and b ≡ c (mod m), then a ≡ c (mod m).

Remark: The above three properties imply that "≡ (mod m)" is an equivalence relation on the set Z.

4. If a ≡ b (mod m) and c ≡ d (mod m), then a + c ≡ b + d (mod m) and a − c ≡ b − d (mod m).

5. If a ≡ b (mod m) and c ≡ d (mod m), then ac ≡ bd (mod m).

6. Assume that a ≡ b (mod m). Let k ≥ 1. Then a k ≡ b k (mod m).

7. Suppose that P(x) is any polynomial with coefficients in Z. Assume that a ≡ b (mod m). Then P(a) ≡ P(b) (mod m).

8. Assume that a ≡ b (mod m). Then gcd(a, m) = gcd(b, m).

9. If a ≡ b (mod m) and n | m, then a ≡ b (mod n).

10. Assume that gcd(m, n) = 1. Assume that a ≡ b (mod m) and that a ≡ b (mod n). Then a ≡ b (mod mn).

11. Suppose that a ∈ Z. Then there exists a unique integer r such that a ≡ r (mod m) and 0 ≤ r ≤ m − 1. This integer r is the remainder when a is divided by m.

12. Assume that ca ≡ cb (mod m) and that (c, m) = 1. Then a ≡ b (mod m).
13. Assume p is a prime. If ab ≡ 0 (mod p), then either a ≡ 0 (mod p) or b ≡ 0 (mod p).
14. Assume that p is a prime and that p - a. Then a p−1 ≡ 1 (mod p).

Euler's totient function:

Euler's Totient Theorem states that if gcd(a,n)=1, then $\boldsymbol{a^{\varphi(n)} \equiv 1 (mod\ n)}$.
The Euler's Totient function $\boldsymbol{\varphi(n)}$ counts the number of positive integers up to a given n that are relative prime to n.
For $\boldsymbol{n = {p_1}^{k_1}{p_2}^{k_2}{p_3}^{k_3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots {p_r}^{k_r}, \varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_r}\right)}$
(Euler's Totient Theorem). For a relatively prime to m, we have $a^{\varphi(m)} \equiv 1\ (mod\ \ m)$.
(Fermat's Little Theorem). For a relatively prime to a prime p, we have $a^{p-1} \equiv 1\ (mod\ p)$.

Fermat's Theorem : Let p be a prime number and a be an integer then $a^p \equiv a\ (mod\ p)$

Wilson's Theorem: Let 'p' be a prime. Then (p - 1)! ≡ $-1(mod\ p)$

Decimal representation and divisibility tests

The decimal representation of integers is the number system that takes 10 as the base. Under this representation system, an n -digit whole number (where n is a non-negative integer)

$\overline{\boldsymbol{a_1 a_2 \ldots a_n}} = a_1 10^{n-1} + a_2 10^{n-2} + \ldots + a_{n-1} 10^1 + a_n$

Decimal Expresion of Whole Numbers with Same Digits or Periodically Changing Digits:

$\underbrace{\overline{\boldsymbol{aaa.\ldots.aa}}}_{\boldsymbol{n\ times}} = a(10^{n-1}+10^{n-2}+\ldots+10^1+1) = \frac{a}{9}\ (10^n - 1)$

$\underbrace{\overline{\boldsymbol{abcabc\ldots abc}}}_{\boldsymbol{n\ times}} = \overline{abc}\ (10^{3(n-1)}+10^{3(n-2)}+\ldots+10^3+1) = \frac{\overline{\boldsymbol{abc}}}{9}\ (10^{3n} - 1)$

Integer Division Results: The introduction of modular arithmetic permits us to illustrate some useful techniques for determining certain integer factors of numbers.

When finding the prime decomposition of an integer n:

• n is divisible by 2 if and only if the unit's digit of n is even.

• n is divisible by 3 if and only if the sum of its digits is divisible by 3.

• n is divisible by 5 if and only if the unit's digit of n is a 0 or a 5.

• n is divisible by 7 if and only if 7 divides the integer that results from first truncating n by removing its units' digit, and then subtracting twice the value of this digit from the truncated integer.

• n is divisible by 9 if and only if the sum of its digits is divisible by 9.

• n is divisible by 11 if and only if the alternating (by positive and negative signs) sum of its digits is divisible by 11.

QUESTIONS

1 Show that the equation $3x^{10} - y^{10} = 1991$ has no any integral solution.

2 Find the remainder when $2^{100} + 3^{100} + 4^{100} + 5^{100}$ is divided by 7.

3 Find the number of solutions in positive integer of the equation

$3x + 5y = 1008$.

4 Find all positive integers n such that $3^{2n} + 3n^2 + 7$ is a perfect square.

5 Integers a , b , c satisfy $a + b - c = 1$, $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$.

6 Let s(n) denote the sum of the digits of a positive integer n in base 10. If s(m)=20 and s(33m) =120. what is the value of s(3m)?

7 Let E(n) denote the sum of the sum of the even digits of n. For example, E(1243)= 2+4=6. What is the value of E(1) + E(2) +E(3) +…………………+ E(100) ?

8 Given that $R \times M \times O = 240, R \times O + M = 46, and\ R + M \times O = 64,$ $what\ is\ the\ value\ of\ R + M + O?$

9 A book is published in 3 volume, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number pages in the third volume is one and a half times that in the second. The sum of page numbers on the first pages of the three volumes is 1709.if n is the last page number, what is the largest prime factor of n?

10 Determine the sum of all possible integers n, the product of whose digits equals n^2 -15n -27.

11 Find x in the following magic square.

x	20	151
38	a	
b	c	D

12 Determine the number of positive integers with exactly three proper divisors which are less than 50.

13 let n be the largest number such that there exists a positive integer such that

$$k.n! = \frac{(((3!)!)!)}{3!}$$

14 Find the least positive integer ‘n’ such that when its leftmost digit is deleted, the resulting integer is equal to $\frac{n}{29}$.

15 Let n be a positive integer. Prove that 3^{2^n}+1 is divisible by 2, but not by 4.

16 How many positive integers n are there such that 3≤n≤100 and $x^{2^n} + x + 1$ is divisible by $x^2 + x + 1$?

17 Let u,v,w be real numbers in geometric progression such that u>v>w. Suppose $u^{40} = v^n = w^{60}$. Find the value of n.

18 Five distinct 2- digit numbers are in a geometric progression. Find the middle term.

19 Let $P(n) = (n + 1)(n + 3)(n + 5)(n + 7)(n + 9)$. What is the largest integer that is a divisor of P(n) for all positive integers n?

20 Let E(n) denotes the sum of the even digits of n. For example, E(1234) = 2+4 = 6. What is the value of $E(1) + E(2) + E(3) + \ldots\ldots\ldots + E(100)$?

21 In the square of an integer a, the tens' digit is 7. What is the unit's digit of a^2?

22 Find the smallest multiple of '15' such that each digit of the multiple is either '0' or '8'.

23 A number 'X' leaves the same remainder while dividing 5814, 5430, 5958. What is the largest possible value of 'X'?

24 Find the largest prime factor of: $3^{12} + 2^{12} – 2.6^6$

25 Without actually calculating, find which is greater: 31^{11} or 17^{14}

26 Let m, n be positive integers such that gcd (m, n) + lcm (m, n) = m + n

Show that one of the two numbers is divisible by the other.

27 Prove that there are infinitely many primes of the form 4k + 3.

28 Find all primes p such that p + 2 and p + 4 are also primes.

29 Let p be a prime. Prove that $x^p - x = x (x - 1) (x - 2) \ldots (x - (p - 1)) \pmod p$ for any x.

30 Find the number of trailing zeros in the 100! .

31 Prove that any natural number n > 6 can be written as the sum of two integers greater than 1, each of the summands being relatively prime.

32 Prove that there are arbitrarily long strings that do not contain a prime number.

33 Prove that the sum of the squares of two odd integers is divisible by 2 but not by 4.

34 Show that the square of any odd number is one more than a multiple of eight.

35 Prove that if a and b are coprime, then a squared and b squared are also coprime

36 Find all integers n such that $n^2+2n + 2$ is a perfect square.

37 Prove that no integer of the form 4n + 3 is a sum of two squares

38 Prove there are infinitely many prime numbers.

39 What is the remainder when 7^{100} is divided by 100?

40 Find the smallest positive integer x such that:

$x \equiv 2 \mod 3$

$x \equiv 3 \mod 4$

$x \equiv 1 \mod 5$

41 Find the smallest number that leaves a remainder of 1 when divided by 2, 3, 4, 5, 6 and is divisible by 7.

42 Find the smallest two-digit number x such that the sum of the distinct prime factors of x is equal to that of x + 1.

43 Find a 4-digit number that is prime and all its digits are also prime.

44 Find the smallest two-digit number x such that both x and its reverse are divisible by 3.

45 Find the smallest two-digit number x such that x and x + 1 are both divisible by exactly two distinct primes.

46 Find the number of positive integer n such that $n^4 + 4$ is prime.

47 Find the number of solutions of solution to the congruence $x^2 \equiv 1 (\mathrm{mod}\ 15)$.

48 Find the smallest positive integer n, the number $n^5 - n$ is divisible by 30.

49 Find the number of positive integer solutions to the equations $x^2 + y^2 = 100$.

50 Find the number of prime numbers less than 100 that can be expressed as the sum of two perfect squares.

51 Find the number of positive integers less than 1000 that are divisible by 3 or 5, but not both.

52 Find the gcd of $2^{100} - 1$ and $2^{120} - 1$.

53 Find the number of positive integers less than 1000 that are divisible by 7 and have a remainder of 3 when divided by 5.

54 Find the number of ways to express 50 as a sum of distinct positive integers.

55 Find the number of prime numbers less than 100 that can be expresses as the sum of two perfect squares in two different ways.

56 Find the remainder when 7^{2023} is divided by 100.

57 Find the remainder when 3^{100} is divided by 100.

58 How many integers between 1 and 1000 are divisible by neither 2,3, nor 5?

59 What is the greatest 3- digit number that leaves a remainder 1 when divided by 2,3,4,5 and 6?

60 Determine all pairs of integers (a, b) that satisfy: ab + 2a +3b =12.

61 A drawer contains 10 red, 8 blue, and 6 green socks. What is the minimum number of socks you must draw (without looking) to be sure of having at least one pair of the same colour?

62 Prove that if you have 51 distinct integers chosen from the set {1, 2, ..., 100}, then there must be two integers in your chosen set such that one divides the other.

63 A tournament with n players has the property that for any two players A and B, either A beats B or B beats A, but not both. Prove that there exists a player who has defeated every other player in the group.

64 Prove that among any 11 integers, there exist two whose difference is divisible by 10.

65 Let S be a set of 11 distinct integers. Prove that there exist two disjoint subsets A, B subsets of S such that the sum of elements in A is equal to the sum of elements in B.

66 Solve $(xy - 7)^2 = x^2 + y^2$, integers $x, y \geq 0$

67 Let $p, q \in \mathbb{N}$, such that p/q = 1 – ½ + 1/3 - …. + 1/1319

Prove that p is divisible by 1979.

68 712! + 1 prime?

69 Find $a, b, c \in \mathbb{Z}$ with $a < b < c$ such that $(a - 1)(b - 1)(c - 1)$ divides $abc – 1$

70 For all $n \geq 1$, prove the following by Mathematical induction:

$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$

71 For all $n \geq 1$, prove the following by Mathematical induction:

$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

72 Show that $\frac{(2n)!}{2^n n!}$ is integer for all $n \geq 0$.

73 Find the number of ordered pairs (a, b) such that $a, b \in \{10, 11, 12, \ldots, 29, 30\}$ and

gcd (a, b) + lcm (a, b) = a + b.

74 The product 55 x 60 x 65 is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?

75 What is the least positive integer by which $2^5.3^6.4^3.5^3.6^7$ should be multiplied so that, the product is a perfect square.

76 Find the remainders of $(\frac{3^{560}}{8})$.

77 If $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, for $n \geq 4$. Prove that $a_n < 2^n$, $n \geq 1$.

78 If $n_1, n_2, \ldots, n_p$ are p positive integers, whose sum is an even number, prove that the number of odd integers, among them, cannot be odd.

79 Prove that the equation $4x^3 – 7y^3 = 2010$.

80 Find the number of 2-digit natural numbers, which, when increased by 11 has the order of digits reversed.

81 Find all pairs of positive integers (a, b) such that, the sum of their sum, difference, product and quotient is 36.

82 Prove that $\sqrt{3\sqrt[3]{3}}$ is irrational.

83 Prove that $107^{90} - 76^{90}$ is divisible by 1891.

84 Show that $30^{99} + 61^{100}$ is divisible by 31.

85 Solve in integers: $(x - y)^2 + 2y^2 = 27$.

86 Show that the expression given by $\sqrt{6+2\sqrt{2}+2\sqrt{3}+3\sqrt{6}}$ - $\sqrt{\frac{1}{\sqrt{5-2\sqrt{6}}}}$ is an integer.

87 Find all triplets of positive integers (x, y, z) satisfying $2^x + 2^y + 2^z = 2336$.

88 Consider the equation in positive integers $x^2 + y^2 = 2000$ with $x < y$. Prove that $31 < y < 45$.

89 Consider the equation in positive integers $x^2 + y^2 = 2000$ with $x < y$. Rule out the possibility that one of x, y is even and the other is odd.

90 Consider the equation in positive integers $x^2 + y^2 = 2000$ with $x < y$. Prove that y is a multiple of 4.

91 If a,b,c are real numbers and $(a+b-5)^2 + (b+2c+3)^2 + (c+3a-10)^2 = 0$, find the integer nearest to $a^3+b^3+c^3$.

92 Find the number of positive integers n such that the highest power of 7 dividing n! is 8.

93 a,b,c are the digits of a nine digits number $\overline{abcabcabc}$. Calculate the quotient when this number is divided by 1001001.

94 A five-digit number $\mathrm{n} = \overline{\mathrm{abcde}}$ is such that when divided respectively by 2, 3, 4, 5, 6 the remainders are a, b, c, d, e. What is the remainder when n is divided by 100?

95 Let a, b, c be three distinct positive integers such that the sum of any two of them is a perfect square and having minimal sum a+b+c. Find this sum.

96 If N,(N+2) and (N+4) are prime numbers, then the number of possible solutions for N are____?

97 If n^2+2n-8 is a prime number where $n \in N$ then n is_____?

98 Find the last two digits of 62^{48}.

99 Given $\overline{1x6y7}$ is a five-digit number divisible by 9. The number of ordered pairs (x,y) satisfying this is____?

100 The number of numbers of the form $\overline{30a0b03}$ that are divisible by 13, where a,b are digits, is ___?

101 The last two digits of 3^{2012} when represented in decimal notation, will be____?

102 find the five least positive integers n for which $n^2-1$is a product of three different primes.

103 The product $55 \times 60 \times 65$ is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?

104 Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number, a + 1 divides b − 1 and b divides $a^2 + a + 2$.

105 Positive integers a, b, c satisfy $\frac{ab}{a-b} = c$. What is the largest possible value of a + b + c not exceeding 99?

106 find all prime solutions p, q, r of the equation $p(p+1) + q(q+1) = r(r+1)$.

107 Find two integers u and v satisfying 20u+63v=1

108 If p and p^2+8 are both prime numbers, find the value of p?

109 If 2^n-1 be a prime, prove that n is prime.

110 Find the number of odd positive divisors of 2700.

ANSWERS

1 Suppose the existence of $x, y \in Z$ such that $3x^{10} - y^{10} = 1991$. Note that $11/1991$. $[/means\ 'divides']$

So, neither x nor y is divisible by 11 for otherwise 11 would divide both.

So we can say,$11^{10} / 3x^{10} - y^{10} = 11^{10}/1991$, an impossibility

Hence x and y are prime to 11 .

So, $x^{10} \equiv y^{10} \equiv 1(mod 11)$

$$1991 = 3x^{10} - y^{10} \equiv 3 - 1 = 2 \text{ , a contradiction.}$$

2 $2^{100} \equiv 2 \; mod\; 7$

$3^{100} \equiv 4 \; mod\; 7$

$4^{100} \equiv 4 \; mod\; 7$

$5^{100} \equiv 2 \; mod\; 7$

so, $2^{100} + 3^{100} + 4^{100} + 5^{100} \equiv 12 \; mod\; 7$ ....(i) as $a \equiv b \; mod\; m$

$c \equiv d \; mod\; m$

$\therefore\; a + c \equiv b + d \; mod\; m$

But,$12 \equiv 5 \; mod\; 7$

So, from (i) we get, $2^{100} + 3^{100} + 4^{100} + 5^{100} \equiv 5 \; mod\; 7$

So, the required remainder is 5 (Ans) .

3 Let $x, y \in N$ such that $3x + 5y = 1008$, then $3/5y \to 3/y \to 3 = 5k$, for some $k \in N$ [/ means 'divides']

Now, $3x + 15k = 1008$

$$x + 5k = 336$$

$$5k \leq 335$$

$$k \leq 67$$

Thus ,any solution pair is given by $(x, y) = (336 - 5k, 3k)$ where $1 \leq k \leq 67$

so, the number of solution is 67 (Ans) .

4 Since $3^{2n} + 3n^2 + 7$ is a perfect square, let $3^{2n} + 3n^2 + 7 = p^2$, for some natural number p ,then $p^2 > 3^{2n}$ so that $p > 3^n$ so $p \geq 3^n + 1$.

Thus, $3^{2n} + 3n^2 + 7 = p^2 \geq (3^n + 1)^2 = 3^{2n} + 2.3^n + 1$ .This shows that

$2.3^n \leq 3n^2 + 6$. If $n \geq 3$,this cannot hold. One can prove this either by induction or by direct argument:

$$2.3^n = 2.(1+2)^n = 2\left[1 + 2n + \frac{n(n-1)}{2}.2^2 + \cdots\right] > 2 + 4n + 4n^2 - 4n$$

$$= 3n^2 + (n^2 + 2) \geq 3n^2 + 11 > 3n^2 + 6\,.$$

Hence,$n = 1\ or\ 2$. If $n = 1$, $3^{2n} + 3n^2 + 7$ will not be a perfect square.

If $n = 2$, $3^{2n} + 3n^2 + 7 = 81 + 12 + 7 = 100 = 10^2$.

Hence $n = 2$ is the only solution to be a perfect square.

5 Given, $a + b - c = 1$, $a^2 + b^2 - c^2 = -1$

$a + b - 1 = c$ …….. (i), now squaring both sides of (i), we get

$a^2 + b^2 + 1 + 2ab - 2(a - b) = c^2 \rightarrow ab = a + b \rightarrow (a-1)(b-1) = 1$.

So, $a - 1 = b - 1 \neq 1 \ \rightarrow a = b = 2\ or\ a = b = 0$.

So, $c = 3\ (when\ a = b = 2)\ or\ \ c = -1\ (when\ a = b = 0)$.

Hence $a^2 + b^2 + c^2 = 17\ or\ 1\ \ \therefore the\ required\ sum = 18$ (Ans).

6 We will take sum of digit base 10 to (mod9)

$Also\ \ s(ab)\ = s(a).s(b)\ \ (mod\ 9)$

$Now\ \ s(m) = 20$

$s(33m)\ = 120\ = s(11)\ X\ s(3m)$

$120 =\ 2\ X\ s(3m)\ \ \ (Since, s(11) =\ 2(mod9))$

$60 =\ s(3m)$

$Hence,\ s(3m)\ \ = 60$

7 E(1)+ E(2)+E(3)+…………………+E(100)

= sum of all even digits from 1 to 100

=sum of all the even digits in list(01+02+03+………………….+98+99+100)

=0 X20+ 2X20+ 4X20+6X20 +8X20

(Since there are 2X100 =200 and each digit appears 200/10=20 times)

=(2+4+6+8) X20 =20X20 =400

8 Given that $R \times M \times O\ = 240\ \ \ldots\ldots\ldots\ldots.(1)$

$$R \times O + M\ = 46 \ldots\ldots\ldots\ldots..(2)$$

$$R + M \times O\ = 64\ \ \ldots\ldots\ldots\ldots.(3)$$

$From\ (2) \times M = R \times M \times O + M \times 2 = 46 \times M$

$= M^2 - 46M + 240 = 0$ (FROM ….1)

= M= 6 or 40

From (3)X R , R^2 +RMO =64 R

= R^2 - 64 R +240 =0

= R= 4 or 60

If R=4 , M=6, then o =10 (from …..1)

= R =4, M=40, then o=3/2 (not positive integer)

= R =60, M=6, then o=2/3 (not positive integer)

= R =60, M=40, then o=1/10 (not positive integer)

Thus , R+M+O =4+6+10 =20

9 : Let the number of pages in volume-1 be x

Then, the number of pages in volume-2 = x+50

Also the number of pages in volume-3 = 3(x+50)/2

Moreover 1+(x+1)+(2x+51) =1709

3x +53=1709

x=552

so n= 552+602+903 =2057

so n= 11^2 X 17

Hence largest prime factor of n=17

10 n^2 -15n -27 is always odd number for all n$\in I$

n, must be of maximum of two-digit number.

Because maximum product of three-digit number is 729 & minimum value of n^2 -15n -27 for 3 digits number is 10000-1500-27 which is greater than 729.

n^2 -15n -27 is increasing function for all n $\in \{8,9,10, \dots \dots \dots \dots .\}$

At n=17, n^2 -15n -27 is equal to 17

At n=19, n^2 -15n -27 is equal to 49

At n=21, n^2 -15n -27 is equal to 99

And maximum product of digits of two-digit number is 81

So, n must be less than 21

Between 1 to 15 , At n=17, n^2 -15n -27 is negative

So n=17 only

Sum of possible number equal to 17

11 : let us name the numbers in other cells.

x	20	151
38	a	
b	c	D

Now based on the definition of magic square we can write

x+20+151 = x + 38 + b which implies b=133

similarly x+38+b = b+a+151 which implies a=x-113

similarly x+38+b = 20+a+c substituting for a and b

x+38+133=20+x-113+c we get c=264 based on above steps we can write

X	20	151
38	x-113	
133	264	D

X+38+133=133+264+d which implies d=x-226

x+20+151 = x+x-113+d and substituting for d = x-226

we get x+20+151 = x+x-113+x-226

finally x=255

12 We need three divisors and we know 1 is divisor for all numbers. Hence the problem reduces to finding two divisors.

The required integers must be a product of prime numbers of the form 'pq' or p^3 to have two divisors.(p,q primes and $p \neq q$)

Case-1: If the integer is of the form pq then the divisors are 1,p,q

Case-2: If it is of the form p^3 then the divisors are 1,p,p^2.

As per the question the divisor must be less than 50 so our p,q,p^2 must be less than 50 and prime. Lets list out such primes which can be p,q,p^2

Prime numbers less than 50 = { 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47} i.e there are 15 primes.

Now Case-1: we can get two primes out of 15 primes in ${}^{15}_{2}C$ ways which is 105 ways. So there are 105 such numbers .

Case-2: 1,p,p^2 : $p^2<50$ we have 4 such primes (since $11^2>50$)

To sum up our findings there are a total of 105+4 = 109 integer which have their divisors less than 50.

13 3! = 3x2x1 = 6 and (3!)! = (6)! = 720

(((3!)!)!) = ((6)!)! = (720)! = $720 \times 719 \times 718 \times 717 \times \ldots\ldots\ldots \times 1 = 720 \times 719!$

Therefore given equation becomes

$$k.n! = \frac{720 \times 719!}{6} = 120 \times 719!$$

Largest possible n is 719.

14 Let the left most digit be a and the number n has d digits.

A can be any digit other than 0 therefore a= {1,2,3,....,9}.

After deletion of 'a' the number 'n' has d-1 digits left and the place value of 'a' will be 10^{d-1} . the face value of a will be $a.10^{d-1}$. Let the number left out after deletion of 'a' be m. This gives two equations

Then $m = \frac{n}{29}$ $and\ n = 29m$ and $n = a.10^{d-1} + m$.

Combining both $29m = a.10^{d-1} + m$

$28m = a.10^{d-1}$

$7.4.m = a.10^{d-1}$

Therefore 7 divides a

From the values of a= {1,2,3,....,9}, there is only possibility i.e a=7

Substituting a = 7 in

$7.4.m = a.10^{d-1}$ we get

$4.m = 10^{d-1}$

For the m value to be integer 10^{d-1} must be greater than 10^2.

Therefore least possible value of d is 3.

And least m possible is 25. (as $4.m = 10^2$)

Therefore required least possible n is

$n = a.10^{d-1} + m$.

$n = 7.10^2 + 25$.

$n = 725$

15 Clearly is 3^{2^n} odd as any power of three is odd.

So $3^{2^n} + 1$ is even and is divisible by 2. That part of proof is complete. Now the task is to show that it is not divisible by 4.

Let us write 3^{2^n} $as\ (3^2)^{2^{n-1}}$ which further can be written as $(9)^{2^{n-1}}$ and $(8+1)^{2^{n-1}}$

Let us use some binomial theorem here.

$(8+1)^{2^{n-1}} = 8^{2^{n-1}} + \binom{2^{n-1}}{1} (8^{2^{n-1}-1})(1)^1 + \ldots\ldots\ldots + 1$

All terms of the expansion except the last one is divisible by 4. Which tells us that when divided by 4 , $(8+1)^{2^{n-1}}$ or 3^{2^n} gives a remainder 1 and hence

3^{2^n}+1 leaves a reminder 2.

Hence proved.

16 $(x^2+x+1) = (x-w)(x-w^2)$ and P(x) = $x^{2^n} + x + 1$

This implies P (w) = 0 and P (w^2) = 0

$w^{2^n} + w + 1$ = 0 and $(w^2)^{2^n} + w^2 + 1$ = 0

We know that $1 + w + w^2$ = 0

This implies that $w^{2^n} = w^2$,

$w^{3k} = 1, w^{3k+1} = w\ and\ w^{3k+2} = w^2$

Then, 2^n must be in form of 3k+2.

If we divide 2^n by 3 then remainder should be 2.

2^n =8 or 32 or 128 ………..(these numbers gives remainder 2 on division by 3)

Thus total number value of n = 49.

17 Given that u > v > w

Then u= x , $v = \frac{x}{r}, w = \frac{x}{r^2}$

$u^{40} = v^n = w^{60}$ then $x^{40} = \frac{x^n}{r^n} = \frac{x^{60}}{r^{120}}$

from first and last $x^{40} = \frac{x^{60}}{r^{120}}$

$x = r^6$

From first two, $x^{40} = \frac{x^n}{r^n}$

$$x^{n-40} = r^n$$

$$(r^6)^{n-40} = r^n$$

6(n-40) = n

n = 48.

18 Let the numbers are a, ar, ar^2, ar^3 and ar^4.

$ar^4 < 100$ and $a \geq 10$

$r \leq 1.5$ and $a < \frac{100}{(1.5)^4}$

a < 18

this implies that a = 16

No are 16, 24,36,54 and 81.

Middle term is 36.

19 Given that P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)

n=0 then P(0) = 1x3x5x7x9

n= 8 then P(8) = 9x11x13x15x24

n= 10 then P(10) = 11x13x15x17x19

n = 20 then P(20) = 21x23x25x27x29

HCF of all these = 15

thus, 15 is the largest number that divides P(n) for all even n.

20 E(1) +E(2)+ E(3) + ………+ E(100) = Sum of all even digits from 1 to 100.

= Sum of all even digits in list (01, 02 , 03, 04, ……………98, 99, 00)

= 0x20 + 2x20 + 4x20 + 6x20 + 8x20

(Since there are 2x100= 200 digits and each digit appears $\frac{200}{10} = 20\ times$.

= (2+4+6+8) x20

= 20x20

= 400.

21 The digit in ten's place of a^2 is 7,

Where a is an integer

If (i) The digit in the unit place of a is either 4 or 6 and

(ii) The digit in the Ten's place of a is 2 then the ten's place of a^2 is 7.

In each case, the digit in the unit place is 6.

22 Smallest multiple of 15, such that each digit of the multiple in either 0 or 8 are

Two & Three digit nos	Four digit and Five digit nos
80 880 808 800	8000 8008 8080 8800 8880 80888 80888 88088

So only possibility for multiple of 15 i.e. divisible by 5
is last digit is 0 i.e.
(i) 2 digits 80
(ii) 3 digits 880, 800
(iii) 4 digits 8000, 8800, 8880, 8080
(iv) 5 digits 88880,80000, 88800, 88000, 88080
As 15 = 5x3
So the number should be divisible by 3 the sum of digit should be divisible by 3.
Hence let us analyse the sum of digits in (i), (ii), (iii) and (iv),
2 digit : not possible
3 digit : not possible
4 digit : with 888
sum in 8+8+8 = 24 that is divisible by 3
But last digit should be 0 and it should contain three numbers of 8. i. e. 8880

23 Let p, q, r and s be any number from the question, if r in remainder.
5814 = p x+ r ………………… (i)
5430 = q x + r ………………… (ii)
5958 = s x + r …………………. (iii)
from (i) & (ii)
384 = (p-q) x
from (ii) & (iii)
5430 – 5958 = (q - s) x
⇒ 528 = (s - q) x
from (iii) & (i)
5814 – 5958 = (p - s) x
⇒ 144 = (s - p) x
so we get three equation
384 = (p - q) x
528 = (s - q) x
144 = (s - p) x
⇒ (p - q) x = 2.2.2.2.2.2.2.3
(s - q) x = 2.2.2.2.3. 11
(s - q) x = 2.2.2.2.3.3

So the HCF of these three numbers

= 2.2.2.2.3

= 48

So the required largest number is 48

Check:

48 x 121 = 5808 then + 6 = 5814

48 x 113 = 5424 then + 6 = 5430

48 x 124 = 5952 than + 6 = 5958

24 $3^{12} + 2^{12} – 2.6^{6}$

= $(3^6)^2 + (2^6)^2 – 2. (3 \times 2)^6$

= $(3^6)^2 + (2^6)^2 – 2. (3^6) \times (2)^6$

= $(3^6 – 2^6)^2$

= $[(3^3)^2-(2^3)^2]^2$

= $[(3^3+2^3) (3^3-2^3)]^2$

= $[(3+2) (3^2 - 3 \times 2 + 2^2) (3-2) (3^2 + 3 \times 2 + 2^2)]^2$

= $[(5) \times (9 - 6 + 4) (1) \times (9 + 6 + 4)]^2$

= $[(5) \times (19) \times (7)]^2$

= $[5 \times 19 \times 7]^2$

= $5^2 \times 19^2 \times 7^2$

∴ The largest prime factor of $(3^{12} + 2^{12} – 2.6^{6})$ =19

25 31^{11} or 17^{14}

Sol 31^{11} or 17^{14}

$31 < 32$

or $31 < 2^5$

or $31^{11} < (2^5)^{11}$

or $31^{11} < 2^{55}$

$17 > 16$

or $17 > 2^4$

or $17^{14} > (2^4)^{14}$

or $17^{14} > 2^{56}$

∵ $31^{11} < 2^{55}$

and $17^{14} > 2^{56}$

and $2^{56} > 2^{55}$

∴ $17^{14} > 31^{11}$

26 Let gcd (m, n) = g

Then: $g + mn/g^2 = m+n$

or, $mn + g^2 = g(m+n)$

or, $(m - g)(n - g) = 0$. Thus, either g = m or g = n, meaning m |n and n |m

27 Assume finitely many such primes: $p_1, p_2, ..., p_n$. Let $N = 4(p_1 \times p_2 \times \ldots \times p_n) - 1$.

Then $N \equiv 3 \pmod 4$, and none of the p divide N. So any prime dividing N must be $\equiv 3 \pmod 4$, contradicting the assumption. Hence, infinitely many such primes exist.

28 Check small primes: 3, 5, 7 work -> (3,5,7). Try further, but either p+2 or p+4 becomes divisible by 3. Only (3,5,7) works.

29 Just see that for any x, one of x,x - 1,...,x-(p-1) is 0 modulo p. Hence, the right side becomes 0 modulo p. What about the left side? Well, that is zero for any residue too by Fermat's Little Theorem! Hence, if we define the polynomials $f(x) = x^p - x$ and $g(x) = x(x-1)... (x - (p - 1))$ then f(x) equivalent g(x) (mod p) for any z.

If you see carefully, this doesn't say that the polynomials f(x), g(x) are the same (i.e. have the same coefficients modulo p), it merely says it would the same value. For instance, x^p -x (mod p) is true for all z value-wise, but the polynomials x^p and are obviously different.

30 Let us divide 100 by 5.

100/5 = 20 and 20/5 = 4

Now, 4 is less than 5. So, we stop the division here.

Also, the number of trailing zeros = 20 + 4 = 24

31 If n is odd, we may choose a = 2,b = n−2. If n is even, then is either of the form 4k or 4k +2. If n = 4k, then take

a = 2k +1,b = 2k −1. These two are clearly relatively prime (why?).

If n = 4k +2,k > 1 take a = 2k +3,b = 2k −1.

32 Let $k \in N, k \geq 2$. Then each of the numbers

k! +2,...,k! +k

is composite.

33 Any odd integer can be represented in the form (2k + 1), where k is any integer.

$(2k + 1)^2 = 4k^2 + 4k + 1$ and $(2j + 1)^2 = 4j^2 + 4j + 1$ (where j is another integer)

$(4k^2 + 4k + 1) + (4j^2 + 4j + 1) = 4k^2 + 4j^2 + 4k + 4j + 2$

$2(2k^2 + 2j^2 + 2k + 2j + 1)$

Since the expression is a multiple of 2, the sum is divisible by 2. However, it cannot be divisible by 4 because of the '+ 1' term.

34 Let odd number be of the form 2n + 1.

Square: (2n + 1) squared = 4n squared + 4n + 1 = 4n(n + 1) + 1.

One of n or n + 1 is even, so n(n + 1) is even.

Thus, 4n(n + 1) is divisible by 8.

So, the square is 8k + 1.

35 If a and b have no common divisors other than one, then any common divisor of a squared and b squared must divide both a and b, which contradicts them being coprime.

Hence, a squared and b squared are also coprime.

36 Let $n^2 + 2n + 2 = k^2$

Then, $(n + 1)^2 + 1 = k^2$

So, $k^2 - (n + 1)^2 = 1$.

Difference of squares: $(k - n - 1)(k + n + 1) = 1$.

Only integer solutions: $k - n - 1 = 1$, $k + n + 1 = 1$.

No integer n satisfies both.

Try small values:

n = 0 → value is 2

n = 1 → 5

n = 2 → 10

n = 3 → 17

No perfect square.

Only value when k squared = (n + 1) squared + 1 is not possible.

So, no solution.

37 Suppose $4n + 3 = a^2 + b^2$.

Then $a^2 + b^2$ mod 4 can be 0, 1, or 2.

Never 3.

Thus, 4n + 3 cannot be a sum of two squares.

38 Assume there are finitely many primes: $p_1, p_2, ..., p_n$.

Let $N = p_1 \times p_2 \times ... \times p_{n+1}$.

N is not divisible by any of these primes, so it's either prime itself or divisible by a new prime.

Contradiction.

So, infinitely many primes exist.

39 Euler's theorem says $7^{40} \equiv 1 \pmod{100}$

So, $7^{100} = (7^{40})^2 \times 7^{20} \equiv 1^2 \times 7^{20} \pmod{100}$

We calculate 7^{20} mod 100, and it turns out:

Answer: 1

40 Try x = 1, 6, 11... (values ≡ 1 mod 5).

At x = 11:

11 ≡ 3 mod 4

11 ≡ 2 mod 3

Answer: 11

41 LCM(2, 3, 4, 5, 6) = 60

So x ≡ 1 mod 60 → x = 60k + 1

Find smallest such x divisible by 7 → Try k = 34

x = 2041 → 2041 ÷ 7 = 291.57

Answer: 2041

42 (Note: Count each prime factor only once, even if it appears multiple times.)

Try small values of x:

x = 24

Prime factors = 2, 3 → Sum = 2 + 3 = 5

x + 1 = 25

Prime factor = 5 → Sum = 5

Sums match!

Answer: x = 24

43 Prime digits: 2, 3, 5, 7

Try 2357 → it's prime.

Answer: 2357

44 x = 12

12 → divisible by 3

21 → divisible by 3

45 x = 14 → 2 × 7

x + 1 = 15 → 3 × 5, Both have two distinct primes.

46 The expression (n^4+4) can be factored as (n^2-2n+2)(n^2+2n+2). For (n^4+4) to be prime, one of these factors must be equal to 1. However, neither as (n^2-2n+2) nor as (n^2+2n+2) is equal to 1 for any integer n greater than 1. Now n^2-2n+2= $(n-1)^2$+1.

When n>1,$(n-1)^2$>0, so $(n-1)^2$+1>1. n^2+2n+2=$(n+1)^2$+1.

Similarly, when n>0, $(n+1)^2$>0, so $(n+1)^2$+1>1.

Therefore, the only possible way for n^4 + 4 to be prime is when n=1, in which case n^4 + 4=1^2+ 2^2=5, which is a prime number.

47 To find the solutions, we first notice that$15 = 3 \times 5$.
We can split the congruence into two separate congruence: $x^2 \equiv 1(mod\ 3)$and $x^2 \equiv 1(mod\ 5)$.
For the first congruence $x^2 \equiv 1(mod\ 3)$, the solutions are $x^2 \equiv 1(mod\ 3)$ & $x^2 \equiv 2(mod\ 3)$
For the second congruence $x^2 \equiv 1(mod\ 5)$,
the solutions are $x^2 \equiv 1(mod\ 5)$ & $x^2 \equiv 4(mod\ 5)$.

48 Find the smallest positive integer n, the number $n^5 - n\ is\ divisible\ by\ 30$
<u>Factor the expression</u>
$n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = n\ (n + 1)(n - 1)(n^2 + 1)$
<u>Check for divisibility by 2 and 3.</u>
The product of three consecutive integers is $(n - 1)\ n\ (n + 1)$ always divisible by 2 & 3. Thus $(n - 1)\ n\ (n + 1)$ is divisible by 6.
<u>Check for divisibility by 5.</u>
If n is divisible by 5, then n^2 – n is divisible by 5.
$If\ n \equiv 1(mod\ 5), then\ n - 1 \equiv 0(mod\ 5)\ so$ n^2 – n is divisible by 5.
$If\ n \equiv 2\ (mod\ 5), then\ n^2 \equiv 4(mod5), so\ n^2 + 1 \equiv 0(mod\ 5)\ so$ n^2 – n is divisible by 5.
$If\ n \equiv 3\ (mod\ 5), then\ n^2 \equiv 9\ (mod5), so\ n^2 + 1 \equiv 0(mod\ 5)\ so$ n^2 – n is divisible by 5.
$If\ n \equiv 4\ (mod\ 5), then\ n^2 \equiv 16\ \ (mod5), so\ n^2 + 1 \equiv 0(mod\ 5)\ so$ n^2 – n is divisible by 5.
Therefore n^2 – n is divisible by 5.
<u>Check for divisibility by 30.</u>
Since n^2 – n is divisible by 2, 3 & 5. It is divisible by 2 x 3 x 5 =30
Thus the smallest positive integer n, the number $n^5 - n\ is\ divisible\ by\ 30 = 2$.

49 There are two positive integer solutions to the equation x^2+y^2 =100. The solutions are (6, 8) and (8, 6).
<u>Explanation.</u>
Recognize the equation:
The equation x^2+y^2 =100 represents a circle entered at the origin with a radius of 10.
Consider positive integers:
We are looking for positive integer solutions, so x and must be greater than 0.
Find possible values for x:
Since x and y are integers, we need to find perfect squares that add up to 100. We know that 6^2=36 and 8^2=64 & 36+64=100
List the solutions:
Therefore, the possible solutions are (6, 8) and (8, 6).

50 A prime number can be expressed as the sum of two squares if and only if it is either 2 or of the form 4k+1, where k is a positive integer.
The primes that can be expressed as the sum of two squares are:
2 = 1^2 + 1^2

$5 = 1^2 + 2^2$
$13 = 2^2 + 3^2$
$17 = 1^2 + 4^2$
$29 = 2^2 + 5^2$
$37 = 1^2 + 6^2$
$41 = 4^2 + 5^2$
$53 = 2^2 + 7^2$
$61 = 5^2 + 6^2$
$73 = 3^2 + 8^2$
$89 = 5^2 + 8^2$
$97 = 4^2 + 9^2$

There are 10 prime numbers less than 100 that can be expressed as the sum of two perfect squares.

These primes are: 2, 5, 13, 17, 29, 37, 41, 53, 61, and 73.

51 Solution:

What you're solving for

Count of positive integers less than 1000 that are divisible by 3 or 5, but not both.

What's given in the problem

The range of integers is 1 to 999.

Divisibility is by 3 or 5, but not both.

How to solve

Count the integers divisible by 3 and 5 separately, subtract the intersection (divisible by both 3 and 5), and then subtract the intersection from the sum of the individual counts.

Step by step explanation

Step 1:

Count integers divisible by 3.

Divide 999 by 3 to find the number of multiples of 3 less than 100.

999/3=333

Step 2:

Count integers divisible by 5.

Divide 999 by 5 to find the number of multiples of 5 less than 100.

999/5=199(ignoring the remainder)

Step 3:

Count integers divisible by both 3 & 5 that is by 15..

Divide 999 by 15 to find the number of multiples of 15 less than 100.

999/15=66 (ignoring the remainder)

Step 4:

Count integers divisible by 3 or 5.

Add the numbers of multiples of 3 & 5, then subtract the multiples of 15.

333+199-2x66=532-132=400

Step 5:

Count integers divisible by 3 or 5 but not both.

Subtract the multiples of 15 from the sum of multiples of 3 & 5.

(333 – 66) + (199 – 66) =267 +133=400

Therefore, the number of positive integers less than 1000 that are divisible by 3 or 5 but not both is 400.

52 The greatest common divisor (GCD) of $2^{100}-1$ and $2^{120}-1$ is $2^{20}-1.$ $This\ is\ because\ the\ GCD\ of\ a^x-1\ and\ a^y-1$ is $a^{gcd(x,y)}-1$. In this case, a=2, x=100 and =120. The GCD of 100 and 120 is 20, therefore the answer is $2^{20}-1$.

53 Solution:

<u>What you're solving for</u>

Find the number of positive integers less than 1000 that are divisible by 7 and leave a remainder of 3 when divided by 5.

. <u>What's given in the problem</u>

The integers are less than 1000.

The integers are divisible by 7.When divided by 5, the integers leave a remainder of 3.

<u>How to solve</u>

Find the general form of the integers satisfying the given conditions and then find how many such integers are less than 1000.

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Step 1:

Find the general form of the integers.

Integers divisible by 7 have the form 7n, where n is a positive integer.

Integers that leave a remainder of 3 when divided by 5 have the form 5m + 35m + 3, where m is a non-negative integer.

Equate the two expressions to find the general form: 7n = 5m + 3

Solve for n in terms of m:

n=(5m+3)/7equals the fraction with numerator 5 m plus 3 and denominator 7 end-fraction

n=5m+37

Since n must be an integer, 5m + 3must be divisible by 7.

Find the smallest non-negative integer m that makes 5m + 3 divisible by 7.

If m = 1, 5m + 3 = 8 (not divisible by 7).

If m = 2, 5m + 3 = 13 (not divisible by 7).

If m = 3, 5m + 3 = 18 (not divisible by 7).

If m = 4, 5m + 3 = 23(not divisible by 7).

If m = 5 , 5m + 3 = 28(divisible by 7, n = 4).

Therefore, the smallest integer satisfying the condition is 7n = 7(4) = 28 and m = 5.

The general form of the integer is 7n = 35k +28, where k is a non-negative integer.

Simplify: n = 5k + 4

Substituting back into 7(5k + 4) = 35k + 28

Step 2:

Find the integers less than 1000.

The integers are of the form 35k + 28, where kis a non-negative integer.

We need 35k + 28 < 1000.

Subtract from both sides: 35k < 972

Divide both sides by k < 972/35

k < 27.77

The possible values for k are 0, 1, 2, ..., 270, 1, 2, ..., 27, which are 28 values in total.

There are 28 such integers.

54 There are 2772 ways to express 50 as a sum of distinct positive integers. The problem asks for the number of partitions of 50 into distinct parts. This is equivalent to the partition function p(n) with the additional constraint of distinctness. The partition function, p(n), counts the number of ways

to write n as a sum of positive integers, where the order of the summands doesn't matter. The distinct partition function, q(n), counts the number of partitions of n into distinct parts. There is no simple closed-form formula for these functions, but they can be calculated using generating functions or recurrence relations. The number of ways to express 50 as a sum of distinct positive integers is given by q (50).

Using generating functions, the generating function for q (n) is:

$\prod_{k=1}^{\infty}(1+x^k)$ =1+x+x^2+2x^3+2x^4+3x^5+...

The coefficient of in this expansion gives q(n). For n=50, q(50) = 2772.

55 Solution :

Let's consider primes less than 100:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

The primes that can be expressed as the sum of two squares are:

$2 = 1^2 + 1^2$

$5 = 1^2 + 2^2$

$13 = 2^2 + 3^2$

$17 = 1^2 + 4^2$

$29 = 2^2 + 5^2$

$37 = 1^2 + 6^2$

$41 = 4^2 + 5^2$

$53 = 2^2 + 7^2$

$61 = 5^2+ 6^2$

$73 = 3^2 + 8^2$

$89 = 5^2 + 8^2$

$97 = 4^2 + 9^2$

There are no prime numbers less than 100 that can be written as the sum of two squares in two different ways.

56 What's given in the problem

We need to find 7^{2023} (mod 100)

Helpful Information

Euler's Totient Theorem states that if gcd(a,n)=1, then $a^{\varphi(n)} \equiv 1(mod\ n)$.

The Euler's Totient function $\varphi(n)$ counts the number of positive integers up to a given n that are relative prime to n.

For $n = {p_1}^{k_1}{p_2}^{k_2}{p_3}^{k_3} \dots \dots \dots \dots \dots \dots \dots {p_r}^{k_r}, \varphi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right) \dots \left(1-\frac{1}{p_r}\right)$

How to Solve

We will use Euler's Totient Theorem to simplify the exponent and then calculate the remainder.

Sep 1. Calculate $\varphi(100)$.

$\varphi(100) = \varphi(2^2.5^2)$.

$\varphi(100) = 100.\left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right)$

$\varphi(100) = 100.\left(\frac{1}{2}\right).\left(\frac{4}{5}\right) = 40$

Sep 2. Apply Euler's Totient Theorem.

Since gcd (7, 100)=1, we have $7^{\varphi(100)} \equiv 1(mod\ 100)$ or $7^{40} \equiv 1(mod\ 100)$

Step 3.

Reduce the exponent 2023 modulo 40.

$2023 \div 40 = 50\ with\ a\ remainder\ of\ 23.$

$2023 = 40.50 + 23$

$So, 2023 \equiv 23\ (mod\ 40)$
Step 4.
Calculate 7^{2023} (mod 100)
$7^{2023} \equiv 7^{40.50+23}(mod\ 100)$
$7^{2023} \equiv (7^{40})^{50}.7^{23}(mod\ 100)$
$7^{2023} \equiv 1^{50}.7^{23}(mod\ 100)$
$7^{2023} \equiv 7^{23}(mod\ 100)$
Step 5.
Calculate 7^{23} (mod 100)
$7^1 \equiv 7(mod\ 100)$
$7^2 \equiv 49(mod\ 100)$
$7^3 \equiv 343 \equiv 43(mod\ 100)$
$7^4 \equiv 7.43 = 301 \equiv 1(mod\ 100)$
$7^4 \equiv 7.43 = 301 \equiv 1(mod\ 100)$
$Since\ 7^4 \equiv 1(mod\ 100), we\ can\ simplify\ \ 7^{23}$
$7^{23} \equiv 7^{4.5+3}(mod\ 100)$
$7^{23} \equiv (7^4)^5.7^3(mod\ 100)$
$7^{23} \equiv 1^5.7^3(mod\ 100)$
$7^{23} \equiv 7^3(mod\ 100)$
$7^{23} \equiv 43(mod\ 100)$
Therefore, the remainder when 7^{2023} is divided by 100 is 43.

57 Similarly, as per previous question, we can answer.

The remainder when 3^{100} is divided by 100 is 1.

58 Solution:
What you're solving for
You are finding the number of integers between 1 and 1000(inclusive) that are not divisible by 2,3, or 5.
What's given in the problem
The range of integers is from 1 to 100.
We need to exclude numbers divisible by 2,3, or 5.
Helpful information
The Principle of Inclusion-Exclusion helps count elements in the union of sets.
The number of integers divisible by n in a range up to N is $\left\lfloor \frac{N}{n} \right\rfloor$
How to solve
Calculate the total numbers divisible by 2,3, or 5 using the Principle of Inclusion-Exclusion, then subtract this from the total number of integers.
Step 1.
Calculate the number of integers divisible by 2,3, and 5 individually.
Let N_2 be the count of numbers divisible by 2: $N_2 = \left\lfloor \frac{N}{n} \right\rfloor = 500$
Let N_3 be the count of numbers divisible by 3: $N_3 = \left\lfloor \frac{N}{n} \right\rfloor = 333$
Let N_5 be the count of numbers divisible by 5: $N_5 = \left\lfloor \frac{N}{n} \right\rfloor = 200$
Step 2.
Calculate the number of integers divisible by pairwise products.
$N_{2\cap 3} = \left\lfloor \frac{1000}{2\times 3} \right\rfloor = \left\lfloor \frac{1000}{6} \right\rfloor = 166$

$N_{3\cap 5} = \left\lfloor \frac{1000}{3\times 5} \right\rfloor = \left\lfloor \frac{1000}{15} \right\rfloor = 66$

$N_{2\cap 5} = \left\lfloor \frac{1000}{2\times 5} \right\rfloor = \left\lfloor \frac{1000}{10} \right\rfloor = 100$

Step 3.

Calculate the number of integers divisible by the product of all three.

$N_{2\cap 5} = \left\lfloor \frac{1000}{2\times 3\times 5} \right\rfloor = \left\lfloor \frac{1000}{30} \right\rfloor = 33$

Step 4.

Apply the Principle of Inclusion-Exclusion to find numbers divisible by at least one of 2,3, or 5.

$N_{divisible} = N_2 + N_3 + N_5 - (N_{2\cap 3} + N_{3\cap 5} + N_{2\cap 5}) + N_{2\cap 3\cap 5}$

$N_{divisible} = 500 + 333 + 200 - (166 + 66 + 100) + 33$

$N_{divisible} = 1033 - 332 + 33 = 734$

Step 5.

Subtract the count of divisible numbers from the total number of integers.

Total integers from 1 to 1000 is 1000.

Numbers not divisible by 2,3, or 5=$1000 - N_{divisible}$

Numbers not divisible= $1000 - 734 = 266$.

Thus, there are 266 integers between 1 and 1000 that are divisible by neither 2,3, nor 5.

59 Solution:

What's given in the problem

The number must be a 3-digit number.

The number leaves a remainder of 1 when divided by 2, 3, 4, 5, and 6.

Helpful information

A number leaving a remainder of 1 when divided by multiple numbers means it is 1 more than a multiple of their Least Common Multiple (LCM).

How to solve

Find the Least Common Multiple (LCM) of the divisors, then find the largest multiple of the LCM that is less than the greatest 3-digit number, and finally add 1 to it.

Step1.

Find the LCM of 2,3,4,5, and 6.

Prime factorization of 2 is 2.

Prime factorization of 3 is 3.

Prime factorization of 4 is 2^2.

Prime factorization of 5 is 5.

Prime factorization of 6 is 2×3.

$LCM = 2^2 \times 3 \times 5 = 4 \times 3 \times 5 = 60.$

Step2.

Find the largest multiple of 60 that is a 3-digit number.

The greatest 3-digit number is 999.

Divide 999 by 60: $\frac{999}{60} \approx 16.65$

The largest integer multiple is 16.

Largest multiple of 60 less than 999 is 60 x 16 =960.

Step3.

Add the remainder to the multiple.

The required number is 960 + 1=961

Solution

The greatest 3-digit number that leaves a remainder 1 when divided by 2,3,4,5, and 6 is 961.

60 The pairs of integers (a, b) that satisfy the equation ab + 2a + 3b = 12 are (3, -1), (0, 4), (-5, 3), and (-8, 2).

Explanation

Rearrange the equation: Start by rearranging the given equation to factor by grouping.

ab + 2a + 3b = 12

ab + 2a + 3b + 6 = 12 + 6

a(b + 2) + 3(b + 2) = 18

(a + 3)(b + 2) = 18

Find all pairs of integers that multiply to 18. These pairs will represent the possible values for (a+3) and (b+2).

The factor pairs of 18 are: (1, 18), (2, 9), (3, 6), (6, 3), (9, 2), (18, 1), (-1, -18), (-2, -9), (-3, -6), (-6, -3), (-9, -2), (-18, -1).

For each factor pair (x, y) where (a + 3) = x and (b + 2) = y, solve for a and b:

If a + 3 = 1 and b + 2 = 18, then a = -2 and b = 16.
If a + 3 = 2 and b + 2 = 9, then a = -1 and b = 7.
If a + 3 = 3 and b + 2 = 6, then a = 0 and b = 4.
If a + 3 = 6 and b + 2 = 3, then a = 3 and b = 1.
If a + 3 = 9 and b + 2 = 2, then a = 6 and b = 0.
If a + 3 = 18 and b + 2 = 1, then a = 15 and b = -1.
If a + 3 = -1 and b + 2 = -18, then a = -4 and b = -20.
If a + 3 = -2 and b + 2 = -9, then a = -5 and b = -11.
If a + 3 = -3 and b + 2 = -6, then a = -6 and b = -8.
If a + 3 = -6 and b + 2 = -3, then a = -9 and b = -5.
If a + 3 = -9 and b + 2 = -2, then a = -12 and b = -4.
If a + 3 = -18 and b + 2 = -1, then a = -21 and b = -3
From the above solutions, the integer pairs are: (-2, 16), (-1, 7), (0, 4), (3, 1), (6, 0), (15, -1), (-4, -20), (-5, -11), (-6, -8), (-9, -5), (-12, -4), (-21, -3).

Therefore, the integer pairs (a, b) that satisfy the equation ab + 2a + 3b = 12 are:
(3, -1), (0, 4), (-5, 3), (-8, 2)

61 This is a classic application of the Pigeonhole Principle. Imagine the colours (red, blue, green) as the pigeonholes. If you draw three socks, it's possible to have one of each colours. However, if you draw a fourth sock, it must match one of the colours already drawn, creating a pair. Therefore, the answer is 4.

62 Write each integer in the form 2^k x m, where m is odd. There are 50 odd numbers between 1 and 100: {1, 3, 5, ..., 99}. Since we have 51 integers, by the Pigeonhole Principle, at least two must have the same odd part (m). Let these integers be 2^a x m and 2^b x m. If a > b, then 2^b x m divides 2^a x m, so one divides the other. If b > a, then 2^a x m divides 2^b x m.

63 This is a classic application of the Pigeonhole Principle combined with a bit of strategy. Let's assume, without loss of generality, that we have ordered the players in a sequence such that player p_1 defeated p_2 who defeated p_3 and so on, until p_k and let p_k be the last player in that

sequence who defeated another player p_{k+1}. This means p_k is defeated by all the players before p_k in that sequence, and p_k defeated all players from p_{k+1} to p_k.

64 Consider the remainders when each integer is divided by 10. There are 10 possible remainders: 0, 1, 2, ..., 9. By PHP, among 11 integers, there exist two integers a and b with the same remainder when divided by 10. Let a = 10q + r and b = 10p + r, where q and p are integers. Then, a - b = 10(q - p), which is divisible by 10.

65 There are 2^{11} = 2048 possible subsets of S. The sum of elements in each subset ranges from 0 (empty set) to the sum of the 11 integers. Notice that the number of possible sums is less than 2048. By PHP, there exist two subsets A and B with the same sum. If $A \cap B \neq \emptyset$, remove the intersection to get two disjoint subsets with the same sum.

66 $(xy - 7)^2 = x^2 + y^2$

$(xy - 6)^2 + 13 = (x + y)^2$

$(xy - 6)^2 - (x + y)^2 = 13$

$(x + y + xy - 6)(x + y - xy + 6) = 13$

$x + y + xy - 6 = 13, 1, -1, -13$

$x + y - xy + 6 = 1, 13, -13, -1$

Adding : $2(x + y) = 14, 14, -14, -14$

$x + y = 7, 7, -7, -7$

Subtracting : $2xy - 12 = 12, -12, 12, -12$

$xy = 12, 0, 12, 0$

Solving : (x , y) = (3, 4), (4, 3), (0, 7), (7, 0), (-3, -4), (-4, -3), (0, -7), (-7, 0)

67 p/q = (1 + 1/3 + 1/5 + … + 1/1319) – (1/2 + ¼ + 1/6 + … + 1/1318)

= (1 + ½ + 1/3 + … + 1/1319) – 2(1/2 + ¼ + … + 1318)

= (1 + ½ + 1/3 + … + 1/1319) – ((1 + ½ + 1/3 + … + 1/659)

= 1/660 + 1/661 + … + 1/1318 + 1/1319

= (1/660 + 1/1319) + (1/661 + 1/1318) + … + (1/989 + 1/990)

$= \sum_{k=660}^{989}\left(\frac{1}{k} + \frac{1}{1979-k}\right)$

$= \sum_{k=660}^{989} \frac{1979}{k(1979-k)}$

$= 1979 \sum_{k=660}^{989} \frac{1}{k(1979-k)}$

= 1979 a/b, for some a, b ∈ N with (a, b) = 1

It is true that 1979 does not divide,

So, p must be divisible by 1979

68 Wilson's theorem: For integers p > 0, (p - 1)! $\equiv -1 \ (mod \ p)$

This implies p is prime.

(712)! $\equiv -1 \ (mod \ 713)$

Now, 712 = 23 x 31

714 is even

715 = 5 x 11 x 13

716 is even

717 = 3 x 239

718 is even

From above, 714, 715, 716, 717, 718 are not prime

719 is prime

Using Wilson theorem, (718)! $\equiv -1 \ (mod \ 719)$

(712!)(713)(714)(715)(716)(717)(718) $\equiv -1 \ (mod \ 719)$

(712!)(-6)(-5)(-4)(-3)(-2)(-1) $\equiv -1 \ (mod \ 719)$

(712!) 720 $\equiv -1 \ (mod \ 719)$

(712!) (1) $\equiv -1 \ (mod \ 719)$

(712!) + 1 $\equiv 0 \ (mod \ 719)$

So, 712! + 1 is not a prime.

69 (a - 1)(b - 1)(c - 1) = abc – ab – bc – ca + a + b + c – 1

Notice $\frac{abc-1}{(a-1)(b-1)(c-1)} < \frac{abc}{(a-1)(b-1)(c-1)}$ = $(\frac{a}{a-1})(\frac{b}{b-1})(\frac{c}{c-1})$ < 2(3/2)(4/3) = 4

[$\therefore \frac{a}{a-1}$ = 1 + $\frac{1}{a-1} \leq 1 + \frac{1}{2-1} = 2 \quad as \ 1 < a < b < c \ so, a \geq 2, b \geq 3, c \geq 4$]

$\frac{abc-1}{(a-1)(b-1)(c-1)}$ = 2 or 3

Notice that if a $\geq 4, then \ \frac{abc-1}{(a-1)(b-1)(c-1)} <$ (4/3)(5/4)(6/3) = 2

So, a ≥ 4 can not hold

i.e. a = 2 or 3

When a = 2, $\frac{2bc-1}{(b-1)(c-1)}$ = 2 or 3

2bc – 1 = 2bc – 2b – 2c + 2

2bc – 1 = 3bc – 3b – 3c + 3

bc – 3b – 3c + 9 = 5

(b - 3)(c - 3) = 5

b – 3 = 1, c – 3 = 5

b = 4, c = 8

(a, b, c) = (2, 4, 8)

When a = 3, $\frac{2bc-1}{(b-1)(c-1)}$ = 2 or 3

3bc – 1 = 4bc -4b – 4c + 4 or 3bc – 1 = 6bc – 6b – 6c + 6

bc – 4b – 4c + 5 = 0

(b - 4)(c - 4) = 11

b – 4 = 1, c – 4 = 11

b = 5, c = 15

70 k= 1 : $\frac{1}{1^2} = 1 \leq 2 - \frac{1}{1} = 1$

For n = k , $\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$, the statement is true

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k}\,\frac{1}{(k+1)^2}$$

Since k ≥ 1, then $k^2 + 2k < k^2 + 2k + 1$

$\frac{k^2+2k}{(k+1)^2} < 1$

$\frac{k(k+2)}{(k+1)^2} < 1$

$\frac{(k+1)+1}{(k+1)^2} < 1/k$

$$\frac{1}{k+1} + \frac{1}{(k+1)^2} < \frac{1}{k}$$

$$\frac{-1}{k} + \frac{1}{(k+1)^2} < -\frac{1}{k+1}$$

$2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$

$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{k^2}\,\frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$

So, the statement is also true for n = k + 1

71 Let T(n) : $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

T(1) : $\frac{1}{2} = 2 - \frac{1+2}{2} = \frac{1}{2}$, which is true

Let T(k) : $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$ be true.

Now, $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{k}{2^k}+\frac{k+1}{2^{k+1}}=2-\frac{k+2}{2^k}+\frac{k+1}{2^{k+1}}$

$= 2 - \frac{-(2k+4)+(k+1)}{2^{k+1}}$

$= 2 - \frac{(k+1)+2}{2^{k+1}}$

So, T(k + 1) is also true.

72 Let T(n) : $\frac{(2n)!}{2^n n!}$ be an integer

T(0) : $\frac{(0)!}{2^0 0!}$ = 1, an integer

Let T(k) : $\frac{(2k)!}{2^k k!}$ be an integer

Now, $\frac{(2(k+1))!}{2^{k+1}(k+1)!} = \frac{(2k)!(2k+2)(2k+1)}{2^k(k)!2(k+1)}$ = integer. (2k+1) = integer

So, T(k+1) is also true.

73 g + l = a + b [∴ gl = ab]

g + ab/g = a + b

$g^2 - (a + b)g + ab = 0$

(g - a)(g - b) = 0 →So, g = a, b

For a = b, there will be 21 cases.

If a = 10, b may be 20 or 30 as well and vice versa

a = 11, b = 22 two ways

a = 12, b = 24 two ways

a = 13, b = 26 two ways

a = 14, b = 28 two ways

a = 15, b = 30 two ways

Total = 21 + 4 + 10 = 35 ways.

74 N = 55 x 60 x 65= 5 x 11 x 5 x 12 x 5 x 13

= 5 x 11 x 13 x 15 x 20

So, the least value of largest of these integers = 20

75 Let M = $2^5.3^6.4^3.5^3.6^7 = 2^5.3^6.2^6.5^3.2^7.3^7 = 2^{18}.\ 3^{13}.5^3$

For M to be a perfect square, M should be multiplied by 3x5 = 15

76 $\frac{3^{560}}{8} = \frac{(3^2)^{280}}{8} = \frac{9^{280}}{8} = \frac{[(9.9.9...280\ times)]}{8}$

Remainder for the above expression = $\frac{[remainder\ for\ (1.1.1\ ...\ 280\ times)]}{8}$ = 1

77 $1 < 2^1$, $2 < 2^2$, $3 < 2^3$

Let $k \geq 4$ and assume $a_k < 2^k$, k = 4, 5, … , k

Then $a_{k+1} = a_k + a_{k-1} + a_{k-2}$

$< 2^k + 2^{k-1} + 2^{k-2}$

$< 2^k + 2^{k-1} + 2^{k-1}$

$= 2^k + 2.2^{k-1} = 2^k + 2^k = 2.2^k = 2^{k+1}$

$\therefore$ $a_{k+1} < 2^{k+1}$

78 Consider, $N = n_1 + n_2 + ... + n_p$, where N is even

Let ‘k’ numbers among ‘p’ members be odd.

Then (p – k) numbers, among them will be even.

For N to be even, ‘k’ odd numbers also must be even

This implies k is even.

But k is odd (given) which is a contradiction.

Hence the number of odd integers among them cannot be odd.

79 $4x^3 - 7y^3 = 2010$

Now, 2010 = 7 x 287 + 1

$4x^3 - 7y^3 = 1 + 7 \times 287$

$4x^3 - 1 = 7(y^3 + 287)$

As x, y are integers

$x^3 \equiv 0, 1, 6 \pmod 7$

$4x^3 - 1 \not\equiv 0 \pmod 7$

Now, $4x^3 - 1 = 7(y^3 + 287)$

7 don’t divide LHS while 7 divides RHS, which is a contraction.

So, the equation has no solution in integers.

80 Let ‘ab’ be the 2 – digit number, where $a \neq 0$ *and* $b \neq o$

Now, ba = ab + 11

10b + a = 10a + b + 11

10(b – a) + (a – b) = 11

11(b – a)- (b – a) + (a - b) = 11

11(b – a) + 2(a – b) = 11

11(b – a) is divisible by 11

So, 2(a – b) is divisible by 11

(a – b) is also divisible by 11

But, this is not possible, as a, b are digits.

The only possibility is a = b = 0

By reversing the digits, we get only the same number. But a and b are distinct.

Thus, there are numbers satisfying the given conditions.

81 (a + b) + (a - b) + ab + a/b = 36

2a + ab + a/b = 36

$2ab + ab^2 + a = 36b$

$a(b^2 + 2b + 1) = 36b$

$a = 36b/(b + 1)^2$

So $(b + 1)^2$ divides 36

b + 1 divides 6

b + 1 = 2 or 3 or 6

b = 1, 2, 5 then a = 9, 8, 5

Since a > b, (a, b) = (9, 1), (8, 2) only.

82 Let $\sqrt{3\sqrt[3]{3}}$ = d, a rational

$(\sqrt{3\sqrt[3]{3}})^2 = d^2$

$3 \times 3^{2/3} = d^2$, which is rational

$3^{2/3}$ is rational

But $\sqrt[3]{8} = 2$ and $\sqrt[3]{27} = 3$

$3^{2/3} = \sqrt[3]{9}$ is irrational

So, $\sqrt{3}\sqrt[3]{3}$ is irrational.

83 $107^{90} - 76^{90} = (107^2)^{45} - (76^2)^{45}$ is divisible by $107^2 - 76^2 = (107 + 76)(107 - 76) = 183 \times 31 = 1891$

84 $30^{99} \equiv (-1)^{99} \equiv (-1)(\text{mod } 31)$

$61^{100} \equiv (-1)^{100} \equiv (-1)(\text{mod } 31)$

$30^{99} + 61^{100} \equiv (-1 + 1)(\text{mod } 31)$

$30^{99} + 61^{100} \equiv 0 \ (\text{mod } 31)$

So, $30^{99} + 61^{100}$ is divisible by 31.

85 $2y^2$ is even and $2y^2 \geq 0$

$(x - y)^2$ is odd as RHS is odd

$(x - y)^2 = 1$ or 9 or 25

$2y^2 = 27 - (x - y)^2$

$2y^2 = 27 - 1, 27 - 9, 27 - 25$

$2y^2 = 26, 18, 2$

$Y^2 = 13, 9, 1$

If $(x - y)^2 = 1$, $x - y = \pm 3$ *and* $y = \pm 3$

If $(x - y)^2 = 25$, $x - y = \pm 5$ *and* $y = \pm 1$

Thus solutions are (0,3),(6, 3), (0, -3), (-6, -3), (6, 1), (-4, 1), (-6, -1), (4, -1).

86 Solution: $\sqrt{6 + 2\sqrt{2} + 2\sqrt{3} + 3\sqrt{6}}$

$= \sqrt{3 + 2 + 1 + 2\sqrt{2} + 2\sqrt{3} + 3\sqrt{6}}$

$= \sqrt{(\sqrt{3} + \sqrt{2} + 1)^2}$

$= \sqrt{3} + \sqrt{2} + 1$

$$\frac{1}{\sqrt{5 - 2\sqrt{6}}}$$

$= \frac{\sqrt{5+2\sqrt{6}}}{1}$

$= \sqrt{3} + \sqrt{2}$

Thus $\sqrt{3} + \sqrt{2} + 1 - (\sqrt{3} + \sqrt{2}) = 1$, a rational number.

87 $2336 = 32 \times 73 = 2^5 \times 73$

$2^x + 2^y + 2^z = 2^5 \times 73$

$2^{x-5} + 2^{y-5} + 2^{z-5} = 73$

RHS is odd and LHS is even

Let $2^{x-5} = 1$

$x - 5 = 0$

$x = 5$

$1 + 2^{y-5} + 2^{z-5} = 73$

$2^{y-5} + 2^{z-5} = 72 = 2^3 \times 3^2$

$2^{y-8} + 2^{z-8} = 9$

Again RHS is odd and LHS is even

$2^{y-8} = 1$

$y - 8 = 0$

$y = 8$

$1 + 2^{z-5} = 9$

$z = 8$

Since the equation is symmetric about x, y, z, the solutions are given by

(x, y, z) = (5, 8, 11) , (5, 11, 8) , (8, 5, 11) , (8, 11, 5) , (11, 5, 8) , (11, 8, 5)

Thus, there are six solutions to this problem.

88 $x^2 + y^2 = 2000$, $x < y$

$y^2 < 2000 < 45^2$

$y < 45$

If $y \leq 31$,

$x < y \leq 31$

$x^2 + y^2 \leq 2(31)^2 = 1922 < 2000$

$y > 31$

Thus $31 < y < 45$

89 $x^2 + y^2 = 2000$

If one of x, y is even and other is odd, then

$x^2 + y^2$ = even + odd

While RHS is even.

So, this is not possible.

90 Let x = 2m, y = 2n for some m, n

So, $m^2 + n^2 = 500$

RHS is a multiple of 4 and m and n themselves must be even.

y must be a multiple of 4.

91 $(a+b-5)^2+(b+2c+3)^2+(c+3a-10)^2 = 0$
Sum of squares = 0 hence each expression in square is 0
a+b-5 = 0
=> a + b = 5
b+2c+3 = 0
=> b + 2c = 3
c+3a-10 = 0
=> c + 3a = 10
=> c = 10 - 3a
b + 2c = 3
=> b + 2(10 - 3a) = 3
=> b + 20 - 6a = 3
=> 6a - b = 17
a + b = 5
6a - b = 17
=> 7a = 22
=> a = 22/7
b = 13/7
c = 4/7
$a^3+b^3+c^3 = (22/7)^3 + (13/7)^3 + (4/7)^3$
$= (22^3 + 12^3 + 4^3)/7^3$
= (10648 + 1728 + 64)/343
= 12440/343
= 36.27
≈36
integer nearest to **$a^3+b^3+c^3$ = 36**

92 highest power of 7 dividing n! is 8.

=> $[n/7] + [n/7^2] + [n/7^3]$ +..+. .+ ..+ .. + = 8

$7^2 = 49$ if n < 49 then

We get only [n/7] where n < 49

Hence [n/7] < 7

We need highest power = 8

Lets check for 49

= [49/7] + [$49/7^2$] + [$49/7^3$] + ..+ ..+ ..+ .. +

= 7 + 1 + 0 + 0 +…

= 8

Hence n = 49 Satisfy this

highest power of 7 dividing 49! is 8

check for 55

$= [55/7] + [55/7^2] + [55/7^3] +.+.+.+.+$…

= 7 + 1 + 0 + 0 +…

= 8

check for 56 $= [56/7] + [56/7^2] + [55/7^3] +.+.+.+.+$ = 8 + 1 + 0 + 0 + = 9

Hence 56! has power of 7 more than 8

From 1! to 55! power of 7 is max 8

from 49! to 55! power of 7 is exactly 8

93 Let the nine-digit number be N. We can write N as:

$$N = \overline{abcabcabc}$$

We can expand this number as follows:

$$N = a \times 100000000 + b \times 10000000 + c \times 1000000 + a \times 100000 + b \times 10000 + c \times 1000 + a \times 100 + b \times 10 + c$$

Grouping the terms with the same digits:

$$N = a \times (100000000 + 100000 + 100) + b \times (10000000 + 10000 + 10) + c \times (1000000 + 1000 + 1)$$

Factor out abc:

$N = (a \times 100000 + b \times 10000 + c \times 1000) * 1000 + (a \times 100 + b \times 10 + c)$ -- This is incorrect for the initial grouping.

Let's use a simpler algebraic approach:

$$abcabcabc = abc \times 1000000 + abc \times 1000 + abc \times 1$$
$$abcabcabc = abc \times (1000000 + 1000 + 1)$$
$$abcabcabc = abc \times 1001001$$

Therefore, when abcabcabc is divided by 1001001, the quotient is simply abc.

94 A number n can be expressed as $n = 100 \times q + r$, where q is the quotient and r is the remainder when n is divided by 100.

Since $n = \overline{abcde}$, we can write $n = 10000a + 1000b + 100c + 10d + e$.

We can factor out 100 from the first three terms:

$n = 100(100a + 10b + c) + 10d + e.$

Therefore, when n is divided by 100, the remainder is $10d + e$, which is represented by de.

95 we set up the equations: a + b = k^2, a + c = m^2, and b + c = n^2.

Solving for a,b,c in terms of k^2,m^2,n^2 , we get:

$$a = \frac{k^2 + m^2 - n^2}{2}$$

$$b = \frac{k^2 - m^2 + n^2}{2}$$

$$c = \frac{-k^2 + m^2 + n^2}{2}$$

We need a, b, c to be distinct positive integers, and we are looking for the minimal sum a+b+c . By trial and error and checking combinations of perfect squares, the combination that yields distinct positive integers with the minimal sum is found by setting k^2=49, m^2=16, and n^2=25.

However, this leads to non-positive values for some variables. A better approach is to look for perfect squares that satisfy the conditions after some manipulation. The smallest set of distinct positive integers that satisfies the conditions are a=5, b=20, and c=44, leading to the sum a+b+c =69.

96 Case 1: N = 3

If N = 3, then N+2 = 5, and N+4 = 7. All three numbers (3, 5, 7) are prime numbers, so N=3 is a valid solution.

Case 2: N > 3

If N is any prime number greater than 3, then N can be expressed in one of two forms: 3k + 1 or 3k + 2.

If N = 3k + 1:

Then N+2 = (3k + 1) + 2 = 3k + 3 = 3(k+1). Since N+2 is a multiple of 3 and greater than 3 (as N > 3), N+2 cannot be a prime number.

If N = 3k + 2:

Then N+4 = (3k + 2) + 4 = 3k + 6 = 3(k+2). Since N+4 is a multiple of 3 and greater than 3 (as N > 3), N+4 cannot be a prime number.

Therefore, for any prime N greater than 3, at least one of (N+2) or (N+4) will be divisible by 3 and thus not prime, meaning N=3 is the only possible.

97 Factor the expression:

The expression $n^2 + 2n - 8$ can be factored as (n + 4) (n - 2). For a product to be prime:

For the product of two integers (n + 4) and (n - 2) to be a prime number, one of the factors must be 1 and the other must be the prime number itself.

Analyse the factors:

If n - 2 = 1, then n = 3. In this case, (n + 4) = (3 + 4) = 7. So, the product is 1 × 7 = 7, which is a prime number.

If n + 4 = 1, then n = -3. This is not a natural number, so it's not a valid solution.

98 Observe that 62^{48}can be written as $2^{48} * 31^{48}$

Or last two digits of 62^{48} = last two digits of 2^{48} × last two digits of 31^{48}

Last two digits of2^{48} =$(2^{10})^4 \times 2^8$

≡244×28

≡76×56≡56

And

Last two digits of 31^{48}≡41

Therefore, the last two digits of 62^{48}=56×41=96.

99 A number is divisible by 9 if the sum of its digits is divisible by 9.
The sum of the digits of 1x6y7 is 1 + x + 6 + y + 7 = 14 + x + y.
For 1x6y7 to be divisible by 9, 14 + x + y must be a multiple of 9.
Since x and y are single digits, the minimum value of x + y is 0 (when x=0, y=0) and the maximum value is 18 (when x=9, y=9).
Therefore, 14 + x + y can range from 14 (when x+y=0) to 32 (when x+y=18).
The multiples of 9 within this range are 18 and 27.

Case 1: 14 + x + y = 18
This implies x + y = 4. The possible ordered pairs (x, y) are (0, 4), (1, 3), (2, 2), (3, 1), (4, 0). This gives 5 pairs.

Case 2: 14 + x + y = 27
This implies x + y = 13. The possible ordered pairs (x, y) are (4, 9), (5, 8), (6, 7), (7, 6), (8, 5), (9, 4). This gives 6 pairs.

Total number of ordered pairs (x, y) is the sum of the pairs from both cases: 5 + 6 = 11.

100 The number 30a0b03 can be written as 3000000 + 100000a + 1000b + 30.
When divided by 13, the expression becomes:
30a0b03 ≡4 + 4a + 12b + 4 (mod 13).
Simplifying, we get:

30a0b03≡ 8 + 4a + 12b (mod 13).
Since 12b ≡ -b (mod 13), we have:
30a0b03 ≡ 8 + 4a - b (mod 13).
For the number to be divisible by 13, we need:
8 + 4a – b ≡ 0 (mod 13).
This can be rewritten as:
4a – b ≡ -8 ≡5 (mod 13).

Now, we need to find pairs of digits (a, b) where a and b are from 0 to 9 that satisfy this congruence.
For example:
If a = 0, then -b ≡5 (mod 13), so b ≡-5≡ 8 (mod 13). Thus, b = 8. So, (0, 8)
If a = 1, then 4 - b ≡5 (mod 13), so -b ≡1 (mod 13), b ≡-1 ≡12 (not a digit).
If a = 2, then 8 - b ≡5 (mod 13), so -b ≡-3, b ≡3 (mod 13). Thus, b = 3. (2, 3)
If a = 3, then 12 - b ≡5 (mod 13), so -b ≡-7, b≡7 (mod 13). Thus, b = 7. (3, 7)
If a = 4, then 16 - b ≡5 (mod 13), so 3 - b ≡5 (mod 13), -b ≡2, b ≡-2 ≡11 (not a digit).
If a = 5, then 20 - b ≡5 (mod 13), so 7 - b ≡5 (mod 13), -b ≡-2, b ≡2 (mod 13). Thus, b = 2. (5, 2)
If a = 6, then 24 - b ≡5 (mod 13), so 11 - b ≡5 (mod 13), -b ≡-6, b ≡6 (mod 13). Thus, b = 6. (6, 6)
If a = 7, then 28 - b ≡5 (mod 13), so 2 - b ≡5 (mod 13), -b ≡3, b ≡-3 ≡10 (not a digit).
If a = 8, then 32 - b ≡5 (mod 13), so 6 - b≡ 5 (mod 13), -b ≡-1, b ≡1 (mod 13). Thus, b = 1. (8, 1)
If a = 9, then 36 - b ≡5 (mod 13), so 10 - b ≡5 (mod 13), -b ≡-5, b ≡5 (mod 13). Thus, b = 5. (9, 5)

101 Identify the modulus: To find the last two digits, we need to calculate the value modulo 100.

Apply Euler's Totient Theorem: Euler's totient function, φ(n), counts the number of positive integers up to n that are relatively prime to n.

For n=100, $\varphi(100)=\varphi(2^2 . 5^5)=\varphi(2^2)\cdot\varphi(5^2)=(2^2-2^1)(5^2-5^1)=(4-2)(25-5)=2*20=40$

. Euler's theorem states that if a and $n$are relatively prime, then

$a^{\phi(n)} \equiv 1 (\text{mod} n)$. In our case, $3^{40} \equiv 1 (\text{mod} 100)$.

Simplify the exponent: We need to find the remainder of the exponent 2012 when divided by 40: 2012=40×50+12.

Calculate the power: Therefore, $3^{2012}=3^{40\times50+12}=(3^{40})^{50}*3^{12}$.

Substitute the congruence: Since $3^{40} \equiv 1 (\text{mod} 100)$

, we have $(3^{40})^{50} \equiv 1^{50} \equiv 1 (\text{mod} 100)$.

Final Calculation: So, $3^{2012} \equiv 1\cdot3^{12} (\text{mod} 100)$

. Now we need to calculate $3^{12} (\text{mod} 100)$:

$3^1 \equiv 3 (\text{mod} 100)$

$_3{}^2 \equiv 9 (\text{mod} 100)$

$3^3 \equiv 27 (mod100)$

$34 \equiv 81 (mod100)$

$3^8 = (3^4)^2 \equiv 81^2 = 6561 \equiv 61 (mod100)$

$_3{}^{12} = 3^8 \cdot 3^4 \equiv 61 \cdot 81 = 4941 \equiv 41 (mod100)$

Therefore, the last two digits of 3^{2012} are 41.

102 $2^2 - 1 = 3$.If for a positive integer n the number$n^2 - 1$

is a product of three different primes, then n>2

$n^2 - 1 = (n-1)(n+1)$. The number n must be even since otherwise

(n -1) (n + 1) will be divisible by 4. Moreover, the numbers n - 1>1

andn+1>1since n>2can not be both composite. Since in this

case (n - 1) (n+ 1) could not be a product of three different primes.

Thus, one of n - 1 and n + 1 must be a prime and the other one

must be a product of two primes. For n = 4, we get n - 1 = 3, n + 1 = 5

and this condition is not satisfied.

Similarly for n = 6, we get n - 1 = 5 and n + 1 = 7;

for n = 8, we have n -1 = 7, n + 1 = $9 = 3^2$.

For n = 10, we have n - 1= $9 = 3^2$ and

for n = 12 we have n - 1 = 11, n + 1 = 13.

For n = 14, we have n -1 = 13, n + 1 = 15 = (3)(5).

thus, the least positive integer n for $n^2 - 1$ is a product of three different primes is n = 14 for which$n^2 - 1 = 3 \times 5 \times 13$.

Then $16^2 - 1 = 3 \times 5 \times 17$

similarly$20^2 - 1 = 3 \times 7 \times 19$

$22^2 - 1 = 3 \times 7 \times 23$

Continuing in this way we find easily the fifth number to be n = 32

for which$32^2 - 1 = 2 \times 11 \times 31$.

Thus the first five integers n for which$n^2 - 1$ is a product of three

different primes are 14,16,20,22and32

103 $$55 \times 60 \times 65$$

$$N = 5 \times 11 \times 12 \times 5 \times 13 \times 5$$

$$N = 5 \times 11 \times 13 \times 15 \times 20$$

So, least value of largest of these integers=20

104 Given a, b ∈ N is a 3digit number,

a + 1 divides b – 1

So, b – 1 = k(a + 1) ; b = k(a + 1) + 1

and b divides $a^2 + a + 2$

So,k(a + 1) + 1 divides $a^2 + a + 2$

⇒ k(a + 1) + 1 divides $k(a^2 + a + 2) – a(k(a + 1) + 1))$

k(a + 1) + 1 divides 2k – a

ka + k + 1 / 2k – a

⇒ a = 2k

ka + k + 1 / 0

b = k(2k + 1) + 1 = $2k^2 + k + 1$ as b is 3digit number

$100 \leq 2k^2 + k + 1 \leq 999$

$$\sqrt{\frac{99}{2} + \frac{1}{16}} \leq k + \frac{1}{4} \leq \sqrt{\frac{998}{2} + \frac{1}{16}}$$

60 ... ≤ k ≤ 220 ...

k ∈ [7, 22]

So, total comes = 16

105 Given a, b, c ∈ Z^+

$\frac{ab}{a-b} = c$

$\Rightarrow \frac{ac}{a+c} = b$

$\frac{1}{b} = \frac{1}{a} + \frac{1}{c}$ .... (1) if a = km, c = kn, then b = kr

From equation (i), we get

$\frac{1}{kr} = \frac{1}{km} + \frac{1}{kn} \Rightarrow \frac{1}{r} = \frac{1}{m} + \frac{1}{n}$ (2)

m = 3, n = 6, r = 2 is one of the solution of the equation (2)

So, a = 3k, c = 6k, b = 2k

Given a + b + c $\le$ 99;

(3 + 6 + 2) k $\le$ 99;

k $\le$ 9

So, a = 27, c = 54 and b = 18

a + b + c = 27 + 54 + 18 = 99

106 Rewrite the equation:

The given equation is $p^2+p+q^2+q=r^2+r$.
This can be rewritten as $p(p+1)+q(q+1)=r(r+1)$.
Notice that $n(n+1)$ is always an even number.

Consider parity:

Since

$p(p+1)$, $q(q+1)$, and $r(r+1)$ are all even, the equation holds true for any prime numbers p, q, r. This doesn't help much with finding specific solutions.

Consider the case when one of the primes is 2:

Case 1: $p=2$

The equation becomes

$2(3)+q(q+1)=r(r+1)$, so $6+q^2+q=r^2+r$.
Rearranging, $r^2-q^2+r-q=6$, which is $(r-q)(r+q)+(r-q)=6$.
Factoring, $(r-q)(r+q+1)=6$.
Since q and r are primes, $r-q$ and $r+q+1$ must be integers.
Possible factor pairs of 6 are (1, 6), (2, 3), (-6, -1), (-3, -2).

If $r-q=1$ and $r+q+1=6$:

From $r-q=1$, $r=q+1$

. Since q and r are primes and differ by 1, the only possibility is

$q=2$, $r=3$.
Substituting into $r+q+1=6$: 3+2+1=6, which is true.
So, (p,q,r) = (2,2,3) is a solution.

If $r-q=2$ and $r+q+1=3$:

Adding the two equations: $2r+1=5 \Longrightarrow 2r=4 \Longrightarrow r=2$.
Then $q=r-2=0$, which is not a prime. So no solution here.

Consider negative factors as well, but since r,q are primes, r+q+1

must be positive. Also, $r-q$ can be negative if q>r.

Case 2: q=2

By symmetry with Case 1, we also get (p,q,r) = (2,2,3) and (p,q,r) = (2,2,3). (This is the same solution as above).

Case 3: r=2

The equation becomes $p(p+1) + q(q+1) = 2(3) = 6$.
Since p, q are primes, the smallest value for $p(p+1)$ or $q(q+1)$

when p, $q \geq 2$is 2(3) =6.
If p=2, then $6+q(q+1)=6 \Longrightarrow q(q+1)=0$, which is not possible for a prime q.
Thus, r cannot be 2.

Consider the case when none of the primes are 2 (i.e., all are odd primes):

If p,q,$r$are all odd primes, then $p(p+1)$, $q(q+1)$, and $r(r+1)$

are all even.
Consider the equation modulo 3.
If $p \equiv 1 \pmod{3}$, $p+1 \equiv 2 \pmod{3}$, so $p(p+1) \equiv 2 \pmod{3}$.
If $p \equiv 2 \pmod{3}$, $p+1 \equiv 0 \pmod{3}$, so $p(p+1) \equiv 0 \pmod{3}$.
If p=3, $p(p+1) = 3(4) = 12 \equiv 0 \pmod{3}$.

If none of p,q,r is 3, then $p,q,r \equiv 1$ or $2 \pmod{3}$.

$p(p+1) + q(q+1) = r(r+1) \pmod{3}$.
Possible values for $n(n+1) \pmod{3}$ are 0 and 2.

If $p,q,r \not\equiv 0 \pmod{3}$, the possible combinations of $p(p+1) \pmod{3}$ and $q(q+1) \pmod{3}$

are:

0+0=0

0+2=2

2+0=2

$2+2=4 \equiv 1 \pmod{3}$. This is not possible for $r(r+1) \pmod{3}$.
This implies that at least one of p,$q$must be 3 if $r \not\equiv 0 \pmod{3}$.

If p=3:

$12+q(q+1) = r(r+1)$

$q(q+1) = r(r+1) - 12$.
If q=5, 5(6) =30. $r(r+1) = 30+12 = 42$. $r^2+r-42=0$.

$(r+7)(r-6)=0$.

$r=6$(not prime).
If $q=7$, $7(8) = 56$. $r(r+1)=56+12=68$. No integer solution for r.

If $r=3$:

$p(p+1) + q(q+1) = 3(4) = 12$.
Since p, q are primes, the only way the $n(n+1)$ sum to 12 is if one of them is 2.
If $p=2$, $2(3) + q(q+1) = 12 \Longrightarrow 6+q(q+1) = 12 \Longrightarrow q(q+1) = 6$.
This means $q=2$.
So, $(p,q,r) = (2,2,3)$ is a solution.

Conclusion:

The only prime solution is (2,2,3)

and its permutations for p and q

. Since the equation is symmetric with respect to p and q, (2,2,3) is the unique set of prime solutions.

The final prime solutions are $(p,q,r)=(2,2,3)$ and (2,2,3)

(which is the same set of values).

The unique set of prime numbers is {2,3}.

Final Answer: The only prime solution set is $p=2$, $q=2$, $r=3$

(and permutations of p,q).

107 By division algorithm,
63= (3×20) + 3;
20 = (6×3) + 2;
3= (1×2) + 1
=> 1= 3–2
=3 - (20– (6×3))
= (7×3) – 20
= 7× {63 - (3×20) – 20
= (7×63) - (22×20).
Hence we may take
1 = (-22) ×20 +7×63
i.e.u=(-22), v=7

108 Explanation:

Case p = 2: $p^2+8=2^2+8=12$. 12 is not prime, so p cannot be 2.

Case p = 3: $p^2+8=3^2+8=17$. Both 3 and 17 are prime numbers.

Case p = 5: $p^2+8=5^2+8=33$. 33 is divisible by 3 and 11, so it is not prime.

Case p = 7: $p^2+8=7^2+8=57$. 57 is divisible by 3 and 19, so it is not prime.

Case p = 3k + 1 (for any integer k): $p^2+8=(3k+1)^2+8=9k^2+6k+1+8=9k^2+6k+9$

. This is divisible by 3 and greater than 3, so it's not prime.

Case p = 3k + 2 (for any integer k):

$p^2+8=(3k+2)^2+8=9k^2+12k+4+8=9k^2+12k+12$

. This is also divisible by 3 and greater than 3, so it's not prime.

Therefore, the only prime number p for which both p and p^2+8

are prime is $p=3$.

109 Assume n is not prime: Let's assume that n is a composite number. This means that $n$can be written as a product of two integers, a and b, where both a and b are greater than 1.

So, n=ab where a >1 and b> 1.

Then, $2^n-1=2^{ab}-1=(2^a)^b-1$.

Using the difference of powers factorization, we can write

$(2^a)^b-1$ as $(2^a-1)\ ((2^a)^{b-1}+(2^a)^{b-2}+...+2^a+1)$.

Since $a>1$, $2^a-1>1$

. Also, since $b>1$

, the second factor is greater than 1. Therefore,

2^n-1 is not prime, contradicting the initial assumption that 2^n-1

is prime. Thus, n must be prime.

110 $2700=2^2 3^3 5^2$

Every positive divisor of 2700 is of the form $2^\alpha 3^\beta 5^\gamma$, where $0 \leq \alpha \leq 2, 0 \leq \beta \leq 3, 0 \leq \gamma \leq 2$.

Therefore, each term in the product of $(1+2+2^2)(1+3+3^2+3^3)(1+5+5^2)$ is a positive divisor of 2700 and conversely.

The odd positive divisors of 2700 are given by the terms of the product $1.(1+3+3^2+3^3)(1+5+5^2)$

The number of odd positive divisors are (3+1)(2+1) i.e, 12.

COMBINATORICS

A. Permutations and Combinations

• **Multiplication Principle of Counting:**

The Fundamental Principle of Counting states that if one event can occur in m ways and another in n ways, then the two events together can occur in m × n ways.

• **Addition Principle of Counting:**

If a work A can be done in *a* ways and work B can be done *b* ways, then either one of the works can be done in *(a+b)* ways. This can be generalized. (But if there are common ways they need to be subtracted)

• **Factorial Notation:**

A factorial, denoted $n!$, is the product of all positive integers up to n. It's used in counting permutations. $(n! = 1.2.3 \ldots \ldots \ldots . n)$

• **Permutations** are arrangements of items in a specific order. $^nP_r = \frac{n!}{(n-r)!}$

• **Combinations** are selections of items where order does not matter. $^nC_r = \frac{n!}{r!(n-r)!}$

• When n is the number of objects and d_1, d_2, d_3, ….. are the no of objects that **repeats more than once** in the word then the total no of arrangements possible is $\frac{n\,!}{d_1!.d_2!.d_3!\ldots\ldots}$

• **Circular permutations** count arrangements around a circle: $(n - 1)!$

• The number of ways of arranging n **objects in a circle** where rotations of the same arrangement aren't considered distinct and **reflections of the same arrangement aren't considered distinct** is $\frac{(n-1)!}{2}$

• **Restricted permutations** involve conditions like fixing or excluding elements from certain positions.

B. Binomial Theorem and Pascal's Triangle

• The binomial theorem provides a formula to expand expressions of the form $(a + b)^n$.

• The general term in the expansion of $(a + b)^n$ is given by $T(k+1) = {}^nC_k \times a^{(n-k)} \times b^k$.

• The middle term depends on whether n is even or odd. For even n, it's the $(\frac{n}{2} + 1)$th term.

• Pascal's Triangle is a triangular array where each number is the sum of the two directly above it.

• Binomial coefficients (^nC_r) follow identities like $^nC_r = {}^nC_{(n-r)}$.

• Combinatorial proofs use counting principles to verify identities such as $^{(n+1)}C_r = {}^nC_r + {}^nC_{(r-1)}$.

Use of Binomial theorem in Combinatorics:

(i) If $p_1 + p_2 + p_3$ things are such that p_1 things are alike, p_2 things are alike and p_3 things all different,

then number of selections of any r things out of $p_1 + p_2 + p_3$ things

= coefficient of x^r in $(x^0 + x + x^2 + \ldots + x^{p1}) \times (x^0 + x + x^2 + \ldots + x^{p2})(x^0 + x^1)^{p3}$

= coefficient of x^r in $(1 + x + x^2 + \ldots + x^{p1})(1 + x + x^2 + \ldots + x^{p2})(1 + x)^{p3}$

(ii) If $p_1 + p_2 + p_3$ things are such that p_1 things are alike of 1st kind, p_2 things alike of 2nd kind and p_3 things alike of 3rd kind, then we have:

(a) Number of selections of any r things out of $p_1 + p_2 + p_3$ things containing at least one thing from p_1 alike things
$= \text{coefficient of } x^r \text{ in } (x + x^2 + x^3 + ... + x^{p1})(x^0 + x + x^2 + ... + x^{p2})(x^0 + x + x^2 + ... + x^{p3})$
$= \text{coefficient of } x^r \text{ in } (x + x^2 + ... + x^{p1})(1 + x + x^2 + ... + x^{p2})(1 + x + x^2 + ... + x^{p3})$
(b) Number of selections of any r things containing at least one thing from p_1 alike and one thing from p_2 alike things is given by
$= \text{coefficient of } x^r \text{ in } (x + x^2 + ... + x^{p1})(x + x^2 + ... + x^{p2})(x^0 + x + x^2 + ... + x^{p3})$
$= \text{coefficient of } x^r \text{ in } (x + x^2 + x^3 + ... + x^{p1})(x + x^2 + ... + x^{p2})(1 + x + x^2 + ... + x^{p3})$
(c) Number of selections of any r things containing at least one thing from p_1 alike and two things from p_2 alike things is given by
$= \text{coefficient of } x^r \text{ in } (x + x^2 + ... + x^{p1})(x^2 + x^3 + ... + x^{p2})(1 + x + x^2 + ... + x^{p3})$

C. Principle of Inclusion-Exclusion/Casework

• Used to count the number of elements in a union of sets where there are overlaps.

• Two sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

• Three sets A, B, and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

- General formula for n sets involves adding sizes of single sets, subtracting pairwise intersections, adding triple-wise intersections, and so on, alternately.
- Use PIE when: You're asked to count how many things satisfy at least one of several properties, but the sets overlap.
- Casework is used when a problem can be divided into non-overlapping cases, and each case is easier to count individually.
  - Divide the problem into distinct, exhaustive cases.
  - Solve each case separately.
  - Add the results of all cases to get the total count.
- Use Casework when:

- Direct counting is complex.
- The problem naturally splits into cases (like different digit conditions, even/odd, etc.).

- Tip: Make sure cases do not overlap and cover all possibilities.

D. Pigeonhole Principle

If $n+1$ objects are distributed into n box, then atleast one box contains two or more of the objects. For example, among 13 people there are 2 who have their birthdays in the same month.

QUESTIONS

1 How many different ways can 5 books be arranged on a shelf?

2 In how many ways can 7 people be seated in a row?

3 From a group of 10 students, how many ways can a team of 4 be chosen?

4 How many 4-letter words can be formed from the letters of the word 'APPLE'?

5 How many 4-digit numbers can be formed using digits 1-9 without repetition?

6 In how many ways can 6 people sit around a round table?

7 How many ways can you arrange the letters in the word 'MISSISSIPPI'?

8 How many different 3-digits even numbers can be formed from the digits 1 to 7?

9 How many permutations of the word 'LEVEL' exist if vowels must be together?

10 How many committees of 3 can be formed from 6 men and 4 women with at least 1 woman?

11 In how many ways can a president and vice-president be selected from 10 candidates?

12 Find the number of permutations of the word 'BANANA'.

13 How many combinations of 5 cards can be selected from a deck of 52?

14 How many ways can you form a committee of 4 out of 8 people?

15 How many ways can 5 people be arranged with a specific person always at the end?

16 In how many ways can 3 red and 2 blue balls be arranged in a row?

17 How many ways can a password of 4 characters be formed from A, B, C, D if repetition is not allowed?

18 How many different ways can 4 people be seated if two must sit together?

19 How many 3-letter combinations can be made from the word 'BRAVE'?

20 A class has 12 students. How many groups of 5 can be formed?

21 Expand $(x + 2)^4$ using the binomial theorem.

22 Find the coefficient of x^3 in $(x + 1)^6$.

23 Find the general term in the expansion of $(3x - 2)^5$.

24 What is the middle term in the expansion of $(a + b)^{10}$?

25 Evaluate the 4th term of the expansion of $(2x - 3)^6$.

26 Use Pascal’s Triangle to expand $(a + b)^4$.

27 In a class of 60 students, 35 like mathematics, 28 like physics, and 20 like both. How many like at least one of the two subjects?

28 How many integers from 1 to 100 are divisible by 2 or 3 or 5?

29 In how many ways can we select 3 digits from {1,2,3,4,5,6,7,8,9} such that no two digits are consecutive?

30 From the digits 1 to 9, how many 3-digit numbers can be formed such that at least one digit is even?

31 In a survey, 70 people liked tea, 60 liked coffee, and 50 liked milk. 30 liked both tea and coffee, 25 liked both tea and milk, 20 liked both coffee and milk, and 10 liked all three. How many people liked at least one drink?

32 How many numbers between 1 and 1000 are divisible by neither 2, 3 nor 5?

33 A password must be 3 letters chosen from A, B, C, D, E with repetition allowed, but it must contain at least one A. How many such passwords exist?

34 In how many ways can 5 people sit in a row such that person A is not adjacent to person B?

35 How many integers from 1 to 1000 are divisible by 7 or 11 but not both?

36 How many 4-digit numbers can be formed using digits 1 to 6 such that at least one digit repeats?

37 There are 12 points in a plane, 5 of which are concyclic and out of remaining 7 points, no three are collinear and none concyclic with previous 5 points. Find the number of circles passing through at least 3 points out of 12 given points.

38 In a plane, a set of 8 parallel lines intersects a set of n other parallel lines, giving rise to 420 parallelograms (many of them overlap with one another). Find the value of n.

39 There are 6 single choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 5 each?

40 How many 5-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word EQUATIONS so that the two consonants occur together in every word?

41 Out of ten people, 5 are to be seated around a round table and 5 are to be seated across a rectangular table. Find the number of ways to do so.

42 10 different toys are to be distributed among 10 children. Find the total number of ways of distributing these toys so that exactly 2 children do not get any toy.

43 In a shooting competition a man can score 5, 4, 3, 2 or 0 points for each shot. Find the number of different ways in which he can score 30 in seven shots.

44 Find the number of ways in which two Americans, two Britishers, one Chinese, one Dutch and one Egyptian can sit on a round table so that persons of the same nationality are separated.

45 Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.

46 Find the number of ways in which 14 identical toys can be distributed among three boys so that each one gets at least one toy and no two boys get equal number of toys.

47 The total number of positive integral solutions for (x, y, z) such that $xyz = 24$ is

(a) 36 (b) 90 (c) 120 (d) None of these

48 The total number of positive integral solutions of $abc = 30$ is

(a) 30 (b) 27 (c) 8 (d) None of these

49 The number of proper divisors of 1800 which are also divisible by 10, is

(a) 18 (b) 27 (c) 34 (d) None of these

50 The number of even proper divisors of 5040, is

(a) 48 (b) 47 (c) 46 (d) None of these

51 The number of odd proper divisors of 5040, is

(a) 12 (b) 10 (c) 11 (d) None of these

52 What is the GCD of $2^{20} - 1$ and $2^{15} - 1$?

53 Find the number of edges in a complete graph with 5 vertices.

54 Find the number of ways to express 5 as the sum of 1's and 2's.

55 How many 5-digit numbers can be formed using the digits 1 to 9 (no zeros), such that no digit is repeated and the number is even?

56 Find the number of solutions to the equation $x_1 + x_2 + x_3 = 7$, where x_1, x_2, x_3 are non-negative integers

57 How many integer solutions are there to the equation

$x_1 + x_2 + x_3 + x_4 = 20$ where $x_1, x_2, x_3, x_4 \geq 2$

58 Find the number of odd integers between 30,000 and 80,000 in which no digit is repeated

59 Prove that if 11 numbers are chosen from the set {1, 2, ..., 20}, then there must be two numbers that differ by 1.

60 How many ways can 7 people sit around a circular table such that two particular people (say A and B) do not sit next to each other?

61 In how many ways can 10 identical candies be distributed among 4 children such that each child gets at least 1 candy and no child gets more than 5?

62 In how many ways can you color 5 identical boxes using 3 colors (Red, Blue, Green) such that each color is used at least once?

63 How many integers between 1 and 1000000 have the sum of the digits equal to 18.

64 Show that the number of rectangles of any size on a chess board is $\sum_{K=1}^{8} k^3$.

65 Find the total number of 5 digit numbers of different digits in which the digit in the middle is the largest.

66 Find the number of 4 digit numbers that can be formed from the digits 5,6,7,8,9 in which at least two digits are identical.

67 Find the total numbers (non-zero) not more than 20 digits that can be formed using the digits 0,1,2,3,4.

68 In how many ways one can select four numbers from 1 to 30 so as to exclude every selection of four consecutive numbers.

69 Find the sum of the digits in unit place of all numbers formed by the numbers 3,4,5,6 taken all at a time.

70 Find the number of ordered pair of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$

71 In the decimal system of numeration of six-digit numbers, in how many numbers the sum of the digits is divisible by 5.

72 Find the number of ways in which one can get a score of 11 by throwing three dice.

73 Find the total number of proper factors of the number 35700. Also find

a) sum of all these factors

b) sum of odd proper divisors

c) The number of proper divisor divisible by 10 and sum of these divisors.

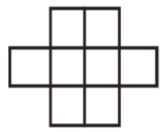
74 A square chessboard has 16 squares (4 rows and 4 columns). One puts 4 checkers in such a way that only one checker can be put in a square. Find the number of ways putting these checkers if there must be exactly one checker per row and column.

75 Find the number of integers from 0 through 9999 that have no two equal neighbouring digits in their decimal representation.

76 If a denotes the number of permutations of (x + 2) things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of (x – 11) things taken all at a time such that a = 182 bc, then the value of x is

(A) 15 (B) 12 (C) 10 (D) 18

77 In the given figure of squares, 6 A's should be written in such a manner that every row contains at least one



'A'. In how many numbers of ways is it possible?

78 The number of ways in which 7 girls can stand in a circle so that they do not have same neighbours in any two arrangements?

(A) 720 (B) 380 (C) 360 (D) none of these

79 The numbers 1, 2, 3, ..., 19, 20 are written on a blackboard. It is allowed to erase any two numbers 'a' and 'b' and write the new number a + b –1. What number will be on the blackboard after 19 such operations?

80 15 points are marked inside the area bounded by a rectangle 6 cm long and 4 cm wide. Show that there are at least two points that are at most $\sqrt{2}$ cm apart.

81 A subset S of the set A = {3, 11, 19, 27, ……, 147, 155} is said to '158-free' if there are no two elements which add up to 158. What is the maximum size of a subset that is '158-free'?

82 How many positive integers of n digits exist such that each digit is 1, 2 or 3? How many of these contain all three of the digits 1, 2 and 3 at least once?

83 In how many ways 6 letters can be placed in 6 envelopes such that:

(i) No letter is placed in its corresponding envelope.

(ii) At least 4 letters are placed in correct envelopes.

(iii) At most 3 letters are placed in wrong envelopes.

84 How many six-digit numbers have at least one even digit?

85 How many necklaces of 12 beads each can be made from 18 beads of various colours?

86 How many terms are there in the expansion of $(a + b + c + d)^{24}$?

87 Find the number of ways of distributing 5 identical balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.

88 Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance is at most $\sqrt{2}$.

89 A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that at least two of them are in the wrong envelopes?

90 In how many ways can 10 identical balls be distributed into 4 distinct boxes such that no box is empty?

91 All the 7-digit numbers containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once and not divisible by 5, are arranged in the increasing order. Find the 2000th number in this list.

92 Find the number of positive integers x which satisfy the condition $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right]$

(Here, $[z]$ denotes, for any real z, the largest integer not exceeding z; e.g., [7/4] = 1.)

93 Three non-zero real numbers a, b, c are said to be in harmonic progression, if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$. Find all three-term harmonic progressions a, b, c of strictly increasing positive integers in which a = 20 and b divides c.

94 Find the number of 4-digit numbers (in base 10) having non-zero digits and which are divisible by 4 but not by 8.

95 Number of 4-digit numbers of the form $N = \overline{abcd}$ which satisfy following three conditions

i) $4000 \leq N < 6000$

ii) N is a multiple of 5

iii) $3 \leq b \leq c \leq 6$

is equal to M, then find the value of $\frac{M.}{3}$

96 Find the number of four–digit odd numbers having digits 1,2,3,4, each occurring exactly once.

97 Find the number of arrangements in which g girls and b boys are to be seated around a table, b ≤ g, so that no two boys are together.

98 Find the number of odd integers between 30,000 and 80,000 in which no digit is repeated.

99 How many ways are there to arrange the letters of the word 'COMBINATORICS' such that all the vowels appear in alphabetical order?

100 A 5×5 grid is to be filled with the numbers 1 to 25 such that each number is used exactly once. How many ways are there to fill the grid such that each row and column contains exactly 5 numbers and the sum of the numbers in each row is the same? (Assume rows and columns are distinguishable.)

101 How many permutations of the numbers 1 through 10 are there such that no number is in its original position (i.e., a derangement), and no two consecutive numbers appear next to each other?

102 Consider all 10-digit numbers made using only the digits 1 and 2, with exactly six 1s and four 2s. How many such numbers are there such that no two 2s are adjacent?

7. Let n be a positive integer. Determine the number of sequences $(a_1, a_2, ..., a_n)$ such that each $a_i \in \{1, 2, 3\}$ and no two consecutive terms are equal.

ANSWERS

1 $5! = 120$

2 $7! = 5040$

3 $C(10, 4) = 210$

4 60 (by considering letter repetitions)

5 $9 \times 8 \times 7 \times 6 = 3024$

6 $(6 - 1)! = 5! = 120$

7 $11! / (4! \times 4! \times 2!) = 34650$

8 90 (only even digits in last place)

9 12

10 Total: 120, All-men: 20 ⇒ Answer: 100

11 $10 \times 9 = 90$

12 $6! / (3! \times 2!) = 60$

13 $C(52, 5) = 2598960$

14 $C(8, 4) = 70$

15 $4! = 24$

16 $5! / (3! \times 2!) = 10$

17 $4! = 24$

18 $3! \times 2! = 12$

19 $C(5, 3) = 10$ (If permutations: 60)

20 $C(12, 5) = 792$

21 $(x + 2)^4 = x^4 + 4x^3 \cdot 2 + 6x^2 \cdot 4 + 4x \cdot 8 + 16 = x^4 + 8x^3 + 24x^2 + 32x + 16$

22 Coefficient = $C(6,3) = 20$

23 $T_{(r+1)} = C(5, r) \cdot (3x)^{(5-r)} \cdot (-2)^r$

24 Middle term (6th term) = $C(10,5) \cdot a^5 \cdot b^5 = 252\, a^5 \cdot b^5$

25 $T_4 = C(6,3) \cdot (2x)^3 \cdot (-3)^3 = 20 \cdot 8x^3 \cdot (-27) = -4320x^3$

26 $(a + b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$

27 $|M \cup P| = 35 + 28 - 20 = 43$

28 Using PIE(Principle of Inclusion and Exclusion): 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74

29 Answer: Use casework or transformation to non-consecutive selections: C(7, 3) = 35

30 Total = 504, All odd = 60 ⇒ At least one even = 504 - 60 = 444

31 Using PIE: 70 + 60 + 50 - 30 - 25 - 20 + 10 = 115

32 Using PIE: 1000 - 734 = 266

33 Total = 125, Without A = 64 ⇒ With at least one A = 61

34 Total = 120, Adjacent = 48 ⇒ Not adjacent = 72

35 142 + 90 - 2×12 = 208

36 Total = 6^4 = 1296, No repeat = 360 ⇒ With repeat = 936

37 Consider Set A consists of 5 concyclic points. Set B consists of remaining 7 points.

Case 1: Circle passes through 3 points of set B

Number of circles = ${}^7C_3$

Case 2: Circle passes through 2 points of set B and one point of set A

Number of circles = ${}^7C_2 \cdot {}^5C_1$

Case 3: Circle passes through 1 point of set B and two points of set A

Number of circles = ${}^7C_1 \cdot {}^5C_2$

Case 4: Circle passes through no point from set B.

Number of circles = 1

All 4 cases are ***exhaustive and mutually exclusive***. So, total number of circles

= ${}^7C_3 + {}^7C_2 \cdot {}^5C_1 + {}^7C_1 \cdot {}^5C_2 + 1$

38 If two lines which are parallel to one another (in one direction) intersect another two lines which are parallel, we get one parallelogram. Thus, we can choose C(8, 2) pairs of parallel lines in one direction and the number of parallel lines intersecting there will be C(n, 2) pairs.

So, the number of parallelograms thus obtained is

C(n, 2) × C(8, 2) = 420

- ⇨ n (n-1) = 30
- ⇨ n = 6 or n = -5 (which is not admissible)

39 Here we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences = 4 × 4 × 4 × 5 × 5 × 5 = 8000.

Note:

1) The number of ways to select and arrange *r* objects from *n* distinct objects such that *p* particular objects are always excluded in the selection = ${}^{n-p}C_r \times r!$

2) The number of ways to arrange *n* distinct objects such that *p* particular objects remain together in the arrangement $(n - p + 1)!\ p!$

40 There are 5 vowels and 3 consonants in EQUATION. To form the words, we will use following steps:

Step 1: Select vowels (3 from 5) in ${}^5C_2$ ways.

Step 2: Select consonants (2 from 3) in ${}^3C_2$ ways.

Step 3: Arrange the selected letters (3 vowels and 2 consonants (always together)) in 4! × 2! ways.

Hence the number of words = ${}^5C_3\ {}^3C_2\ 4!\ 2! = 10 \times 3 \times 24 \times 2 = 1440$.

41 First select 5 people out of 10, those who sit around the table. This can be done in ${}^{10}C_5$ ways.

Number of ways in which these 5 people sit around the round table = 4!

Remaining 5 people sit across a rectangular table in 5! ways. Total number of arrangements = ${}^{10}C_5 \times 4! \times 5!$

42 It is possible in two mutually exclusive cases;

Case 1: 2 children get none, one child gets three and all remaining 7 children get one each.

No of ways = $\left(\frac{10!}{(0!)^2\ 2!3!(1!)^7 7!}\right)(10!)$

Case 2: 2 children get none, 2 children get 2 each and all remaining 6 children get one each.

No of ways = $\left(\frac{10!}{(0!)^2\ 2!(2!)^2 2!(1!)^6 6!}\right)(10!)$

Thus, total no of ways = $(10!)^2\left(\frac{1}{3!7!2!} + \frac{1}{(2!)^4 6!}\right)$

43 Let $x_1, x_2, x_3, x_4, \ldots, x_7$ be the scores in 7 shots. As total score of 30 is Sum of scores in 7 shots = 30

$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 30$ [where $x_i \in \{0, 2, 3, 4, 5\}$ i = 1, 2, …, 7]

Number of solutions of above equation

Number of ways of making 30 in 7 shots to be taken, Coefficient of x^{30} in $(x^0 + x^2 + x^3 + x^4 + x^5)^7$.

$\Rightarrow$ Coefficient of x^{30} in $\{(x^0 + x^2 + x^3) + x^4 (x + 1)\}^7$

$\Rightarrow$ Coefficient of x^{30} in $\{x^{28} (x + 1)^7 + {}^7C_1 x^{24}.(x + 1)^6.(1 + x^2 + x^3) + {}^7C_2\, x^{20} (x + 1)^5 (x^3 + x + 1)^2 + \cdots\}$
[using Binomial theorem]

Thus, Number of ways to score 30

$\Rightarrow {}^7C_2 + {}^7C_1 ({}^6C_3 + {}^6C_2 + {}^6C_0) + {}^7C_2 ({}^5C_1 + 2)$

$\Rightarrow 21 + 252 + 147 = 420.$

44 Total = 6!

$n(A)$ = when $A_1\ A_2$ together = 5 ! 2 ! = 240

$n(B)$ = when $B_1\ B_2$ together = 5 ! 2 ! = 240

$\Rightarrow n(A \cup B) = n(A) + n(B) - n(A \cap B) = 240 + 240 - 96 = 384$

Hence $n(A \cap B)' = \text{Total} - n(A \cup B)$

$= 6! - 384$

$= 720 - 384$

$= 336.$

45 Let x_1, x_2 and x_3 be the number of balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.

The number of ways of distributing 5 balls into Boxes 1, 2 and 3 is the number of integral solutions of the equation $x_1 + x_2 + x_3 = 5$ subjected to the following conditions on x_1, x_2, x_3 (1)

Conditions on x_1, x_2 and x_3:

According to the condition that the boxes should contain at least one ball, we can find the range of x_1, x_2 and x_3, *i.e.*,

Min (x_i) = 1 and Max (x_i) = 3 for i = 1, 2, 3 [using: Max (x_1) = 5 - Min (x_2) - Min(x_3)]
or $1 \le x_i \le 3$ for i = 1, 2, 3
So, number of ways of distributing balls

= Number of integral solutions of (1)

= Coefficient of x^5 in the expansion of $(x + x^2 + x^3)^3$

= Coefficient of x^5 in $x^3 (1 - x^3) (1 - x)^{-3}$

= Coefficient of x^2 in $(1 - x^3) (1 - x)^{-3}$

= Coefficient of x^2 in $(1 - x)^{-3}$ [as x^3 cannot generate x^2 terms]

$= {}^{3+2-1}C_2 = {}^4C_2 = 6.$

Alternate solution:

The number of ways of dividing n identical objects into r groups so that no group remains empty

$= {}^{n-1}C_{r-1}$

$= {}^{5-1}C_{3-1} = {}^{4}C_{2} = 6.$

46 Let the boys get a, $a + b$ and $a + b + c$ toys respectively.

$$a + (a + b) + (a + b + c) = 14, a \geq 1, b \geq 1, c \geq 1$$

$$\Rightarrow 3a + 2b + c = 14, a \geq 1, b \geq 1, c \geq 1$$

∴ The number of solutions

= Coefficient of t^{14} in $\{(t^3 + t^6 + t^9 + \ldots)(t^2 + t^4 + \cdots)(t + t^2 + \cdots)\}$

= Coefficient of t^8 in $\{(1 + t^3 + t^6 + \cdots)(1 + t^2 + t^4 + \ldots)(1 + t + t^2 + \cdots)\}$

= Coefficient of t^8 in $\{(1 + t^2 + t^3 + t^4 + t^5 + 2t^6 + t^7 + 2t^8)(1 + t + t^2 + \cdots + t^8)\}$

= 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 = 10.

Since, three distinct numbers can be assigned to three boys in 3! ways. So, total number of ways = 10 × 3! = 60.

47 We have, $24 = 2 \times 4 \times 3$

$$= 2 \times 2 \times 6$$

$$= 1 \times 6 \times 4$$

$$= 1 \times 3 \times 8$$

$$= 1 \times 2 \times 12$$

$$= 1 \times 1 \times 24$$

Hence, the total number of positive integral solutions for (x, y, z) such that $xyz = 24$ is

$$3! + \frac{3!}{2!} + 3! + 3! + 3! + \frac{3!}{2!} = 6 + 3 + 6 + 6 + 6 + 3 = 30$$

48 We have, $24 = 2 \times 3 \times 5$

$$= 2 \times 15 \times 1$$

$$= 10 \times 3 \times 1$$

$$= 1 \times 5 \times 6$$

$$= 1 \times 1 \times 30$$

Hence, the total number of positive integral solutions for (x, y, z) such that $xyz = 24$ is

$$3! + 3! + 3! + 3! + \frac{3!}{2!} = 6 + 6 + 6 + 6 + 3 = 27$$

49 We have, $1800 = 2^3 \times 3^2 \times 5^2$

Clearly, the required number of proper divisors is same as the number of ways of selecting at least one 2 and at least one 5 out of 3 identical $2's$, 2 identical $3's$ and 2 identical $5's$.

Required number of proper divisors $= 3 \times (2 + 1) \times 2 = 18$

50 We have, $5040 = 2^4 \times 3^2 \times 5 \times 7$

Number of even proper divisors = Number of ways of selecting at least one 2 and any number of $3's$, $5's$ and $7 = 4 \times (2 + 1) \times (1 + 1) \times (1 + 1) - 1 = 47$

51 We have, $5040 = 2^4 \times 3^2 \times 5 \times 7$

Number of odd proper divisors = Number of ways of selecting any number of $3's$ out of two $3's$, any number of $5's$ and $7's = (2 + 1) \times (1 + 1) \times (1 + 1) - 1 = 11$

52 We know, $\gcd(2^m - 1, 2^n - 1) = 2^{\gcd(m,n)} - 1$

$$\gcd(2^{20} - 1, 2^{15} - 1) = 2^{\gcd(20,15)} - 1$$

$$= 2^5 - 1 = 32 - 1 = 31$$

53 A complete graph with n vertices has n(n-1)/2 edges.

For n = 5, the number of edges is 5(5-1)/2 = 10.

54 The possible combinations are:

1 + 1 + 1 + 1 + 1 = 5 (and its permutations) = 1 ways

1 + 1 + 1 + 2 = 5 (and its permutations) = 4 ways

1 + 2 + 2 = 5 (and its permutations) = 3 ways

Total ways = 1 + 4 + 3 = 8

55 For a number to be even, its last digit must be even.

From 1 to 9, the even digits are: 2, 4, 6, 8 → 4 choices

We fix the last digit (even), then choose 4 more digits from the remaining 8 digits (excluding the one used at the end):

- Choose 4 digits from remaining 8 → C (8 4).

- Arrange them in 4! ways.

So total =4·C (8 4) · 4! =4·70·24=6720.

Solution: 6720

56 The number of non-negative integer solutions to the equation:

$x_1 + x_2 + \cdots + x_k = n$ is given by:

C (n + k - 1, k - 1) where:

- n is the total sum,
- k is the number of variables.

Solution: C (7 + 3 - 1, 3 - 1) = C (9, 2) = 36

57 Let $y_i = x_i - 2 \Rightarrow y_i \geq 0$

Then:

$y_1 + y_2 + y_3 + y_4 = 20 - 8 = 12$

Now use stars and bars:

Number of solutions = C (12+4−1   4−1) = C (15 3) = 455

58 **:** Let $\overline{abcde}$ be the required odd integers. a can be chosen from 3, 4, 5, 6 and 7 and e can be chosen from 1, 3, 5, 7, 9.

Note that 3, 5 and 7 can occupy both the positions a and e.

So, let us consider the case where one of 3, 5, 7 occupies the position a.

Case 1: If a gets one of the values 3, 5, 7, then there are 3 choices for a, but then, e has just four choices as repetition is not allowed.

Thus, a and e can be chosen in this case in 3 × 4 = 12 ways.

The 3 positions b, c, d can be filled from among the remaining 8 digits in 8 × 7 × 6 ways.

Total number of ways in this case = 12 × 8 × 7 × 6 = 4,032.

Case 2: If a takes the values 4 or 6, then there are two choices for a and there are five choices for e.

There are again eight choices altogether for the digits b, c, d which could be done in

8 × 7 × 6 ways.

Therefore, in this case, the total numbers are 2 × 5 × 8 × 7 × 6 = 3,360.

Hence, total number of odd numbers between 30,000 to 80,000, without repetition of digits is 4032 + 3360 = 7392

59 Step 1: Construct subsets of S that avoid consecutive numbers

To avoid choosing two numbers that differ by 1, we must not pick any two consecutive integers**.**

Let's try to construct the largest possible subset of {1, 2, ..., 20} that contains no two consecutive numbers.

Suppose we pick every second number. That is, pick:

{1,3,5,7,9,11,13,15,17,19}

This gives us 10 numbers, and none of them are consecutive**.**

So this is the largest possible subset of {1, ..., 20} with no two numbers that differ by 1. You can't add an 11th number without causing at least one pair to differ by 1.

Step 2: Use the Pigeonhole Principle

Now suppose we try to select 11 numbers from {1, 2, ..., 20}.

- We know that it's only possible to pick 10 numbers at most that are non-consecutive (as shown above).
- So, picking 11 numbers must force at least one pair to be consecutive, i.e., to differ by 1.

60

- Total circular permutations of 7 people: (7−1)! =6!
- Ways in which A and B sit together**:**
  Treat A and B as a single unit → now we have 6 units to arrange in a circle:

  (6−1)! × 2! = 5!×2 .

  (Multiply by 2 for A and B order within the pair.)
- Hence, number of ways in which A and B do not sit together**:**

  6!−5! ×2 = 720−120×2 = 720−240 = 480.

61 We need number of integer solutions to:
$x_1 + x_2 + x_3 + x_4 = 10$
where $1 \le x_i \le 5$.

Let $y_i = x_i - 1 \Rightarrow y_i \ge 0$.

Then:
$y_1 + y_2 + y_3 + y_4 = 6$
.and $y_i \le 4$ (because $x_i \le 5 \Rightarrow y_i \le 4$)

Now count non-negative integer solutions to $y_1 + y_2 + y_3 + y_4 = 6$ with $y_i \le 4$

This is done via inclusion-exclusion:

- Total unrestricted solutions = C (6+4−1 4−1)= C (9 3)=84

- Subtract cases where any variable > 4

Let's compute number of solutions where any y_i ≥5

Let's say $y_1 \geq 5 \Rightarrow y_1' = y_1 - 5 \geq 0$.

Then:
$y_1' + y_2 + y_3 + y_4 = 1 \Rightarrow$ C (1+4−1   3)=C (4 3)=4

There are 4 such variables, so subtract 4 × 4 = 16

- No overlaps possible (because total is 6)

Solution: 84−16= 68

62 Total colorings (each box can be one of 3 colors):

$=3^5=243$

Subtract cases where some color is missing.

- Only 2 colors used: C(3 2)·(25−2)=3·(32−2)=3·30=90
- Only 1 color used: 3 ways

So total invalid = 90 + 3 = 93

Valid = 243−93=150

63 Integers between 1 and 1000000 will be, 1,2, 3, 4, 5 or 6-digits numbers, and given sum of digits = 18

Thus we need to obtain the number of solutions of the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$$

Where $0 \leq x_i \leq 9$, i = 1, 2, 3, 4, 5, 6 Therefore, the number of solutions of the Eq. (1), will be

= Coefficient of x^{18} in $(x^0+x^1+x^2+x^3+\ldots\ldots\ldots+x^9)$

= Coefficient of x^{18} in $\left(\frac{1-x^{10}}{1-x}\right)^6$

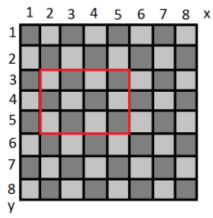
= Coefficient of x^{18} in $(1-x^{10})^6 (1-x)^{-6}$

= Coefficient of x^{18} in $(1-6x^{10}) (1-x)^{-6}$

$= {}^{6+18-1}C_{18} - 6.\ {}^{8+8-1}C_8$

= 25927

64 A rectangle can be fixed on the chess board if and only if we fix two points on x-axis and two points on y-axis. For example, in order to fix the rectangle in red colour, we have to choose two lines horizontally and two lines vertically. There are 9 vertical and 9 horizontal lines.



Hence total number of rectangles on the chess board is the number of ways of choosing two points on x-axis (which can be done in ${}^9C_2$ ways) and two points on y-axis (which can also be done is ${}^9C_2$ ways). Hence require number is $({}^9C_2)^2 = \sum_{K=1}^{8} k^3$

65

Middle digit	Digit available for remaining four places	Number of ways of filling remaining four places
4	0,1,2,3	$3 \times P(3,3)$
5	0,1,2,3,4	$4 \times P(4,3)$
6	0,1,2,3,4,5	$5 \times P(5,3)$
7	0,1,2,3,4,5,6	$6 \times P(6,3)$
8	0,1,2,3,4,5,6,7	$7 \times P(7,3)$
9	0,1,2,3,4,5,6,7,8	$8 \times P(8,3)$

Adding we get,

18+96+300+720+1470+2688=5292

66 Total numbers when repetitions are allowed=5^4

Total numbers when repetitions are not allowed=$5 \times 4 \times 3 \times 2$

So, total numbers when at least one digit is repeated=$5^4 - 120 = 625 - 120 = 505$

67 Total number of one digit number=4

Total number of two-digit numbers=4×5

Total number of three-digit numbers =4×5^2

Total number of twenty-digit numbers=4×5^{19}

Hence, total numbers = 4 + 4×5 + 4×5^2+----+4×5^{19}

=$5^{20} - 1$

68 The number of ways of selecting 4 numbers from 30 numbers without any restriction=C(30,4)

The number of ways of selecting 4 consecutive numbers=27 [eg: (1,2,3,4),(2,3,4,5)…..]

Required number= C (30,4)-27

69 Required sum=$4!\,(3+4+5+6)=24\times 18=432$

70 $x^2+6x+y^2=4$

$\Longrightarrow (x+3)^2+y^2=13$

$\Longrightarrow (x+3)=\pm 2, y=\pm 3$ Or $(x+3)=\pm 3, y=\pm 2$

So, there are 8 possibilities

71 Considering different possibilities-

Sum of the digits in first five places	Digit in unit place
5k	0 or 5
5k+1	4 or 9
5k+2	3 or 8
5k+3	2 or 7
5k+4	1 or 6

The last place can be filled always in 2 ways

Hence, total numbers = $9\times 10\times 10\times 10\times 10\times 2=180000$

72 The possibilities are(1,4,6),(2,3,6),(2,4,5),(1,5,5),(3,3,5),(3,4,4)

Hence, required numbers =$3!+3!+3!+\frac{3!}{2!}+\frac{3!}{2!}+\frac{3!}{2!}=27$

73

$$35700=2^2\times 3^1\times 5^2\times 7^1\times 17^1$$

Hence, a) number of proper factors=$3\times 2\times 3\times 2\times 2-2=70$

Sum of these proper factors=

$(1+2+2^2)(1+5+5^2)(1+3)(1+7)(1+17)-1-35700=89291$

b) sum of odd proper divisors

=$(1+3)(1+5+5^2)(1+7)(1+17)-1=17855$

c) Number of proper divisor divisible by 10=$2\times 2\times 2\times 2\times 2-1=31$

sum of proper divisors divisible by 10

=$(2+2^2)(5+5^2)(1+3)(1+7)(1+17)-35700=67980$

74 Choose one square from last row, say numbered 1, in 4 ways. Then, delete the column corresponding to square numbered 1.

10	9	8	7
11	16	15	6
12	13	14	5
1	2	3	4

Next, choose second square from square numbered 13, 14, 15 in 3 ways and delete the column accordingly. Continuing this, we have total no. of ways = 4 × 3 × 2 × 1 = 24

75 There are 10 one-digit integers satisfying the condition. The first digit (from left) of a two-digit integer can be any digit but 0. The second digit can be any of 9 digits which differ from the first one. Therefore, there are 9^2 two-digit integers with two different digits.

There are 9^3 three-digits integers which satisfy the condition, because there are 9 digits (any digit but the one used as the second one) to choose from for the third place. Continuing in the same way, we get the required number of integers: $10 + 9^2 + 9^3 + 9^4$

76 ${}^{x+2}P_{x+2} = a \quad \Rightarrow a = (x + 2)!$

${}^{x}P_{11} = b \quad \Rightarrow b = \frac{x!}{(x-11)!}$

And ${}^{x-11}P_{x-11} = c \quad \Rightarrow c = (x - 11)!$

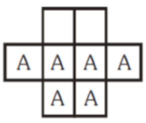
Now, $a = 182bc$

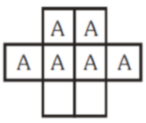
$\Rightarrow (x + 2)! = 182 . \frac{x!}{(x-11)!} . (x - 11)!$

$\Rightarrow (x + 2)(x + 1) = 182 = 14 \times 13$

$\therefore \quad x + 1 = 13 \Rightarrow x = 12$

77 There are 8 squares and 6 'A' in given figure. First, we can put 6 'A' in these 8 squares by ${}^8C_6$ number of ways.

(I) 

(II) 

According to question, atleast one 'A' should be included in each row. So after subtracting these two cases, number of ways are = $({}^8C_6 - 2) = 28 - 2 = 26$.

78 Seven girls can stand in a circle by $\frac{(7-1)!}{2!}$=360 number of ways, because there is no difference in anticlockwise and clockwise order of their standing in a circle.

79 For any collection of n numbers on the blackboard we consider the following quantity X: the sum of all the numbers minus n.

If the sum of all the numbers except ‘a’ and ‘b’ equals s, then before transformation,

X = s + a + b – n, and after transformation,

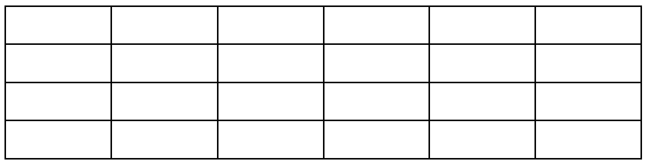
$X = s + (a + b - 1) - (n - 1) = s + a + b - n$.

So the value of X is same : it is invariant.

Initially, we have $X = (1 + 2 + \ldots . + 19 + 20) - 20 = 190$

After 19 operations, when there will be only one number on the blackboard, X will be equal to 190
The last number = X + 1 = 191

80 Mark out |x| (cm) squares in the area of rectangle as shown below:



There will be 24 of these ‘pigeonholes’ i.e. boxes and we take points to be the objects. By | x | square. These can be separated by at most the length of diagonal

$$\sqrt{1^2 + 1^2} = \sqrt{2}\, cm$$

81 Consider the pairing of elements as follows: (3, 155), (11, 147), ……, (75, 83)

Each element of A appears just once, and we have to such pairs.

If we take all the left-hand members or all the right-hand members our set will be 158 free.

But by the Pigeonhole Principle, if we extract 10 + 1 = 11 element of A we have atleast two belonging to one such pair. Hence largest possible membership is 10.

82 There are three digits: 1, 2, 3, and an n-digit number is to be formed, repetitions allowed.

Thus, the number of possibilities is:

$3 \times 3 \times 3 \times \cdots \times 3 = 3^n$

For the second part of the question:

In (1), we include the possibility that all the n digits consist of:
(a) 1 only,
(b) 2 only,
(c) 3 only,
and again in (2), we include the possibility that the n digits consist of only:
(i) 1 and 2,
(ii) 2 and 3,
(iii) 1 and 3.

The number of n-digit numbers all of whose digits are 1 or 2 or 3 is: 3^n

i) The number of n-digit numbers all of whose digits are 1 and 2 (each of 1 and 2 occurring at least once) is: $2^n - 2$

ii) The number of n-digit numbers all of whose digits are 2 and 3 (each of 2 and 3 occurring at least once) is again: $2^n - 2$

iii) The number of n-digit numbers all of whose digits are 1 and 3 (each of 1 and 3 occurring at least once) is: $2^n - 2$

Thus, the total numbers made up of the digits 1, 2 and 3 is:

$3^n - 3(2^n - 2) - 3$

$= 3^n - 3 \times 2^n + 6 - 3$

$= 3^n - 3 \times 2^n + 3$.

The required count is:

$3^n - 3 \times 2^n + 3$.

83 (i) Using derangement formulae:

Number of ways to place 6 letters in 6 envelopes such that all are placed in wrong envelopes:

$$= 6! \times \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^6}{6!}\right]$$

$$= 720 \times [1 - 1 + 1/2 - 1/6 + 1/24 - 1/120 + 1/720]$$

$$= 720 \times [0 + 0.5 - 0.1667 + 0.0417 - 0.0083 + 0.0014 = 720 \times 0.3681 \approx 265.$$

(ii) Number of ways to place letters such that at least 4 letters are placed in correct envelopes:

= (4 letters correct and 2 wrong) + (5 letters correct and 1 wrong) + (All 6 letters correct)

= C(6,4) × D(2) + C(6,5) × D(1) + 1

= 15 × 1 + 6 × 0 + 1 = 16.

(iii) Number of ways to place 6 letters in 6 envelopes such that at most 3 letters are placed in wrong envelopes:

= (All correct) + (1 letter wrong) + (2 letters wrong) + (3 letters wrong)

= 1 + C(6,1) × D(1) + C(6,2) × D(2) + C(6,3) × D(3)

= 1 + 6 × 0 + 15 × 1 + 20 × 2

= 1 + 0 + 15 + 40 = 56.

84 Required numbers = Total no of six-digit numbers – No of six digit numbers with all digits odd

$= 9 \times 10^5 - 5^6$

85 assuming all 18 beads are distinct (each bead has a unique colour), the number of possible ways to choose 12 beads is= C(18 12)

For 12 distinct beads, the number of unique arrangements around a necklace is:

$$(12-1)!\ /2=11!\ /2$$

11! Accounts for circular permutations (rotations).

Divide by 2 because flipping the necklace over doesn't produce a new necklace (reflection symmetry).

So, total number of distinct 12-bead necklaces you can make from 18 distinct beads is

$$= C(18\ \ 12).\ 11!\ /2$$

86 A typical term is a^{k1} ,a^{k2} ,a^{k3} ,a^{k4} where k_1, k_2, k_3, k_4 are non-negative integers whose sum = 24.

The number of terms is the same as the number of distributions of 24 identical balls in four distinguishable cells, empty cell allowed. This is C(24+4-1 , 24)= C(27 , 24).

87 The number of ways of dividing n identical objects into r groups so that no group remains empty = $^{(n-1)}C_{(r-1)} = {}^{(5-1)}C_{(3-1)} = {}^4C_2 = 6$.

88 Divide the square into 9-unit squares as given in the figure.

Out of the 10 points distributed in the big square, at least

one of the small squares must have at least two points by

the Pigeon hole principle. These two points being in a unit

square, are at the most $\sqrt{2}$. Units distance apart as $\sqrt{2}$. Is

the length of the diagonal of the unit square.

89 The number of all the possible ways of putting 6 letters into 6 envelopes is 6!.

There is only one way of putting all the letters correctly into the corresponding envelopes.

Hence if there is a mistake, at least 2 letters will be in the wrong envelope.

Hence the required answer is 6! – 1 = 719.

90 This is a classic integer partition problem with positive integers. The number of integer solutions to x1 + x2 + x3 + x4 = 10 with xi ≥ 1 is C(9, 3) = 84.

91 The number of 7-digit numbers with 1 in the left most place and containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once is 6! =720.

But 120 of these end in 5 and hence are divisible by 5. Thus, the number of 7-digit numbers with 1 in the left most place and containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once but not divisible by 5 is 720-120=600.

Similarly, the number of 7-digits numbers with 2 and 3 in the left most place and containing each of the digits 1, 2, 3, 4, 5, 6, 7 exactly once but not divisible by 5 is also 600 each.

These account for 1800 numbers.

Hence, 2000th number must have 4 in the left most place.

Again the number of such 7-digit numbers beginning with 41, 42 and not divisible by 5 is

120 -24= 96 each and these account for 192 numbers. This shows that 2000th number in

the list must begin with 43.

The next 8 numbers in the list are 4312567, 4312576, 4312657, 4312756, 4315267,

4315276, 4315627 and 4315672.

Thus, 2000th number in the list is 4315672.

92 We observe that $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = 0$,if and only if $x \in \{1,2,3, \ldots \ldots, 98\}$ and there are 98 such numbers.

If we want $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = 1$, then x should lie in the set {101, 102,...,197}, which accounts for 97 numbers.

In general, if we require $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = k$, where $k \geq 1$, then x must be in the set $\{101k, 101k+1, \ldots \ldots, 99(k+1)-1\}$ and there are 99 -2k such numbers.

Observes that this set is not empty only if $99(k+1) - 1 \geq 101k$ and this requirement is met only if $k \leq 49$.

Thus, the total number of positive integers x for which $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right]$ is given by

$$98 + \sum_{k=1}^{49} (99 - 2k) = 2499$$

93 Since 20,b,c are in the harmonic progression, we have

$$\frac{1}{20} + \frac{1}{\mathrm{c}} = \frac{2}{\mathrm{b}}$$

which reduces to $bc + 20b - 40c = 0$.

This may also be written in the form $(40 - b)(c + 20) = 800$

Thus, we must have $20 < b < 40$ or equivalently, $0 < 40 - b < 20$.

Let us consider the factorization of 800 in which one term is less than 20

$$(40-b)(c+20) = 800 = 1 \times 800 = 2 \times 400 = 4 \times 200 = 5 \times 160 = 8 \times 100$$
$$= 10 \times 80 = 16 \times 50$$

We thus get the pairs

(b,c) = (39, 780) (38, 380), (36, 180), (35, 140),(32, 80), (30, 60), (24, 30).

Among these 7 pairs, we see that only 5 pairs (39, 780), (38, 380), (36, 180), (35, 140),

(30. 60) fulfill the condition of divisibility: b divides c. Thus, there are 5 triples satisfying

the requirement of the problem.

94 We divide the even 4-digit numbers having non-zero digits into 4 classes those ending in 2, 4, 6, 8.

(A) Suppose a 4-digit number ends in 2. Then the second right digit must be odd in order to be divisible by 4. Thus the last 2 digits must be of the form 12, 32, 52, 72 or 92. If a number ends in 12, 52 or 92, then the previous digits must be even in order not to be divisible by 8 and we have 4 admissible even digits. Now the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways and we get $9 \times 4 \times 3 = 108$ such numbers. If a number ends in 32 or 72, then the previous digit must be odd in order not to be divisible by 8 and we have 5 admissible odd digits. Here again the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways, and we get 9 x 5 × 2= 90 such numbers.

Thus, the number of 4-digit numbers having non-zero digits, ending in 2, divisible by 4 but not by 8 is 108 + 90 = 198.

(B) If the number ends in 4, then the previous digit must be even for divisibility by 4.

Thus the last two digits must be of the form 24, 44, 54, 84. If we take numbers ending with 24 and 64, then the previous digit must be odd for non-divisibility by 8 and the left most digit can be any non-zero digit. Here we get $9 \times 5 \times 2 = 90$ such numbers. If the last two digits are of the form 44 and 84, then previous digit must be even for non-divisibility by 8. And the left most digit can take 9 possible values. We thus get $9 \times 4 \times 2 = 72$ numbers.

Thus the admissible numbers ending in 4 is 90 + 72 = 162.

I If a number ends with 6, then the last two digits must be of the form 16, 36, 56, 76, 96

For numbers ending with 16, 56, 76 the previous digit must be odd. For numbers ending with 36, 76, the previous digit must be even. Thus we get here $(9 \times 5 \times 3) + (9 \times 4 \times 2) = 135 + 72 = 207$ numbers.

(D) If a number ends with 8, then the last two digits must be of the form 28, 48, 68, 88.

For numbers ending with 28, 68, the previous digit must be even. For numbers ending with 48, 88, the previous digit must be odd. Thus we get$(9 \times 4 \times 2) + (9 \times 5 \times 2) = 72 + 90 = 162$ numbers.

Thus the number of 4-digit numbers, having non-zero digits, and divisible by 4 but not by 8 is 198 + 162 + 207 + 162 = 729.

95 We have N=abcd

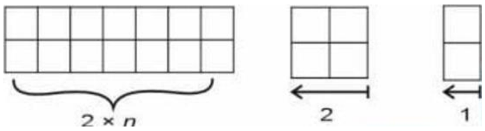
First place a can be filled in 2 ways(i.e.4,5)

For b and c total possibilities are 6 $(3 \leq b \leq c \leq 6)$ (i.e.34,35,36,45,46,56)

Last place d can be filled in 2 ways i.e. 0,5

Hence,total numbers = $2 \times 6 \times 2 = 24$. So, M/3=8

96 Let, abcd be the odd number.



Therefore d can take values as 1 or 3

=> 2 posibility

a can take 3 values

b can take 2 values

c can take 1 value

Total values $= 2 \times 3 \times 2 \times 1 = 12$ numbers

1. $a_n = 2a_{n-2} + a_{n-1}$

   $a_1 = 1, a_2 = 3$

   $a_3 = 2a_1 + a_2 = 5$

   $a_4 = 2a_2 + a_3 = 11$

   $a_5 = 2a_3 + a_4 = 21$

   $a_6 = 43, a_7 = 85$

97 g girls can be seated around a table in (g – 1)!

This positioning of g girls create g gaps for b boys to be seated.

B boys in those g gaps can be seated in C (g , b). b! ways.

Total number of arrangements = (g – 1)! × C (g , b). b!

98 Let abcde be the required odd integers.

A can be chosen from 3, 4, 5, 6 and 7 and e can be chosen from 1, 3, 5, 7, 9. Note that 3, 5 and 7 can occupy both the positions a and e.

So, let us consider the case where one of 3, 5, 7 occupies the position a.

Case 1: If a gets one of the values 3, 5, 7, then there are 3 choices for a, but then, e has just four choices as repetition is not allowed. Thus, a and e can be chosen in this case in 3 × 4 = 12 ways.

The 3 positions b, c, d can be filled from among the remaining 8 digits in 8 × 7 × 6 ways. Total number of ways in this case = 12 × 8 × 7 × 6 = 4,032.

Case 2: If a takes the values 4 or 6, then there are two choices for a and there are five choices for e.

There are again eight choices altogether for the digits b, c, d which could be done in 8 × 7 × 6 ways.

Therefore in this case, the total numbers are 2 × 5 × 8 × 7 × 6 = 3,360.

Hence, total number of odd numbers between 30,000 to 80,000, without repetition of digits is 4,032 + 3,360 = 7,392.

99 The vowels in 'COMBINATORICS' are O, I, A, O, I (i.e., A, I, I, O, O). After arranging them alphabetically as A, I, I, O, O and choosing positions for vowels, the answer is (13 choose 5) × (5! / (2!×2!)) × (8!), accounting for vowel positions, vowel arrangement, and consonant permutations.

100 Since we are using numbers from 1 to 25 once, the total sum is (25×26)/2 = 325. Each row must sum to 65 (325 ÷ 5). The number of such magic square-type arrangements is highly nontrivial and requires enumeration techniques, typically handled via computational methods. Exact count not known in closed form.

101 This is a constrained derangement problem.

First, total derangements D(10) ≈ 10!/e ≈ 1,334,961.

Then subtract arrangements where any two consecutive numbers appear next to each other. This is very complex; approximate count is difficult without recursion or programmatic inclusion-exclusion.

102 Place 6 ones first; there are 7 gaps between them. Choose 4 gaps to place 2s so that no two are adjacent. The number of such arrangements is C(7, 4) = 35.

7. Answer: This is a recurrence relation. Let f(n) be the number of such sequences. Then f(1) = 3, f(2) = 6, and $f(n) = 2 \times f(n-1) + 2 \times f(n-2)\ for\ n \geq 3$.

INEQUALITY

A. Order in the real numbers

Property : *Every real number x has one and only one of the following properties:*

(i) $x = 0$,

(ii) $x \in P$ (that is, $x > 0$),

(iii) $-x \in P$ (that is, $-x > 0$).

Property : *If x, y ∈ P, then x+y ∈ P (in symbols* $x > 0, y > 0 \Rightarrow x+y > 0$*).*

Property : *If x, y ∈ P, then xy ∈ P (in symbols* $x > 0, y > 0 \Rightarrow xy > 0$*).*

Ex 1.(i) *If* $a < b$ *and c is any number, then* $a + c < b + c$.

(ii) *If* $a < b$ *and* $c > 0$, *then* $ac < bc$.

In fact, to prove (i) we see that $a + c < b + c \Leftrightarrow (b + c) - (a + c) > 0 \Leftrightarrow b - a > 0 \Leftrightarrow a < b$

To prove (ii), we proceed as follows: $a < b \Rightarrow b - a > 0$

and since $c > 0$, then $(b - a)c > 0$, therefore $bc - ac > 0$ and then $ac < bc$.

Ex 2. Prove the following assertions.

(i) $a < 0, b < 0 \Rightarrow ab > 0$

(ii) $a < 0, b > 0 \Rightarrow ab < 0$

(iii) $a < b, b < c \Rightarrow a < c$

(iv) $a < b, c < d \Rightarrow a + c < b + d$

(v) $a < b \Rightarrow -b < -a$

(vi) $a > 0 \Rightarrow \frac{1}{a} > 0$

(vii) $a < 0 \Rightarrow \frac{1}{a} < 0.$

(viii) $a > 0, b > 0 \Rightarrow \frac{a}{b} > 0$

(ix) $0 < a < b, 0 < c < d \Rightarrow ac < bd$

(x) $a > 1 \Rightarrow a^2 > a$

(xi) $0 < a < 1 \Rightarrow a^2 < a$

Property : (i) $a > 0, b > 0 \text{ and } a^2 < b^2 \Rightarrow a < b$.

(ii) If $b > 0, \frac{a}{b} > 1 \text{ if and only if } a > b$

Q. N. 4. For any real numbers *x*, *a* and *b*, the following hold.

(i) $|x| \geq 0$, and is equal to zero only when $x = 0$.

(ii) $|-x| = |x|$.

(iii) $|x|^2 = x^2$.

(iv) $|ab| = |a||b|$

(v) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \quad b \neq 0$

Property (Triangle inequality): *The triangle inequality states that for any pair of real numbers a and b,*

$$|a+b| \leq |a| + |b|$$

Moreover, the equality holds if and only if ab ≥ 0.

Property 7. Let *x*, *y*, *a*, *b* be real numbers, prove that

(i) $|x| \leq b \Leftrightarrow -b \leq x \leq b,$

(ii) $\big||a| - |b|\big| \leq |a-b|,$

(iii) $x^2 + xy + y^2 \geq 0,$

(iv) $x > 0, y > 0 \Rightarrow x^2 - xy + y^2 > 0.$

B. The Quadratic Function $ax^2 + 2bx + c$

One very useful inequality for the real numbers is $x^2 \geq 0$, which is valid for any real number. The use of this inequality leads to deducing many other inequalities. In particular, we can use it to find the maximum or minimum of a Quadratic function $ax^2 + 2bx + c$. These Quadratic functions appear frequently in optimization problems or in inequalities.

One common question consists in proving that if $a > 0$, the Quadratic function $ax^2 + 2bx + c$ will have its minimum at $x = \frac{-b}{a}$ and the minimum value is $c - \frac{b^2}{a}$. In fact, $ax^2 + 2bx + c = a\left(x^2 + 2\frac{b}{a}x + \frac{b^2}{a^2}\right) + c - \frac{b^2}{a} = a\left(x + \frac{b}{a}\right)^2 + c - \frac{b^2}{a}$

Since $\left(x + \frac{b}{a}\right)^2 \geq 0]$ and the minimum value of this expression, zero, is attained when $x = \frac{-b}{a}$ we conclude that the minimum value of the Quadratic function is $c - \frac{b^2}{a}$.

If a<0, the Quadratic function $ax^2 + 2bx + c$ will have a maximum at $x = \frac{-b}{a}$ and its value at this point is $c - \frac{b^2}{a}$. In fact, since $ax^2 + 2bx + c = a\left(x + \frac{b}{a}\right)^2 + c - \frac{b^2}{a}$ and since $a\left(x + \frac{b}{a}\right)^2 \leq 0$ (because $a < 0$), the greatest value of this last expression is zero, thus the Quadratic function is always less than or equal to $c - \frac{b^2}{a}$ and assumes this value at the point $x = \frac{-b}{a}$.

Ex 3. *If x, y are positive numbers with* $x + y = 2a$, *then the product xy is maximal when* $x = y = a$.

If $x + y = 2a$, then $y = 2a - x$. Hence, $xy = x(2a - x) = -x^2 + 2ax = -(x-a)^2 + a^2$ has a maximum value when $x = a$, and then $y = x = a$.

This can be interpreted geometrically as *"of all the rectangles with fixed perimeter, the one with the greatest area is the square"*. In fact, if *x*, *y* are the lengths of the sides of the rectangle, the perimeter is $2(x + y) = 4a$, and its area is *xy*, which is maximized when $x = y = a$.

Ex 4. *If x, y are positive numbers with* $xy = 1$, *the sum x+y is minimal when* $x = y = 1$.

If $x\,y = 1$, then $y = \frac{1}{x}$. It follows that x+$y = x + \frac{1}{x} = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2$,

$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)$=0 and then $x + y$ is minimal when = 0, that is, when x = 1. Therefore, $x = y = 1$.

This can also be interpreted geometrically in the following way, *"of all the rectangles with area* 1*, the square has the smallest perimeter"*. In fact, if *x*, *y* are the lengths of the sides of the rectangle, its area is *xy* = 1 and its perimeter is $2(x + y) = 2\left(x + \frac{1}{x}\right) =$

$2\left\{(\sqrt{x} - \frac{1}{\sqrt{x}})^2 + 2\right\} \geq 4.$

Moreover, the perimeter is 4 if and only if $\sqrt{x} - \frac{1}{\sqrt{x}} =$ = 0, that is, when $x = y = 1$.

Ex 5. If a, b > 0*, then* $\frac{a}{b} + \frac{b}{a} \geq 2$ *, and the equality holds if and only if a = b.*

It is enough to consider the previous Ex 4 with $x = \frac{a}{b}$

C. A fundamental inequality

Arithmetic Mean-Geometric Mean

The first inequality that we consider, fundamental in optimization problems, is the inequality between the arithmetic mean and the geometric mean of two nonnegative numbers *a* and *b*, which is expressed as

$\frac{a+b}{2} \geq \sqrt{ab}$ (AM≥GM).

Moreover, the equality holds if and only if *a* = *b*.

The numbers $\frac{a+b}{2}$ and √*ab* is known as the arithmetic mean and the geometric mean of *a* and *b*, respectively. To prove the inequality, we only need to observe that

$$\frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0.$$

And the equality holds if and only if $\sqrt{a} = \sqrt{b}$, that is, when *a* = *b*.

D. (The AM-GM inequality)

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

E. A Wonderful Inequality :

The rearrangement inequality consider two collections of real numbers in increasing order,

$$a_1 \leq a_2 \leq \cdots \leq a_n \quad and \quad b_1 \leq b_2 \leq \cdots \leq b_n$$

For any permutation $(a'_1, a'_2, a'_3, , \cdots, a'_n)$ of $\left(a_{1,}a_{2\,,} \cdots, a_n\right)$., it happens that

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a'_1b_1 + a'_2b_2 + \cdots + a'_nb_n \geq a_nb_1 + a_{n-1}b_2 + \cdots + a_1b_n \quad \textbf{(1.4)}$$

Ex 6. (IMO, 1964) *Suppose that a, b, c are the lengths of the sides of a triangle. Prove that*

$$a^2(b + c - a) + b^2(a + c - b) + c^2(a + b - c) \leq 3abc.$$

Since the expression is a symmetric function of *a*, *b* and *c*, we can assume, without loss of generality, that *b*(*a* + *c* − *b*) ≤ *c*(*a* + *b* − *c*).. In this case, *a*(*b* + *c* − *a*) ≤ *c*(*a* + *b* − *c*).
For instance, the first inequality is proved in the following way: $a(b + c - a) \leq$
$b(a + c - b) \leq c(a + b - c)$

$\Leftrightarrow\ ab + ac - a^2 \le ab + bc - b^2$
$\Leftrightarrow (a - b)c \le (a + b)(a - b)$
$\Leftrightarrow (a - b)(a + b - c) \ge 0.$

By (1.3) of the rearrangement inequality, we have

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c)$$
$$\le ba(b + c - a) + cb(c + a - b) + ac(a + b - c)a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c)$$
$$\le ca(b + c - a) + ab(c + a - b) + bc(a + b - c)$$
$$2[a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c)] \le 6abc$$

Therefore, $[a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c)] \le 3abc]$

Ex 7. (IMO, 1983) *Let a, b and c be the lengths of the sides of a triangle. Prove that* $a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \ge 0$.

Consider the case $c \le b \le a$ (the other cases are similar).

As in the previous Q. N. , we have that $a(b+c-a) \le b(a+c-b) \le c(a + b - c)$ and since $\frac{1}{a} \le \frac{1}{b} \le \frac{1}{c}$, using Inequality (1.2) leads us to

$$\frac{1}{a}a(b + c - a) + \frac{1}{b}b(c + a - b) + \frac{1}{c}c(a + b - c)$$
$$\ge \frac{1}{c}a(b + c - a) + \frac{1}{a}b(c + a - b) + \frac{1}{b}c(a + b - c)$$

Therefore,

$$a + b + c \ge \frac{a(b - a)}{c} + \frac{b(c - b)}{a} + \frac{c(a - c)}{b} + a + b + c$$

It follows that $\frac{a(b-a)}{c} + \frac{b(c-b)}{a} + \frac{c(a-c)}{b} \le 0$. Multiplying by abc, we obtain

$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \ge 0$.

<u>F. (Cauchy-Schwarz inequality):</u>

For real numbers $x_1, ..., x_n, y_1, ..., y_n$, the following inequality holds:

$(\sum_{i=1}^{n} x_i y_i)^2 \le (\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2)$.

The equality holds if and only if there exists some $\lambda \in$ R with $x_i = \lambda y_i$ for all $i = 1,2,...,n$.

<u>G. Nesbitt's inequality:</u> *For a, b, c ∈ R^+, we have*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}.$$

Ex 8. (IMO, 1995) *Let a, b, c be positive real numbers with* $abc = 1$

Prove that

$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}$.

Without loss of generality, we can assume that $c \le b \le a$. Let $x = \frac{1}{a}$, $y = \frac{1}{b}$ and $z = \frac{1}{c}$ thus

$$S = \frac{1}{a^3(b + c)} + \frac{1}{b^3(c + a)} + \frac{1}{c^3(a + b)}$$

$$= \frac{x^3}{\frac{1}{y}+\frac{1}{z}} + \frac{y^3}{\frac{1}{z}+\frac{1}{x}} + \frac{z^3}{\frac{1}{x}+\frac{1}{y}} = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}$$

Since $x \le y \le z$, we can deduce that $x + y \le z + x \le y + z$ and also that $\frac{x}{y+z} \le \frac{y}{z+x} \le \frac{z}{x+y}$. Using the rearrangement inequality (1.2), we show that

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \ge \frac{xy}{y+z} + \frac{yz}{z+x} + \frac{zx}{x+y},$$

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \ge \frac{xz}{y+z} + \frac{yx}{z+x} + \frac{zy}{x+y}$$

which in turn leads to $2S \ge x + y + z \ge 3\sqrt[3]{xyz} = 3$. Therefore, $S \ge \frac{3}{2}$.

<u>H.Tchebyshev's inequality</u> : *Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$, then*

$\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} \ge \frac{a_1+a_2+\cdots+a_n}{n} \cdot \frac{b_1+b_2+\cdots+b_n}{n}$.

<u>I.Holder's inequality</u>

Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be positive numbers and $a, b > 0$ such that $\frac{1}{a} + \frac{1}{b} = 1$, then

$$\sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} x_i^a\right)^{1/a} \left(\sum_{i=1}^{n} y_i^b\right)^{1/b}.$$

<u>J. Bernoulli's inequality</u>

(i) For any real number $x > -1$ and for every positive integer n, we have $(1 + x)^n \ge 1 + nx$

(ii) Use this inequality to provide another proof of the AM-GM inequality.

<u>K. Schur's inequality :</u>

If x, y, z are positive real numbers and n is a positive integer, we have

$$x^n(x-y)(x-z) + y^n(y-z)(y-x) + z^n(z-x)(z-y) \ge 0$$

For the case $n = 1$, the inequality can take one of the following forms:

(a) $x^3 + y^3 + z^3 + 3xyz \ge xy(x+y) + yz(y+z) + zx(z+x)$

(b) $xyz \ge (x+y-z)(y+z-x)(z+x-y)$

(c) $x + y + z = 1, \quad 9xyz + 1 \ge 4(xy + yz + zx)$

<u>L. Power mean inequality:</u>

Let $x_1, x_2, \ldots, x_n$ be positive real numbers and let $t_1, t_2, \ldots, t_n$ be positive real numbers adding up to 1. Let r and s be two nonzero real numbers such that $r > s$. then $(t_1x_1^r + \cdots + t_nx_n^r)^{\frac{1}{r}} \ge (t_1x_1^s + \cdots + t_nx_n^s)^{\frac{1}{s}}$ with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Ex 9. (IMO, 1995) *Let a, b, c be positive real numbers such that* $abc = 1$. *Prove that* $\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \ge \frac{3}{2}$.

Observe that $\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} = \frac{\frac{1}{a^2}}{a(b+c)} + \frac{\frac{1}{b^2}}{b(c+a)} + \frac{\frac{1}{c^2}}{c(a+b)}$

$\ge \frac{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}{2(ab+bc+ca)} = \frac{ab+bc+ca}{2(abc)} \ge \frac{3\sqrt[3]{(abc)^2}}{2} = \frac{3}{2}$.

M. The AM-GM inequality:

For real positive numbers $y_1, y_2, \ldots, y_n$,

$$\frac{y_1 + y_2 + \cdots + y_n}{n} \geq \sqrt[n]{y_1 y_2 \cdots y_n}$$

Note that the AM-GM inequality is equivalent to $\frac{1}{n}\sum_{i=1}^{n} x_i^n \geq x_1 x_2 \cdots x_n$ Where $x_i = \sqrt[n]{y_i}$.

N. Popoviciu's inequality: If *I* is an interval and *f* : *I* → R is a convex function, then for *a*, *b*, *c* ∈ *I* the following inequality holds:

$$\frac{2}{3}\left[f\left(\frac{a+b}{2}\right) + f\left(\frac{b+c}{2}\right) + f\left(\frac{c+a}{2}\right)\right] \leq \frac{f(a)+f(b)+f(c)}{3} + f\left(\frac{a+b+c}{3}\right).$$

Property : *For any positive number x, we have* $x + \frac{1}{x} \geq 2$.

Observe that $x + \frac{1}{x} = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2 \geq 2$. Moreover, the equality holds if and only if $\sqrt{x} - \frac{1}{\sqrt{x}} = 0$, that is, when *x* = 1.

Ex 10. $\boldsymbol{If\ x > 0, prove\ that\ x + \frac{1}{x} \geq 2.}$

$Sol: \ By\ AM - GM\ inequality\ \ x + \frac{1}{x} \geq 2\sqrt{x \times \frac{1}{x}} = 2.$

Ex 11. $\boldsymbol{Prove\ that\ for\ real\ numbers\ a, b: a^2 + b^2 \geq 2ab.}$

$\boldsymbol{Sol: a^2 + b^2 - 2ab = (a - b)^2}$

Ex 12. $If\ a, b, c\ are\ positive\ reals\ such\ that\ abc = 1, prove\ that\ a + b + c \geq ab + bc + ca.$

$Sol: Use\ symmetry\ and\ standard\ inequality\ techniques\ like\ AM - GM\ and\ substitution$

Ex 13. $\boldsymbol{If\ a + b + c = 1\ and\ a, b, c > 0, prove\ that \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.}$

$Sol: By\ AM - HM\ inequality: \ \frac{a + b + c}{3} \geq \frac{3}{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}$

$$\Rightarrow \frac{1}{3} \geq \frac{3}{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}$$

$$\Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.$$

Ex 14. $\boldsymbol{If\ x, y, z > 0, prove\ that\ x^2 + y^2 + z^2 \geq xy + yz + zx.}$

$Sol: x^2 + y^2 + z^2 - xy - yz - zx = ½[(x - y)^2 + (y - z)^2 + (z - x)^2] \geq 0.$

Ex 15. **If a, b ,c are positive integers and $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=3$, then find the value of $a^3+b^3+c^3$.**

Sol : Using AM≥ GM , we get $\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \geq \sqrt[3]{\frac{a}{b}.\frac{b}{c}.\frac{c}{a}}$

$\Rightarrow \frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3} \geq 1 \Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$

Given , $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=3$ occurs when $\frac{a}{b}=\frac{b}{c}=\frac{c}{a}$

Hence a=b=c= k=1

Therefore, least value of $a^3+b^3+c^3$ *is* 3.

Ex16. **It is given that** $\frac{x}{(2y+1)}+\frac{y}{(x+1)}=1$ ***and*** $\frac{(y+1)}{(x+1)}+\frac{1}{(2y+1)}=1$,
Find the value of $x+2y$.

Sol: *Adding* (i) *and* (ii):

$$\frac{(x+1)}{(2y+1)}+\frac{(2y+1)}{(x+1)}=2$$

Using AM $\geq$ *GM*:

$$\frac{\left[\frac{(x+1)}{(2y+1)}+\frac{(2y+1)}{(x+1)}\right]}{2} \geq 1 \Rightarrow \frac{(x+1)}{(2y+1)}+\frac{(2y+1)}{(x+1)} \geq 2$$

$$\Rightarrow \frac{(x+1)}{(2y+1)}=\frac{(2y+1)}{(x+1)}=1 \Rightarrow x+1=2y+1 \Rightarrow x=2y$$

From equation (ii):

$$\frac{(y+1)}{(x+1)}+\frac{1}{(2y+1)}=1$$

Substitute $x=2y$:

$$\frac{(y+1)}{(2y+1)}+\frac{1}{(2y+1)}=1 \Rightarrow \frac{(y+2)}{(2y+1)}=1$$

Ex 17. **Given: It is given that** $\frac{a}{2b+1}+\frac{2b}{3c+1}+\frac{3c}{a+1}=1$ ***and*** $\frac{1}{a+1}+\frac{1}{2b+1}+\frac{1}{3c+1}=2$**, Find**

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Sol:

$$\frac{a}{2b+1}+\frac{2b}{3c+1}+\frac{3c}{a+1}=1 \ldots\ldots\ldots(i)$$

$$\frac{1}{a+1}+\frac{1}{2b+1}+\frac{1}{3c+1}=2\ldots\ldots\ldots(ii)$$

Adding the given two equations, we get:

$$\frac{3c+1}{a+1}+\frac{a+1}{2b+1}+\frac{2b+1}{3c+1}=3$$

$Using\ AM \geq GM\ inequality$:

$$\frac{\left[\frac{3c+1}{a+1}+\frac{a+1}{2b+1}+\frac{2b+1}{3c+1}\right]}{3} \geq 1 \Rightarrow \frac{3c+1}{a+1}+\frac{a+1}{2b+1}+\frac{2b+1}{3c+1} \geq 3$$

$\Rightarrow\ Equality\ holds, so$:

$$\frac{3c+1}{a+1}=\frac{a+1}{2b+1}=\frac{2b+1}{3c+1}=1$$

$From\ above$:

$$3c+1=a+1=2b+1=k$$

$From\ equation\ (ii)$:

$$\frac{1}{a+1}+\frac{1}{2b+1}+\frac{1}{3c+1}=\frac{3}{k}=2 \Rightarrow k=\frac{3}{2}$$

$Solving\ for\ a, b, c$:

$$a+1=\frac{3}{2} \Rightarrow a=\frac{1}{2}, v2b+1=\frac{3}{2} \Rightarrow b=\frac{1}{4}, 3c+1=\frac{3}{2} \Rightarrow c=\frac{1}{6}$$

$Final\ Value$:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{\frac{1}{2}}+\frac{1}{\frac{1}{4}}+\frac{1}{\frac{1}{6}}=2+4+6=12.$$

Ex 18. **Solve: $2x^2 + x - 1 \geq 0$**

Sol: Step 1: Factor the Quadratic expression

$2x^2 + x - 1 \geq 0. \Rightarrow (2x - 1)(x + 1) \geq 0$

Case I: $(2x - 1) \geq 0$ and $(x + 1) \geq 0$

$\Rightarrow x \geq 1/2$ and $x \geq -1$

$\Rightarrow x \geq ½ \Rightarrow x \in [1/2, \infty)$

Case II: $(2x - 1) \leq 0$ and $(x + 1) \leq 0$

$\Rightarrow x \leq 1/2$ and $x \leq -1$

$\Rightarrow x \leq -1 \Rightarrow x \in (-\infty, -1]$

So the final solution is, $x \in (-\infty, -1] \cup [1/2, \infty)$.

Ex 19. **Solve $|x^2 - 4x + 3| < 2$.**

Sol: $|x^2 - 4x + 3| < 2 \Rightarrow -2 < x^2 - 4x + 3 < 2$

(Compound Quadratic Inequality)

Case-I: $-2 < x^2 - 4x + 3$

$\Rightarrow x^2 - 4x + 5 > 0$

$\Rightarrow (x - 2)^2 + 1 > 0$

$\Rightarrow x \in \mathbb{R}$

Case-II: $x^2 - 4x + 3 < 2$

$\Rightarrow x^2 - 4x + 1 < 0$

$\Rightarrow x \in (2 - \sqrt{3}, 2 + \sqrt{3})$

So, solution is $x \in (2 - \sqrt{3}, 2 + \sqrt{3})$.

Ex 20: **Solve $|x^2 - 9| + |x^2 - 16| < 47$**

Sol: This expression changes sign at $x = \pm 3, \pm 4$

Breaking the number line into intervals, we get:

$(-\infty, -4), [-4, -3), [-3, 3), [3, 4), [4, \infty)$

Case-I: $x < -4$

$|x^2 - 9| + |x^2 - 16| < 47$

$\Rightarrow (x^2 - 9) + (x^2 - 16) < 47$

$\Rightarrow 2x^2 - 25 < 47 \Rightarrow 2x^2 < 72$

$\Rightarrow x^2 < 36 \Rightarrow x \in (-6, 6)$

Since $x < -4 \Rightarrow x \in (-6, -4)$

Case-II: $-4 \le x < -3$

$(x^2 - 9) + (16 - x^2) = 7 < 47$ which is always true.

$\Rightarrow x \in [-4, -3)$

Case-III: $3 \le x < 3$

$-(x^2-9)-(x^2-16) = 25 - 2x^2 < 47$

$\Rightarrow -2x^2 < 22 \Rightarrow x^2 > -11$, which is always true

$\Rightarrow x \in [-3, 3)$

Case-IV: $3 \le x < 4$

$(x^2-9)+(16-x^2) = 7 < 47$ which is Always true

$\Rightarrow x \in [3, 4)$

Case-V: $x \ge 4$

$|x^2-9| + |x^2-16| < 47$

$\Rightarrow (x^2-9)+(x^2-16) < 47$

$\Rightarrow 2x^2 - 25 < 47 \Rightarrow 2x^2 < 72$

$\Rightarrow x^2 < 36 \Rightarrow x \in (-6, 6)$. Since $x \ge 4$

$\Rightarrow x \in [4, 6)$

From all cases we get solutions are:

$x \in (-6,-4), x \in [-4,-3), x \in [-3, 3), x \in [3, 4), x \in [4, 6)$

Hence the solution is $x \in (-6,6)$.

Ex 21. **Prove that $n! < \left(\frac{n+1}{2}\right)^n$, for n ≥ 2**

Sol: Using A.M ≥ G.M (Arithmetic Mean ≥ Geometric Mean):

A.M = $\frac{(1+2+3+\dots+n)}{n} = \frac{n(n+1)}{(2n)}$, $G.M = (1 \cdot 2 \cdot 3 \cdot \dots \cdot n)^{\frac{1}{n}} = (n!)^{\frac{1}{n}}$

$$So, \frac{n(n+1)}{(2n)} > (n!)^{\frac{1}{n}} \Rightarrow \frac{n+1}{2} > (n!)^{\frac{1}{n}} \Rightarrow \left(\frac{n+1}{2}\right)^n > n! \Rightarrow n! < \left(\frac{n+1}{2}\right)^n$$

Ex 22. $\boldsymbol{Let, y = \frac{1}{\left(x + \frac{1}{x} + 5\right)}, x \neq 0. \; Find\ the\ maximum\ and\ minimum\ value\ of\ y.}$

$Sol: Given, y = \dfrac{1}{\left(x + \frac{1}{x} + 5\right)}$

$Using\ A.M \geq G.M: \; x + \dfrac{1}{x} \geq 2, for\ x > 0$

$\Rightarrow \; Minimum\ value\ of\ x + \dfrac{1}{x} + 5\ is\ 2 + 5 = 7$

$\Rightarrow \; Maximum\ value\ of\ y\ is \dfrac{1}{7}\ When\ x < 0, then\ -x, -\dfrac{1}{x} > 0$

$Again\ using\ A.M \geq G.M: -x - \dfrac{1}{x} \geq 2 \Rightarrow x + \dfrac{1}{x} \leq -2$

$\Rightarrow \; Minimum\ value\ of\ y\ is\ \frac{1}{(-2+5)} = \frac{1}{3}.$

Ex 23. **For any positive x and y, find the minimum value of**

$$\left(\boldsymbol{x} + \frac{\boldsymbol{1}}{\boldsymbol{x}}\right)^{\boldsymbol{2}} + \left(\boldsymbol{y} + \frac{\boldsymbol{1}}{\boldsymbol{y}}\right)^{\boldsymbol{2}}$$

$Sol: Given\ x\ and\ y\ are\ positive\ Using\ A.M \geq G.M\ we\ get,$

$$\frac{\left[\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2\right]}{2} \geq \sqrt{\left[\left(x + \frac{1}{x}\right)^2 \times \left(y + \frac{1}{y}\right)^2\right]}$$

$$\Rightarrow \frac{\left[\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2\right]}{2} \geq xy + \frac{1}{xy} + \frac{x}{y} + \frac{y}{x} \geq 2$$

$Adding\ 2\ on\ both\ sides:$

$$\Rightarrow \frac{\left[\left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2\right]}{2} \geq 4$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 + \left(y + \frac{1}{y}\right)^2 \geq 8$$

$Hence, the\ minimum\ value\ of\ the\ expression\ is\ 8.$

Ex 24. $If\ a, b, and\ c\ are\ positive\ real\ numbers\ such\ that\ a + b + c = 3, find\ the\ minimum\ value\ of\ the\ expression: S = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$Sol:$

$Given: a, b, and\ c\ are\ positive\ real\ numbers, and\ a + b + c = 3$

$Using\ the\ inequality\ A.M \geq H.M, we\ get:$

$$\frac{a + b + c}{3} \geq \frac{3}{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} \Rightarrow 1 \geq \frac{3}{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} \Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3$$

$Hence, the\ minimum\ value\ of\ the\ expression\ S\ is\ 3.$

Ex 25. $If\ x, y, and\ z\ are\ positive\ such\ that: x^2 + y^2 + z^2 = 12\ \&\ x + y + z = 6\ .Then\ find\ the\ value\ of\ x^3 + y^3 + z^3.$

Sol:

$Using\ p = 2\ and\ q = 1\ in\ the\ power\ mean\ inequality\ for\ x, y, and\ z, we\ get:$

$$\left(\frac{(x^2 + y^2 + z^2)}{3}\right)^{\frac{1}{2}} \geq \left(\frac{(x + y + z)}{3}\right)$$

$$\Rightarrow x^2 + y^2 + z^2 \geq \frac{(x + y + z)^2}{3}$$

$$\Rightarrow x^2 + y^2 + z^2 \geq \frac{(6)^2}{3} \Rightarrow x^2 + y^2 + z^2 \geq 12$$

$But\ given, x^2 + y^2 + z^2 = 12$

$Hence, power\ mean\ equality\ holds \Rightarrow x = y = z = 2.$

Ex 26. Find the minimum value of √(a² + b²) where 3a + 4b = 15.

Sol: Using Cauchy-Schwarz inequality

$$|3a + 4b| \leq \sqrt{a^2 + b^2} \times \sqrt{3^2 + 4^2} \Rightarrow 15 \leq 5\sqrt{a^2 + b^2}$$

$$\Rightarrow 3 \leq \sqrt{a^2 + b^2} \Rightarrow a^2 + b^2 \geq 9$$

Hence, minimum value = 3.

QUESTIONS

Q. N. 1. For real numbers *a*, *b*, *c*, prove that
$|a| + |b| + |c| - |a + b| - |b + c| - |c + a| + |a + b + c| \geq 0.$

Q. N. 2. Let *a*, *b* be real numbers such that $0 \leq a \leq b \leq 1$. Prove that

$(i) 0 \leq \frac{b - a}{1 - ab} \leq 1$

(ii) $0 \leq \frac{a}{1+b} + \frac{b}{1+a} \leq 1$

(iii) $0 \leq ab^2 - ba^2 \leq \frac{1}{4}$

Q. N. 3. Prove that if if *n*, $-$ *m* are positive integers,

$$\frac{m}{n} < \sqrt{2} \text{ if and only if } \sqrt{2} < \frac{m + 2n}{m + n}$$

Q. N. 4. If $a \geq b, x \geq y \Rightarrow a\,x + b\,y \geq a\,y + b\,x.$

Q. N. 5. If *x*, *y* > 0, then $\sqrt{\frac{x^2}{y}} + \sqrt{\frac{y^2}{x}} \geq \sqrt{x} + \sqrt{y}.$

Q. N. 6. $If\ a, b, c > 0\ and\ a + b + c = 1, prove\ that\ ab + bc + ca < \frac{1}{3}.$

Q. N. 7. (IMO, 1960) For which real values of *x* the following inequality holds:

$$\frac{4x^2}{\left(1 - \sqrt{1 + 2x}\right)^2} < 2x + 9$$

Q. N. 8. Prove that for any positive integer *n*, the fractional part of $\sqrt{4n^2 + n} < \frac{1}{4}$.

Q. N. 9. (Short list IMO, 1996) Let *a*, *b*, *c* be positive real numbers such that *abc* = 1. Prove that

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \leq 1.$$

Q. N. 10. $If\ a, b, c\ are\ positive\ reals\ such\ that\ abc = 1, prove\ that\ a + b + c \geq ab + bc + ca.$

Q. N. 11. Suppose the polynomial $ax^2 + bx + c$ satisfies the following: $a > 0, a + b + c \geq 0, a - b + c \geq 0,\ a - c \geq 0,\ b^2 - 4ac \geq 0$. Prove that the roots are real and that they belong to the interval $-1 \leq x \leq 1.$

Q. N. 12. If *a*, *b*, *c* are positive numbers, prove that it is not possible for the inequalities $a(1 - b) > \frac{1}{4}, b(1 - c) > \frac{1}{4}, c(1 - a) > \frac{1}{4}$ to hold at the same time.

Q. N. 13. For $x \geq 0$, prove that $1 + x \geq 2\sqrt{x}.$

Q. N. 14. For $x > 0$, prove that $x + \frac{1}{x} \geq 2.$

Q. N. 15. For *x*, *y* ∈ R⁺, prove that $x^2 + y^2 \geq 2xy$.

Q. N. 16. For *x*, *y* ∈ R⁺, prove that $2(x^2 + y^2) \geq (x + y)^2$.

Q. N. 17. For *x*, *y* ∈ R⁺, prove that $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$.

Q. N. 18. For *a*, *b*, *x* ∈ R⁺, prove that $ax + \frac{b}{x} \geq 2\sqrt{ab}$.

Q. N. 19. If *a*, *b* > 0, then $\frac{a}{b} + \frac{b}{a} \geq 2$

Q. N. 20. If 0 < *b* ≤ *a*, then $\frac{1}{8}\frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8}\frac{(a-b)^2}{b}$.

Q. N. 21. For *x*, *y*, *z* ∈ R⁺, $(x + y)(y + z)(z + x) \geq 8xyz$.

Q. N. 22. For *x*, *y*, *z* ∈ R, $x^2 + y^2 + z^2 \geq xy + yz + zx$.

Q. N. 23. For *x*, *y*, *z* ∈ R⁺, $xy + yz + zx \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$.

Q. N. 24. For *x*, *y* ∈ R, $x^2 + y^2 + 1 \geq xy + x + y$.

Q. N. 25. For $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{1}{\sqrt{xy}} + \frac{1}{\sqrt{yz}} + \frac{1}{\sqrt{zx}}$.

Q. N. 26. For $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq x + y + z$.

Q. N. 27. For $x^2 + y^2 + z^2 \geq x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2}$.

Q. N. 28. For *x*, *y* ∈ R, $x^4 + y^4 + 8 \geq 8xy$

Q. N. 29. For $(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16$

Q. N. 30. For $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$

Q. N. 31. *Find the maximum value of* $x(1 - x^3)$ *for* $0 \leq x \leq 1$.

Q. N. 32. Let $x_i > 0$, $i = 1,\dots,n$. Prove that

$$\left[(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2\right]$$

Q. N. 33. If $\{a_1,\dots,a_n\}$ is a permutation of $\{b_1,\dots,b_n\} \subset$ R⁺, then

$$\left[\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n\right]$$

and $\left[\frac{b_1}{a_1} + \frac{b_2}{a_2} + \cdots + \frac{b_n}{a_n} \geq n\right]$

Q. N. 34. If *a* > 1, then $\left[a^n - 1 > n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right)\right]$

Q. N. 35. If *a*, *b*, *c* > 0 and $[(1 + a)(1 + b)(1 + c) = 8]$, then $a\,b\,c \leq 1$.

Q. N. 36. If *a*, *b*, *c* > 0, then $\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq ab + bc + ca$.

Q. N. 37. For non-negative real numbers *a*, *b*, *c*, prove that $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$.

Q. N. 38. If *a*, *b*, *c* > 0, then $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

Q. N. 39. If *a*, *b*, *c* > 0 satisfy that *abc* = 1, prove that $\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ac}{1+c} \geq 3$.

Q. N. 40. If *a*, *b*, *c* > 0, prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geq \frac{9}{a+b+c}.$$

Q. N. 41. If $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, prove that $n(n + 1)^{\frac{1}{n}} < n + H_n$ for *n* ≥ 2.

Q. N. 42. Let $x_1, x_2, \ldots, x_n > 0$ such that $\frac{1}{1+x_1} + \cdots + \frac{1}{1+x_n} = 1$. Prove that $x_1 x_2 \cdots x_n \geq (n-1)^n$.

Q. N. 43. (Short list IMO, 1998) Let $a_1, a_2, \ldots, a_n$ be positive numbers with $a_1 + a_2 + \cdots + a_n < 1$, prove that $\frac{a_1 a_2 \cdots a_n[1-(a_1+a_2+\cdots+a_n)]}{(a_1+a_2+\cdots+a_n)(1-a_1)(1-a_2)\cdots(1-a_n)} \leq \frac{1}{n^{n+1}}$.

Q. N. 44. Let $a_1, a_2, \ldots, a_n$ be positive numbers such that 1. Prove that $\frac{1}{1+a_1} + \cdots + \frac{1}{1+a_n} = 1$

$$\sqrt{a_1} + \cdots + \sqrt{a_n} \geq (n-1)\left(\frac{1}{\sqrt{a_1}} + \cdots + \frac{1}{\sqrt{a_n}}\right).$$

Q. N. 45. (APMO, 1991) Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive numbers with $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Prove that

$$\frac{a_1^2}{a_1+b_1} + \cdots + \frac{a_n^2}{a_n+b_n} \geq \frac{1}{2}(a_1 + \cdots + a_n).$$

Q. N. 46. Let a, b, c be positive numbers, prove that $\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}$.

Q. N. 47. Let a, b, c be positive numbers with $a + b + c = 1$, prove that

$$\left(\frac{1}{a}+1\right)\left(\frac{1}{b}+1\right)\left(\frac{1}{c}+1\right) \geq 64.$$

Q. N. 48. Let a, b, c be positive numbers with $a + b + c = 1$, prove that

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq 8.$$

Q. N. 49. (Czech and Slovak Republics, 2005) Let a, b, c be positive numbers that satisfy $a\,b\,c = 1$, prove that

$$\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)} \geq \frac{3}{4}.$$

Q. N. 50. Let a, b, c be positive numbers for which $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1$

Prove that $a\,b\,c \geq 8$.

Q. N. 51. Let a, b, c be positive numbers, prove that

$$\frac{2ab}{a+b} + \frac{2bc}{b+c} + \frac{2ca}{c+a} \leq a+b+c$$

Q. N. 52. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive numbers, prove that

$$\sum_{i=1}^{n} \frac{1}{a_i b_i} \sum_{i=1}^{n} (a_i + b_i)^2 \geq 4n^2.$$

Q. N. 53. (Russia, 1991) For all non-negative real numbers x, y, z, prove that

$$\frac{(x+y+z)^2}{3} \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$$

Q. N. 54. (Russia, 1992) For all positive real numbers x, y, z, prove that

$$x^4 + y^4 + z^2 \geq \sqrt{8}xyz$$

Q. N. 55. (Russia, 1992) For any real numbers $x, y > 1$, prove that

$$\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 8$$

Q. N. 56. Any three positive real numbers a, b and c satisfy the following inequality :

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$$

Q. N. 57. Any three positive real numbers *a*, *b* and *c*, with $abc = 1$, satisfy

$$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \geq 2(a^2b + b^2c + c^2a).$$

Q. N. 58. Any three positive real numbers *a*, *b* and *c* satisfy

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

Q. N. 59. Any three positive real numbers *a*, *b* and *c* satisfy

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{a+b+c}{abc}.$$

Q. N. 60. If *a*, *b* and *c* are the lengths of the sides of a triangle, prove that

$$\frac{a}{b+c-a} \quad \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3$$

Q. N. 61. If $a_1, a_2, \ldots, a_n \in R^+$ and $s = a_1 + a_2 + \cdots + a_n$, then $\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n} \geq \frac{n}{n-1}$

Q. N. 62. If $a_1, a_2, \ldots, a_n \in R^+$ and $s = a_1 + a_2 + \cdots + a_n$, then

$$\frac{s}{s-a_1} + \frac{s}{s-a_2} + \cdots + \frac{s}{s-a_n} \geq \frac{n^2}{n-1}$$

Q. N. 63. If $a_1, a_2, \ldots, a_n \in R^+$ and $a_1 + a_2 + \cdots + a_n = 1$ then

$$\frac{a_1}{2-a_1} + \frac{a_2}{2-a_2} + \cdots + \frac{a_n}{2-a_n} \geq \frac{n}{2n-1}.$$

Q. N. 64. (Quadratic mean-arithmetic mean inequality) Let $x_1, \ldots, x_n \in R^+$, then

$$\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}} \geq \frac{x_1+x_2+\cdots+x_n}{n}.$$

Q. N. 65. For positive real numbers *a*, *b*, *c* such that $a + b + c = 1$, prove that $ab + bc + ca \leq \frac{1}{3}$.

Q. N. 66. (Harmonic, geometric and arithmetic mean) Let $x_1, \ldots, x_n \in R^+$, prove that

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}} \leq \sqrt[n]{x_1x_2\cdots x_n} \leq \frac{x_1+x_2+\cdots+x_n}{n}.$$

And the equalities hold if and only if $x_1 = x_2 = \cdots = x_n$.

Q. N. 67. Let $a_1, a_2, \ldots, a_n$ be positive numbers with $a_1a_2\cdots a_n = 1$. Prove that

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} \geq \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}.$$

Q. N. 68. (China, 1989) Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$\frac{a_1}{\sqrt{1-a_1}} + \cdots + \frac{a_n}{\sqrt{1-a_n}} \geq \frac{1}{\sqrt{n-1}}\left(\sqrt{a_1} + \cdots + \sqrt{a_n}\right).$$

Q. N. 69. Let *a*, *b* and *c* be positive numbers such that $a + b + c = 1$. Prove that

$i. \sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} < 5,$

$ii. \sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} \leq \sqrt{21}.$

Q. N. 70. Let $a, b, c, d \in R^+$ with $ab + bc + cd + da = 1$, prove that

$$\frac{a^3}{b+c+d} + \frac{b^3}{a+c+d} + \frac{c^3}{a+b+d} + \frac{d^3}{a+b+c} \geq \frac{1}{3}.$$

Q. N. 71. Let *a*, *b*, *c* be positive numbers with $abc = 1$, prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a+b+c$$

Q. N. 72. Show that $19^{93} > 13^{99}$ without using tables/calculation. .(IOQM 2019-20)

Q. N. 73.(i) For *a*, *b* ∈ R⁺, with $a + b = 1$, prove that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\geq\frac{25}{2}.$$

(ii) For *a*, *b*, *c* ∈ R⁺, with $a + b + c = 1$, prove that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2\geq\frac{100}{3}.$$

Q. N. 74. For 0 ≤ *a*, *b*, *c* ≤ 1, prove that

$$\frac{a}{b+c+1}+\frac{b}{c+a+1}+\frac{c}{a+b+1}+(1-a)(1-b)(1-c)\leq 1$$

.

Q. N. 75. Let a, b be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$.(IOQM 2022-23)

Q. N. 76. Prove that the function *f*(*x*) = sin*x* is concave in the interval [0,*π*].

Use this to verify that the angles *A,B,C* of a triangle satisfy sin*A* +sin*B*+sin*C* ≤3$\frac{\sqrt{3}}{2}$

Q. N. 77. If *A*, *B*, *C*, *D* are angles belonging to the interval [0,*π*], then

(i) sinA sin B≤ $sin^2\left(\frac{A+B}{2}\right)$ and the equality holds if and only if *A* = *B*,

(ii) $\sin A\sin B\,\sin C\sin D\leq(\sin(\frac{A+B+C+D}{4}))^4$,

(iii) $\sin A\sin B\sin C\leq(\sin(\frac{A+B+C}{3}))^3$,

Moreover, if *A,B,C* are the internal angles of a triangle, then

(iv) $\sin A\sin B\sin C<\frac{3}{8}\sqrt{3}$,

(v) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}<\frac{1}{8}$,

(vi) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

Q. N. 78. (Canada, 1992) For any three non-negative real numbers *x*, *y* and *z* we have

$x(x-z)^2+y(y-z)^2\geq(x-z)(y-z)(x+y-z)$.

Q. N. 79. If *a*, *b*, *c* are positive real numbers, prove that

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2}\geq\frac{9}{4(a+b+c)}$$

Q. N. 80. Let *a*, *b* and *c* be positive real numbers, prove that

$$1+\frac{3}{ab+bc+ca}\geq\frac{6}{a+b+c}.$$

Moreover, if $a\,b\,c = 1$, prove that

$$1+\frac{3}{a+b+c}\geq\frac{6}{ab+bc+ca}.$$

Q. N. 81. (Kazakhstan, 2008) Let *x*, *y*, *z* be positive real numbers such that $xyz = 1$. Prove that $\frac{1}{yz+z}+\frac{1}{zx+x}+\frac{1}{xy+y}\geq\frac{3}{2}$

Q. N. 82. Let *a*, *b*, *c* be positive real numbers. Prove that

$\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right) \geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$.

Q. N. 83. For real numbers *x*, *y*, *z*, prove that

$$x^2+y^2+z^2 \geq |xy+yz+zx|$$

Q. N. 84. For positive real numbers *a*, *b*, *c*, prove that

$\frac{a^2+b^2+c^2}{abc} \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

Q. N. 85. If *x*, *y*, *z* are real numbers such that *x* < *y* < *z*, prove that

$$(x-y)^3+(y-z)^3+(z-x)^3>0$$

Q. N. 86. Let *a*, *b*, *c* be the side lengths of a triangle. Prove that

$\sqrt[3]{\frac{a^3+b^3+c^3+3abc}{2}} \geq a, b, c.$

Q. N. 87. (Romania, 2007) For non-negative real numbers *x*, *y*, *z*, prove that

$\frac{x^3+y^3+z^3}{3} \geq xyz+\frac{3}{4}[(x-y)(y-z)(z-x)]$.

Q. N. 88. (UK, 2008) Find the minimum of $x^2+y^2+z^2$, where *x*, *y*, *z* are real numbers such that $x^3+y^3+z^3-3xyz=1$

Q. N. 89. (South Africa, 1995) For *a*, *b*, *c*, *d* positive real numbers, prove that

$\frac{1}{a}+\frac{1}{b}+\frac{4}{c}+\frac{16}{ad} \geq \frac{64}{a+b+c+d}$.

Q. N. 90. Let *a* and *b* be positive real numbers. Prove that

$$8(a^4+b^4) \geq (a+b)^4$$

Q. N. 91. Let *x*, *y*, *z* be positive real numbers. Prove that

$\frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x} \geq \frac{9}{x+y+z}$.

Q. N. 92. Let *a*, *b*, *x*, *y*, *z* be positive real numbers. Prove that

$\frac{x}{ay+bz}+\frac{y}{az+bx}+\frac{z}{ax+by} \geq \frac{3}{a+b}$.

Q. N. 93. Let *a*, *b*, *c* be positive real numbers. Prove that

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \geq a+b+c$$

Q. N. 94. (i) Let *x*, *y*, *z* be positive real numbers. Prove that

$$\frac{x}{x+2y+3z}+\frac{y}{y+2z+3x}+\frac{z}{z+2x+3y} \geq \frac{1}{2}.$$

(ii) (Moldova, 2007) Let *w*, *x*, *y*, *z* be positive real numbers. Prove that

$\frac{w}{x+2y+3z}+\frac{x}{y+2z+3w}+\frac{y}{z+2w+3x}+\frac{z}{w+2x+3y} \geq \frac{2}{3}$.

Q. N. 95. (Croatia, 2004) Let *x*, *y*, *z* be positive real numbers. Prove that $\frac{x^2}{(x+y)(x+z)}+\frac{y^2}{(y+z)(y+x)}+\frac{z^2}{(z+x)(z+y)} \geq \frac{3}{4}$.

Q. N. 96. For *a*, *b*, *c*, *d* positive real numbers, prove that

$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \geq 2.$

Q. N. 97. Let *a*, *b*, *c*, *d*, *e* be positive real numbers. Prove that

$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+e}+\frac{d}{e+a}+\frac{e}{a+b} \geq \frac{5}{2}$.

Q. N. 98. (i) Prove that, for all positive real numbers *a*, *b*, *c*, *x*, *y*, *z* with *a* ≥ *b* ≥ *c* and *z* ≥ *y* ≥ *x*, the following inequality holds:

$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq 1\frac{(a+b+c)^3}{3(x+y+z)}$.

(ii) (Belarus, 2000) Prove that, for all positive real numbers *a*, *b*, *c*, *x*, *y*, *z*, the following inequality holds:

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}.$$

Q. N. 99. Prove that any three positive real numbers *a*, *b* and *c* satisfy

$a^5 + b^5 + c^5 \geq a^3bc + b^3ca + c^3ab.$

Q. N. 100. Let *x*, *y*, *z* be positive real numbers. Prove that

$$\frac{x^3}{x^3+2y^3} + \frac{y^3}{y^3+2z^3} + \frac{z^3}{z^3+2x^3} \geq 1$$

SOLUTIONS

Solution 1. If *a*, *b* or *c* is zero, the equality follows. Then, we can assume |*a*| ≥ |*b*| ≥ |*c*| >0. Dividing by |*a*|, the inequality is equivalent to

$$1 + \left|\frac{b}{a}\right| + \left|\frac{c}{a}\right| - \left|1 + \frac{b}{a}\right| - \left|\frac{b}{a} + \frac{c}{a}\right| + \left|1 + \frac{b}{a} + \frac{c}{a}\right| \geq 0.$$

Since $\left|\frac{b}{a}\right| \leq 1$ and $\left|\frac{c}{a}\right| \leq 1$,

We can deduce that $\left|1 + \frac{b}{a}\right| = 1 + \frac{b}{a}$ and $\left|1 + \frac{c}{a}\right| = 1 + \frac{c}{a}$.

$$Thus, it\ is\ sufficient\ to\ prove: \left|\frac{b}{a}\right| + \left|\frac{c}{a}\right| - \left|\frac{b}{a} + \frac{c}{a}\right| - \left(1 + \frac{b}{a} + \frac{c}{a}\right) + \left|1 + \frac{b}{a} + \frac{c}{a}\right| \geq 0.$$

Solution 2. (i) Use that 0 ≤ *b* ≤ 1 and 1 + *a* >0 in order to see that

$$0 \leq b(1+a) \leq 1 + a \Rightarrow 0 < b - a \leq 1 - ab \Rightarrow 0 \leq \frac{b-a}{1-ab} \leq 1.$$

(ii) The inequality on the left-hand side is clear. Since 1 + *a* ≤ 1 + *b*, it follows that $\frac{1}{1+b} \leq \frac{1}{1+ab}$, and then prove that

$$\frac{a}{1+b} + \frac{b}{1+a} \leq \frac{a}{1+a} + \frac{b}{1+a} = \frac{a+b}{1+a} \leq 1$$

.

(iii) For the inequality on the left-hand side, use that $ab^2 - ba^2 = ab(b-a)$ is the product of non-negative real numbers. For the inequality on the right-hand side, note that $b \leq 1 \Rightarrow b^2 \leq b \Rightarrow -b \leq -b^2$, and then

$$ab^2 - ba^2 \leq ab^2 - b^2a^2 = b^2(a-a^2) \leq a - a^2 = 1/4 - (1/2 - a)^2 \leq 1/4.$$

Solution 3. Prove in general that $x < \sqrt{2} \Longrightarrow 1 + \frac{1}{1+x} > \sqrt{2}$.

Solution 4. $ax + by \geq ay + bx \Leftrightarrow (a-b)(x-y) \geq 0$.

Solution 5. We can assume that $x \geq y$. Then, use the previous Q. N.4 substituting with and $\sqrt{x^2}$, $\sqrt{y^2}$,$\frac{1}{\sqrt{y}}$ and $\frac{1}{\sqrt{x}}$.

Solution 6.

$$(ab + bc + ca) < \frac{(a + b + c)^2}{3} = \frac{1^2}{3} = \frac{1}{3}.$$

Solution 7. In order for the expressions in the inequality to be well defined, it is necessary that x$\geq -\frac{1}{2}$ and x$\neq$ 0. Multiply the numerator and the denominator by $(1 +\sqrt{2x+1})^2$. Perform some simplifications and show that $\sqrt{2x+1}$ <7; then solve for x.

Solution 8. Since $4n^2 < 4n^2 + n < 4n^2 + 4n + 1$, we can deduce that $2n < \sqrt{4n^2+n} < 2n+1$. Hence, its integer part is $2n$ and then we have to prove that$\sqrt{4n^2+n} < 2n + \frac{1}{4}$

, this follows immediately after squaring both sides of the inequality.

Solution 9. Since $(a^3 - b^3)(a^2 - b^2) \geq 0$, we have that $a^5 + b^5 \geq a^2b^2(a + b)$, then

$$\frac{ab}{a^5+b^5+ab} \leq \frac{ab}{a^2b^2(a+b)+ab} = \frac{abc^2}{a^2b^2c^2(a+b)+abc^2}=\frac{c}{a+b+c}.$$

Similarly, $\frac{bc}{c^5+b^5+bc} \leq \frac{a}{a+b+c}$ and $\frac{ac}{c^5+a^5+bc} \leq \frac{b}{a+b+c}$.

Hence, $\frac{ab}{a^5+b^5+ab}+, \frac{bc}{c^5+b^5+bc}+\frac{bc}{c^5+b^5+bc} \leq \frac{c}{a+b+c}+\frac{a}{a+b+c}+\frac{b}{a+b+c}=\frac{a+b+c}{a+b+c}=1$

Solution 10. $Use\ symmetry\ and\ standard\ inequality\ techniques\ like\ AM - GM\ and\ substitution.$

Solution 11. Consider $p(x) = a x^2 + b x + c$, using the hypothesis, $p(1) = a + b + c$ and $p(-1) = a - b + c$ are not negative. Since a >0, the minimum value of p is attained at $-\frac{b}{2a}$ and its value is $\frac{4ac-b^2}{4a} < 0. If\ x_1, x_2$ are the roots of p, we can deduce that $\frac{b}{a}$ =-$(x_1 + x_2)$ and $\frac{c}{a}$ =x_1x_2, Therefore

$\frac{a+b+c}{a}$ =(1 -x_1)$(1 - x_2)$, $\frac{a-b+c}{a}$ =(1 +x_1)$(1 + x_2)$ and $\frac{a-c}{a} = 1 - x_1x_2$.

Observe that, (1 -x_1)$(1 - x_2) \geq 0$, $(1 + x_1)(1 + x_2) \geq 0$ and $1 - x_1x_2 \geq 0$ imply that $-1 \leq x_1, x_2 \leq 1$

Solution 12. If the inequalities are true, then a, b and c are less than 1. On the other hand, since $x(1-x) \leq \frac{1}{4}$ for $0 \leq x \leq 1$

Solution 13. Use the AM-GM inequality with $a = 1$, $b = x$.

Solution 14. Use the AM-GM inequality with $a = x, b = \frac{1}{x}$.

Solution 15. Use the AM-GM inequality with $a = x^2$, $b = y^2$.

Solution 16. In the previous Q. N. add $x^2 + y^2$ to both sides.

Solution 17. Use the AM-GM inequality with a=$\frac{x+y}{x}$, b= a=$\frac{x+y}{y}$ and also use the AM-GM inequality for x and y.

Solution 18. Use the AM-GM inequality with ax *and* b/x.

Solution 19. Use the AM-GM inequality with a/b and b/a

Solution 20. simplify and find the bounds using $0 < b \leq a$.

Solution 21. $\frac{a+b}{2} - \sqrt{ab} = \left(\frac{\sqrt{a}-\sqrt{b}}{2}\right)^2$ $and\ x + y \geq 2\sqrt{xy}.$

Solution 22. $x^2 + y^2 \geq 2xy$.

Solution 23. $xy + zx \geq 2x\sqrt{yz}.$

Solution 24. See Solution 22.

Solution 25. Use $\frac{1}{x} + \frac{1}{y} \geq \frac{2}{\sqrt{xy}}$

Solution 26. $\frac{xy}{2} + \frac{yz}{x} \geq 2\sqrt{\frac{xy^2z}{zx}} = 2y.$

Solution 27. $\frac{x^2+(y^2+z^2)}{2} \geq x\sqrt{y^2 + z^2}.$

Solution 28. $.x^4 + y^4 + 8 = x^4 + y^4 + 4 + 4 \geq \sqrt{x^4y^4 16}$=8x.

Solution 29. (a+b+c+d)$\geq 4\sqrt[4]{abcd}$, $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 4\sqrt[4]{\frac{1}{abcd}}.$

Solution 30. $\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}\right) \geq 4\sqrt[4]{\frac{a}{b}\frac{b}{c}\frac{c}{d}\frac{d}{a}}$=4

Solution 31. The idea of the proof is to exchange the product for another one in such a way that the sum of the elements involved in the new product is constant. If $y = x(1 - x^3)$, it is clear that the right side of $3y^3 = 3x^3(1 - x^3)(1 - x^3)(1 - x^3)$, expressed as the product of four numbers $3x^3$, $(1-x^3)$, $(1-x^3)$ and $(1-x^3)$, has a constant sum equal to 3. The AM-GM inequality for four numbers tells us that $3y^3 \leq \left(\frac{3x^3+3(1-x^3)}{4}\right)^4 = \left(\frac{3}{4}\right)^4$.

Thus $y \leq \frac{3}{4\sqrt[3]{4}}$. Moreover, the maximum value is reached using $3x^3 = 1 - x^3$, that is, if $x = \frac{1}{\sqrt[3]{4}}$.

Solution 32. $x_1 + x_2 + +x_n \geq n\sqrt[n]{x_1 . x_2 \ldots x_n}$, $\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \geq n\sqrt[n]{\frac{1}{x_1} . \ldots . \frac{1}{x_n}}$

Solution 33. $\left(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}\right) \geq n\sqrt[n]{\frac{a_1}{b_1} \ldots \frac{a_n}{b_n}}$=n.

Solution 34.

$$a^{n-1} > n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right) \Leftrightarrow (a - 1)(a^{n-1} + \cdots + 1) > n\, a^{\frac{n-1}{2}}\ (a - 1)$$

$$\Leftrightarrow \frac{a^{n-1}+\cdots+a+1}{n} > a^{\frac{n-1}{2}} \text{ , but } \frac{1+a+\cdots a^{n-1}}{n} >. \sqrt[n]{a^{\frac{n(n-1)}{2}}} = a^{\frac{n-1}{2}}$$

Solution 35. $1 = \left(\frac{1+a}{2}\right)\left(\frac{1+b}{2}\right)\left(\frac{1+c}{2}\right) . \geq \sqrt{a}\sqrt{b}\sqrt{c} = \sqrt{abc}$

Solution 36. Using the AM-GM inequality , we obtain

$$\frac{a^3}{b} + \frac{b^3}{c} + bc \geq 3\sqrt[3]{\frac{a^3}{b} . \frac{b^3}{c} \text{bc}} = 3ab.$$

Similarly, $\frac{b^3}{c}+\frac{c^3}{a}+ca \geq 3\sqrt[3]{\frac{b^3}{c}.\frac{c^3}{a}.ca}$ =3bc and $\frac{c^3}{a}+\frac{a^3}{b}+ab \geq$ $3\sqrt[3]{\frac{c^3}{a}.\frac{a^3}{b}.ab}$ ≥3ac.

Therefore,

$2(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a})+(ab\ +\ bc+\ ca) \geq\ 3(ab\ +\ bc\ +\ ca).$

Solution 37. If *abc* = 0, the result is clear. If *abc*>0, then we have

$$(ab)/c\ +\ (bc)/a\ +\ (ca)/b$$
$$=\left(\frac{1}{2}\right)\left[a\ \left(\frac{b}{c}+\frac{c}{b}\right)+\ b\ \left(\frac{c}{a}+\frac{a}{c}\right)+\ c\ \left(\frac{a}{b}+\frac{b}{a}\right)\right]$$
$$\geq\ \left(\frac{1}{2}\right)(2a\ +\ 2b\ +\ 2c),$$

and the result is evident.

Solution 38. Apply the AM-GM inequality twice over, $a^2b + b^2c + c^2a \geq 3abc$, $ab^2 + bc^2 + ca^2 \geq 3abc$.

Solution 39.

$$\frac{1\ +\ ab}{1\ +\ a}=\frac{abc\ +\ ab}{1\ +\ a}=\ ab\ \left(\frac{1\ +\ c}{1\ +\ a}\right)$$

$$\frac{1\ +\ ab}{1\ +\ a}+\frac{1\ +\ bc}{1\ +\ b}+\frac{1\ +\ ca}{1\ +\ c}$$
$$=\ ab\ *\ \left(\frac{1\ +\ c}{1\ +\ a}\right)+\ bc\ *\ \left(\frac{1\ +\ a}{1\ +\ b}\right)+\ ca\ *\ \left(\frac{1\ +\ b}{1\ +\ c}\right)$$
$$\geq\ 3\ \sqrt[3]{(abc)^2}\ =\ 3.$$

Solution 40. $(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a})(a+b+c)$ ≥9

is equivalent to

$$(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a})(a+b+b+c+c+a) \geq 9$$

which follows from Q. N. 32. For the other inequality use $\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$

Solution 41. Note that

$$\frac{n+H_n}{n}=\frac{1\ +\ \frac{1}{2}+\frac{1}{3}+\ \cdots+\left(1+\frac{1}{n}\right)}{n}.$$
$$Evaluate\ the\ expression.$$

Now, apply the AM-GM inequality .

Solution 42. Setting $y_i\ =\frac{1}{1+x_i}$, then $x\ =\frac{1}{y_i}-1$. Observe that $y_1+y_2+y_3+\ldots+y_n=1$

implies that $1-y_i=\sum_{j\neq i} y_i$,then $\sum_{\{j\neq i\}} y_j \geq (n-1)\left(\prod_{\{j\neq i\}} y_j\right)^{\frac{1}{n-1}}$ and

$$\prod_i x_i = \prod_i \left(\frac{1 - y_i}{y_i}\right)$$
$$= \frac{\prod_i \left(\sum_{\{j \neq i\}} y_j\right)\}}{\prod_i y_i}$$
$$\geq \frac{\{\prod_i \left(\prod_{\{j \neq i\}} y_j\right)^{\frac{1}{n-1}}\}}{\{\prod_i y_i\}}$$
$$= (n - 1)^n$$

Solution 43. Define $a_{n+1} = 1-(a_1+\cdots+a_n)$ and $xi = \frac{1-a_i}{a_i}$ $for\ i = 11,2,3, \ldots, n + 1$Apply Solution Q. N. 42 directly.

Solution 44. $\sum_{i=1}^{n} \frac{1}{(1 + a_i)} = 1 \Rightarrow \sum_{i=1}^{n} \frac{a_i}{(1+ a_i)} = n - 1$

.

Observe that

$$\sum_{i=1}^{n} \sqrt{a_i} - (n - 1) \sum_{i=1}^{n} \frac{1}{\sqrt{a_i}} = \sum_{i=1}^{n} \frac{1}{(1 + a_i)} \sum_{i=1}^{n} \sqrt{a_i} - \sum_{i=1}^{n} \frac{a_i}{(1 + a_i)} \sum_{i=1}^{n} \frac{1}{\sqrt{a_i}}$$

$$= \sum_{i=1}^{n} \frac{(a_i - a_j)}{\left((1 + a_j)\sqrt{a_i}\right)} = \sum_{i=1}^{n} \frac{\left(\left(\sqrt{a_i\sqrt{a_j}} - 1\right)\left(\sqrt{a_i} - \sqrt{a_j}\right)^2\left(\sqrt{a_i} + \sqrt{a_j}\right)\right)}{\left((1 + a_i)(1 + a_j)\sqrt{a_i\sqrt{a_j}}\right)}.$$

Since $1 \geq \frac{1}{(1 + a_i)} + \frac{1}{(1 + a_j)} = \frac{(2 + a_i + a_j)}{(1 + a_i + a_j + a_i a_j)}$

, we can deduce that $a_i a_j \geq 1$. Hence the terms of the last sum are positive.

Solution 45. Let $S_a = \sum_{i=1}^{n} \frac{a_i^2}{a_i + b_i}$ and $S_a = \sum_{i=1}^{n} \frac{b_i^2}{a_i + b_i}$. Then

$S_a - S_b = \sum_{i=1}^{n} \frac{(a_i^2 - b_i^2)}{a_i + b_i} = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = 0,$

thus $S_a = S_b = S$. Hence, we have

$$2S = \sum_{i=1}^{n} \frac{(a_i^2 + b_i^2)}{a_i + b_i} \geq \left(\frac{1}{2}\right) \sum_{i=1}^{n} \frac{(a_i + b_i)^2}{a_i + b_i} = \sum_{i=1}^{n} a_i$$

,

where the inequality follows after using Solution of Q. N. 16.

Solution 46. Since the inequality is homogeneous[16] we can assume that $abc= 1$. Setting $x = a^3$, $y = b^3$ and $z = c^3$, the inequality is equivalent to

$$\frac{1}{(x + y + 1)} + \frac{1}{(y + z + 1)} + \frac{1}{(z + x + 1)} \leq 1$$

.

Let $A = x + y + 1$, $B = y + z + 1$ and $C = z + x + 1$, then

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} \leq 1 \Leftrightarrow (A - 1)(B - 1)(C - 1) - (A + B + C) + 1 \geq 0$$

$\Leftrightarrow (x + y)(y + z)(z + x) - 2(x + y + z) \geq 2$

$\Leftrightarrow (x + y + z)(xy+ yz+ zx- 2) \geq 3.$

Now, use that, $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}, \frac{xy+yz+zx}{3} \geq \sqrt[3]{(xyz)^2}, \frac{1}{a^3+b^3+abc} \geq \frac{1}{abc(a+b+c)}$

Solution 47. Note that $abc \leq \left(\frac{a+b+c}{3}\right)^3 = \frac{1}{27}.$

$$\left(\frac{1}{a}+ 1\right)\left(\frac{1}{b}+ 1\right)\left(\frac{1}{c}+ 1\right) = 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} + \frac{1}{abc}$$

$$\geq 1 + \frac{3}{\sqrt[3]{abc}} + \frac{3}{\sqrt[3]{(abc)^2}} + \frac{1}{abc}$$

$$= \left(1 + \frac{1}{\sqrt[3]{abc}}\right)^3 \geq 4^3.$$

Solution 48. The inequality is equivalent to $\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right) \geq \mathbf{8}$.

Now, we use the AM-GM inequality for each term of the product and the inequality follows immediately.

Solution 49. Notice that. $\frac{a}{(a+1)(b+1)} + \frac{b}{(b+1)(c+1)} + \frac{c}{(c+1)(a+1)}$

$= \frac{(a+1)(b+1)(c+1)-2}{(a+1)(b+1)(c+1)} = 1 - \frac{2}{(a+1)(b+1)(c+1)} \geq \frac{3}{4}$

if and only if $(a+1)(b+1)(c+1) \geq 8$, and this last inequality follows immediately from the inequality $\left(\frac{a+1}{2}\right)\left(\frac{b+1}{2}\right)\left(\frac{c+1}{2}\right) \geq \sqrt{a}\sqrt{b}\sqrt{c}=1$

Solution 50. Observe that this Solution is similar to Q. N. 48.

Solution 51. Apply the inequality between the arithmetic mean and the harmonic mean to get

$$\frac{(2ab)}{(a + b)} = \frac{2}{\left(\frac{1}{a}+\frac{1}{b}\right)} \leq \frac{(a + b)}{2}$$

We can conclude that equality holds when $a = b = c$.

Solution 52. First use the fact that $(a + b)^2 \geq 4ab$, and then consider that

$$\sum_{i=1}^{n} \frac{1}{a_i b_i} \geq 4\sum_{i=1}^{n} \frac{1}{(a_i + b_i)^2}$$

Now, use Q. N. 32 to prove that

$$\sum_{i=1}^{n}(a_i + b_i)^2 \sum_{i=1}^{n} \frac{1}{(a_i + b_i)^2} \geq n^2$$

.

Solution 53. Using the AM-GM inequality leads to $xy + yz \geq 2y\sqrt{xz}$. Adding similar results we get $2(xy + yz + zx) \geq 2\left(x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}\right)$. Once again, using AM-

GM inequality, we get $x^2 + x^2 + y^2 + z^2 \geq 4x\sqrt{yz}$ .Adding similar results once more, we obtain $x^2 + y^2 + z^2 \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$.

Now adding both results, we reach the conclusion $\frac{(x+y+z)^2}{3} \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$. .

Solution 54. Using the AM-GM inequality takes us to $x^4+y^4 \geq 2x^2y^2$.

Applying AM-GM inequality once again shows that $2x^2y+z^2 \geq \sqrt{8}xyx$. .

Or, directly we have that

$$x^4 + y^4 + \frac{z^2}{2} + \frac{z^2}{2} \geq 4\sqrt[4]{\frac{x^4y^4z^4}{4}} = \sqrt{8}xyx..$$

Solution 55. Use the AM-GM inequality to obtain

$$\frac{x^2}{y-1} + \frac{y}{x-1} \geq 2\frac{xy}{\sqrt{(x-1)(y-1)}} \geq 8$$

.

The last inequality follows from $\frac{x}{\sqrt{x-1}} \geq 2$, since $(x-2)^2 \geq 0$.

Second solution. Let $a = x - 1$ and $b = y - 1$, which are positive numbers, then the inequality we need to prove is equivalent to $\frac{(a+1)^2}{b} + \frac{(b+1)^2}{a} \geq 8$.

Now, by the AM-GM inequality we have $(a + 1)^2 \geq 4a$ and $(b + 1)^2 \geq 4b$.

Then, $\frac{(a+1)^2}{b} + \frac{(b+1)^2}{a} \geq 4(\frac{a}{b} + \frac{b}{a}) \geq 8$.

The last inequality follows from Solution of Q. N. 19.

Solution 56. Observe that (a,b,c) and (a^2,b^2,c^2) have the same order, then use Wonderful Inequality .

Solution 57. By the previous Question.

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a.$$

Observe that $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$ and $(\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2})$ can be ordered in the same way.

Then, use inequality (1.2) to get $(ab)^3 + (bc)^3 + (ca)^3 = \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq \frac{1}{a^2}\frac{1}{c} + \frac{1}{b^2}\frac{1}{a} + \frac{1}{c^2}\frac{1}{b}$

$$\geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c} = a^2b + b^2c + c^2a$$

Adding these two inequalities leads to the result.

Solution 58. Use Wonderful Inequality with $(a_1, a_2, a_3) = (b_1, b_2, b_3) = (\frac{a}{b}, \frac{b}{c}, \frac{c}{a})$ and $(a_1', a_2', a_3') = (\frac{b}{c}, \frac{c}{a}, \frac{a}{b})$.

Solution 59. Use Wonderful Inequality with and $(a_1, a_2, a_3) = (\frac{1}{b}, \frac{1}{c}, \frac{1}{a})$.

Solution 60. Assume that $a \leq b \leq c$, and consider $(a_1, a_2, a_3) = (a,b,c)$, then use the rearrangement Wonderful Inequality twice over with $(a_{1'}, a_{2'}, a_{3'}) = (b,c,a)$ and (c,a,b), respectively.

Note that we are also using

$(b_1, b_2, b_3) = (\frac{1}{(b+c-a)}, \frac{1}{(c+a-b)}, \frac{1}{a+b-c})$

.

Solution 61. Use the same idea as in the previous Solution, but with n variables.

Solution 62. Turn to the previous Solution and the fact that $\frac{s}{s-a^1,} = 1 + \frac{a^1,}{s-a^1,}$.

Solution 63. Apply Solution 61 to the sequence$a_1, \ldots, a_n, a_1, \ldots, a_n$.

Solution 64. Apply Solution Tchebyshev's inequality.

Solution 65. Note that 1 = $(a^2 + b^2 + c^2) + 2(ab+bc+ca)$, and use the previous Q. N. as follows:

$$\frac{1}{3} = \frac{a + b + c}{3} \leq \sqrt{\frac{\sqrt{a^2} + b^2 + c^2}{3}}$$

.

Therefore $\frac{1}{3} \leq (a^2 + b^2 + c^2)$. Hence, 2(ab+bc+ca)$\leq \frac{2}{3}$, and the result is evident.

Second solution. The inequality is equivalent to $3(ab + bc + ca) \leq (a + b + c)^2$, which can be simplified to $ab + bc + ca \leq a^2 + b^2 + c^2$.

Solution 66. Let $G = \sqrt[n]{x^1 \cdot x^2 \cdot \ldots \cdot x_n}$

be the geometric mean of the given numbers and

$(a_1, a_2, \ldots, a_n) = (x_1/G, x_1 \cdot x_2/G^2, \ldots, x_1 \cdot x_2 \cdot \ldots \cdot x_n/G^n)$

Using Corollary 1.4.2, we can establish that

$n \leq \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} = \frac{G}{x_2} + \frac{G}{x_3} + \cdots + \frac{G}{x_n} + \frac{G}{x_1}$,

thus

$$\frac{n}{\left(\frac{1}{x^1} + \cdots + \frac{1}{x_n}\right)} \leq G$$

Also, using Corollary 1.4.2,

$n \leq \frac{a^1}{a_n} + \frac{a^2}{a^1} + \cdots + \frac{a_n}{a_n{}^{-1}} = \frac{x^1}{G} + \frac{x^2}{G} + \cdots + \frac{x_n}{G}$

Then $G \leq \frac{(x^1 + x^2 + \cdots + x_n)}{n}$

.

The equalities hold if and only if $a_1 = a_2 = \cdots = a_n$, that is, if and only if $x_1 = x_2 = \cdots = x_n$.

Solution 67. The inequality is equivalent to

$a_1{}^{n-1} + a_2{}^{n-1} + \cdots + a_n{}^{n-1} \geq \frac{a_1 \cdot a_2 \cdot \ldots \cdot a_n}{a^1} + \frac{a_1 \cdot a_2 \cdot \ldots \cdot a_n}{a^2} + \cdots + \frac{a_1 \cdot a_2 \cdot \ldots \cdot a_n}{a_n}$

which can be verified using the rearrangement inequality several times over.

Solution 68. $\sum_{i=1}^{n} \frac{a_i}{\sqrt{1-a_i}} \leq \sum_{i=1}^{n} \frac{1}{\sqrt{1-a_i}} - \sum_{i=1}^{n} \sqrt{1-a_i}$

Use the AM-GM inequality to obtain

$$\left(\frac{1}{n}\right)\sum_{i=1}^{n}\frac{1}{\sqrt{1-a_i}} \geq \sqrt[n]{\prod_{i=1}^{n}\left(\frac{1}{\sqrt{1-a_i}}\right)} = \sqrt{\left[\frac{1}{\sqrt[n]{\prod_{i=1}^{n}\left(\frac{1}{\sqrt{1-a_i}}\right)}}\right]}$$

$$\geq \sqrt{\left[\frac{1}{\left(\frac{1}{n}\right)^{n}\sqrt{\sum_{i=1}^{n}\left(\frac{1}{\sqrt{1-a_i}}\right)}}\right]} = \sqrt{\frac{n}{(n-1)}}$$

Moreover,the Cauchy-Schwarz inequality serves to show that

$\sum_{i=1}^{n}\sqrt{1-a_i} \leq \sum_{i=1}^{n}(1-ai)\sqrt{n} = \sqrt{n(n-1)}$ and $\sum_{\{i=1\}}^{n}\sqrt{a_i} \leq \sqrt{n}$

Solution 69.(i) $\sqrt{4a+1} < \frac{4a+1+1}{2} = 2a+1.$

(ii) Use the Cauchy-Schwarz inequality with $u = \sqrt{4a+1},\ \sqrt{4b+1}\ ,\ \sqrt{4c+1}$ and v = (1,1,1).

Solution 70. Suppose that $a \geq b \geq c \geq d$ (the other cases are similar). Then, if $A = b + c + d$, $B = a + c + d$, $C = a + b + d$ and $D = a + b + c$, we can deduce that $\frac{1}{A} \geq \frac{1}{B} \geq \frac{1}{C} \geq \frac{1}{D}$.

Apply the Tchebyshev inequality twice over to show that

$$\frac{a^3}{A}+\frac{b^3}{B}+\frac{c^3}{C}+\frac{d^3}{D} \geq \left(\frac{1}{4}\right)(a^3+b^3+c^3+d^3)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}\right)$$

$$\geq \left(\frac{1}{16}\right)(a^2+b^2+c^2+d^2)(a+b+c+d)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}\right)$$

$$= \left(\frac{1}{16}\right)(a^2+b^2+c^2+d^2)\left(\frac{A+B+C+D}{3}\right)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}\right)$$

.

Now, use the Cauchy-Schwarz inequality to derive the result

$$a^2 + b^2 + c^2 + d^2 \geq ab + bc + cd + da = 1$$

and the inequality $(A + B + C + D)(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}) \geq 16$.

Solution 71. Apply the rearrangement inequality to

$$(a_1,a_2,a_3) = \left(\sqrt[3]{\frac{a}{b}},\sqrt[3]{\frac{b}{c}},\sqrt[3]{\frac{c}{a}}\right),(b_1,b_2,b_3) = \left(\sqrt[3]{\left(\frac{a}{b}\right)^2},\sqrt[3]{\left(\frac{b}{c}\right)^2},\sqrt[3]{\left(\frac{c}{a}\right)^2}\right)$$

And the permutation

$(a_1',a_2',a_3') = \left(\sqrt[3]{\frac{b}{c}},\sqrt[3]{\frac{c}{a}},\sqrt[3]{\frac{a}{b}}\right),(b_1,b_2,b_3) = \left(\sqrt[3]{\left(\frac{a}{b}\right)^2},\sqrt[3]{\left(\frac{b}{c}\right)^2},\sqrt[3]{\left(\frac{c}{a}\right)^2}\right)$ to derive

$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \sqrt[3]{\frac{a^2}{bc}}+\sqrt[3]{\frac{b^2}{ca}}+\sqrt[3]{\frac{c^2}{ab}}.$

Finally, use the fact that $abc = 1$.

Second solution. The AM-GM inequality and the fact that $abc = 1$ imply that

$$\frac{1}{3}(\frac{a}{b}+\frac{a}{b}+\frac{b}{c}) \geq \sqrt[3]{\frac{a}{b}\frac{a}{b}\frac{b}{c}}=\sqrt[3]{\frac{a^2}{bc}} = \sqrt[3]{\frac{a^3}{abc}}=a$$

Similarly, $\frac{1}{3}(\frac{b}{c}+\frac{b}{c}+\frac{c}{a}) \geq b$, and $\frac{1}{3}(\frac{c}{a}+\frac{c}{a}+\frac{a}{b}) \geq c$

and the result follows.

Solution 72 $\left(\frac{19}{13}\right)^2 = \frac{361}{169} > 2$, $\left(\frac{19}{13}\right)^8 > 2^4 > 13$,

$19^8 > 13^9, 19^{88} > 13^{99}$,

$and\ Thus\ 19^{93} > 13^{99}$

Solution 73. The function f(x)=$(x+\frac{1}{x})^2$ is convex in R+.

Solution 74. The function

F(a,b,c)=$\frac{a}{b+c+1}+\frac{b}{a+c+1}+\frac{c}{b+a+1}$+(1-a)(1-b)(1-c)

is convex in each variable, therefore its maximum is attained at the endpoints.

Solution 75. $a^3 - b^3 - ab = 25$ for $a = 4, b = 3$.

Because for, any greater number $a^3 - b^3 - ab > 25$

To prove this if $a > b$, then $a^3 - b^3 - ab$

$= (b+t)^3 - b^3 - b(b+t), t > 0$

$= (3t-1)b^2 + (3t^2 - t)b + t^3$ is always > 4,

Then $b \geq 3$

So, $a^2 + b^3 = 4^2 + 3^3 = 43$.

Answer: 43

Solution 76. Find f ''(x)

Solution 77. Use log(sinx) or the fact that

SinAsinB= Sin($\frac{A+B}{2}+\frac{A-B}{2}$) Sin($\frac{A+B}{2}-\frac{A-B}{2}$).

Solution 78. Notice that $x(x-z)^2+y(y-z)^2-(x-z)(y-z)(x+y-z) \geq 0$ if and only if $x(x-z)(x-y)+y(y-z)(y-x)+z(x-z)(y-z) \geq 0$. The inequality now follows from Schur's inequality . Alternatively, we can see that the last expression is symmetric in x, y and z, then we can assume $x \geq z \geq y$, and if we return to the original inequality , it becomes clear that

$$x(x-z)^2 + y(y-z)^2 \geq 0 \geq (x-z)(y-z)(x+y-z).$$

Solution 79. The inequality is homogeneous, therefore we can assume that $a + b + c = 1$. Now, the terms on the left-hand side are of the form $\frac{x}{(1-x)^2}$ and the function f(x)=$\frac{x}{(1-x)^2}$ is convex, since f(x)=$\frac{4+2x}{(1-x)^4}$ >0.

By Jensen's inequality it follows that $\frac{a}{(1-a)^2}+\frac{b}{(1-b)^2}+\frac{c}{(1-c)^2} \geq 3f(\frac{a+b+c}{3}) = 3f\left(\frac{1}{3}\right) = (\frac{3}{2})^2$.

Solution 80. Since $(a+b+c)^2 \geq 3(ab+bc+ca)$, we can deduce that $1+\frac{3}{ab+bc+ca} \geq 1+\frac{9}{(a+b+c)^2}$

Thus, the inequality will hold if $1+\frac{9}{(a+b+c)^2} \geq \frac{6}{(a+b+c)^1}$.

But this last inequality follows from $\left(1-\frac{3}{a+b+c}\right)^2 \geq 0$

Now, if $abc = 1$, consider $x = \frac{1}{a}$. $y = \frac{1}{b}$ and $z = \frac{1}{c}$; it follows immediately that $xyz = 1$. Thus, the inequality is equivalent to $1+\frac{3}{xy+yz+zx} \geq \frac{6}{(x+y+z)^1}$

which is the first part of this solution.

Solution 81: . With the substitution $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$, the inequality takes the form

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}$$

which is Nesbitt's inequality.

Solution 82 Apply Popoviciu's inequality to the convex function $f(x) = x + \frac{1}{x}$

We will get the inequality $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{9}{a+b+c} \geq \frac{4}{b+c}+\frac{4}{c+a}+\frac{4}{a+b}$. Then multiply both sides by $(a + b + c)$ to finish the proof.

Solution 83 Observe that by using (1.8), we obtain $x^2+y^2+z^2-|x||y|-|y||z|-|z||x| = \frac{1}{2}(|x|-|y|)^2+\frac{1}{2}(|y|-|z|)^2+\frac{1}{2}(|z|-|x|)^2$

,

which is clearly greater than or equal to zero. Hence

$$|xy+yz+zx| \leq |x||y| + |y||z| + |z||x| \leq x^2+y^2+z^2.$$

Second solution. Apply Cauchy-Schwarz inequality to (x,y,z) and (y,z,x).

Solution 84 The inequality is equivalent to $ab + bc + ca \leq a^2 + b^2 + c^2$, which we know is true. See solution 22.

Solution 85. Observe that if $a + b + c = 0$, then it follows from (1.7) that $a^3 + b^3 + c^3 = 3abc$.

Since $(x - y) + (y - z) + (z - x) = 0$, we can derive the following factorization:

$$(x-y)^3 + (y-z)^3 + (z-x)^3 = 3(x-y)(y-z)(z-x).$$

Solution 86. Assume, without loss of generality, that $a \geq b \geq c$. We need to prove that

$$-a^3 + b^3 + c^3 + 3abc \geq 0.$$

Since $-a^3 + b^3 + c^3 + 3abc = (-a)^3 + b^3 + c^3 - 3(-a)bc$,

the latter expression factors into $\frac{1}{2}(-a+b+c)((a+b)^2+(a+c)^2+(b-c)^2)$

The conclusion now follows from the triangle inequality , $b + c > a$.

Solution 87. Let $p = |(x-y)(y-z)(z-x)|$. Using AM-GM inequality on the right-hand side of identity (1.8), we get $x^2+y^2+z^2-xy-yz-xz \geq \frac{3}{2}\sqrt[3]{p^2}$ (87.1)

Now, since $|x - y| \leq x + y$, $|y - z| \leq y + z$, $|z - x| \leq z + x$, it follows that

$$2(x + y + z) \geq |x - y| + |y - z| + |z - x|. \qquad (87.2)$$

Applying again the AM-GM inequality leads to

$$2(x + y + z) \geq 3\sqrt[3]{p},$$

and the result follows from inequalities(87.1) and (87.2).

Solution 88. Using identity (1.7), the condition $x^3 + y^3 + z^3 - 3xyz = 1$ can be factorized as

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = 1. \qquad (4.3)$$

Let $A = x^2 + y^2 + z^2$ and $B = x + y + z$. Notice that $B^2 - A = 2(xy + yz + zx)$.

By identity of Helpful Inequality , we have that $B > 0$. Equation (4.3) now becomes

$B\left(A - \frac{B^2-A}{2}\right) = 1$,

therefore $3A = B^2 + \frac{2}{B}$, Since $B > 0$, we may apply the AM-GM inequality to obtain

$3A = B^2 + \frac{2}{B} = B^2 + \frac{1}{B} + \frac{1}{B} \geq 3$,,

that is, $A \geq 1$. For instance, the minimum $A = 1$ is attained when $(x,y,z) = (1,0,0)$.

Solution 89. Cauchy-Schwarz Inequality helps to establish

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{(1+1+2+4)^2}{a+b+c+d} = \frac{64}{a+b+c+d}$$

.

Solution 90. Apply inequality (1.11) twice over to get

$$(a^4 + b^4) = \frac{a^4}{1} + \frac{b^4}{1} \geq \frac{(a^2 + b^2)^2}{2} \geq \frac{\left(\frac{(a^2 + b^2)^2}{2}\right)^2}{2} = \frac{(a + b)^4}{8}$$

Solution 91. Express the left-hand side as $\frac{\sqrt{2}^2}{x+y} + \frac{\sqrt{2}^2}{y+z} + \frac{\sqrt{2}^2}{z+x}$

and use Cauchy-Schwarz Inequality.

Solution 92. Express the left-hand side as

$$\frac{x^2}{axy + bxz} + \frac{y^2}{ayz + byx} + \frac{z^2}{azx + bzy}$$

and then use Cauchy-Schwarz Inequality to get

$$\frac{x^2}{axy + bxz} + \frac{y^2}{ayz + byx} + \frac{z^2}{azx + bzy} \geq \frac{(x+y+z)^2}{(a+b)(xy+yz+xz)} \geq \frac{3}{a+b}$$

,

where the last inequality follows from Helpful inequality.

Solution 93. Rewrite the left-hand side as

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{a+c} + \frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{a+c}$$

and then apply Cauchy-Schwarz Inequality.

Solution 94 (i) Express the left-hand side as

$$\frac{x^2}{x^2 + 2xy + 3xz} + \frac{y^2}{y^2 + 2yz + 3yx} + \frac{z^2}{z^2 + 2zx + 3yz}$$

and apply Cauchy-Schwarz Inequality to get

$$\frac{x}{x+2y+3z}+\frac{y}{y+2z+3x}+\frac{z}{z+2x+3y}\geq\frac{(x+y+z)^2}{x^2+y^2+z^2+5(xy+yz+zx)}$$

.

Now it suffices to prove that

$$\frac{(x+y+z)^2}{x^2+y^2+z^2+5(xy+yz+zx)}\geq\frac{1}{2}$$

but this is equivalent to $x^2+y^2+z^2 \geq xy+yz+zx$.

(ii) Proceed as in part (i), expressing the left-hand side as

$$\frac{w^2}{xw+2yw+3zw}+\frac{x^2}{xy+2xz+3xw}+\frac{y^2}{yz+2yx+3yz}+\frac{z^2}{zx+2zy+3xz}$$

then use Cauchy-Schwarz Inequality to get

$$\frac{w}{x+2y+3z}+\frac{x}{y+2z+3w}+\frac{y}{z+2x+3y}+\frac{z}{w+2x+3y}$$

$$\geq\frac{(w+x+y+z)^2}{4(wx+xy+yz+zw+wy+xz)}$$

Then, the inequality we have to prove becomes

$$\frac{(w+x+y+z)^2}{4(wx+xy+yz+zw+wy+xz)}\geq\frac{2}{3},$$

which is equivalent to $3(w^2+x^2+y^2+z^2) \geq 2(wx+xy+yz+zw+wy+xz)$. This follows by using the AM-GM inequality six times under the form $x^2+y^2 \geq 2xy$.

Solution 95. We again apply Cauchy-Schwarz Inequality to get

$$\frac{x^2}{(x+y)(x+z)}+\frac{y^2}{(y+z)(y+x)}+\frac{z^2}{(z+x)(z+y)}\geq\frac{(x+y+z)^2}{x^2+y^2+z^2+3(xy+yz+zx)}$$

Also, the inequality

$$\frac{(x+y+z)^2}{x^2+y^2+z^2+3(xy+yz+zx)}\geq\frac{3}{4}$$

is equivalent to

$$x^2+y^2+z^2 \geq xy+yz+zx.$$

Solution 96. We express the left-hand side as

$$\frac{a^2}{\big(a(b+c)\big)}+\frac{b^2}{\big(b(c+d)\big)}+\frac{c^2}{\big(c(d+a)\big)}+\frac{d^2}{\big(d(a+b)\big)}$$

and apply inequality (1.11) to get

$$\frac{a^2}{\big(a(b+c)\big)}+\frac{b^2}{\big(b(c+d)\big)}+\frac{c^2}{\big(c(d+a)\big)}+\frac{d^2}{\big(d(a+b)\big)}$$

$$\geq\frac{(a+b+c+d)^2}{\big(a(b+2c+d)+b(c+d)+d(b+c)\big)}$$

On the other hand, observe that

$$\frac{(a+b+c+d)^2}{(ac+bd)+(ab+ac+ad+bc+bd+cd)}$$

$$=\frac{a^2+b^2+c^2+d^2+2ab+2ac+2bc+2cd}{(ac+bd)+(ab+ac+ad+bc+bd+cd)}.$$

To prove that this last expression is greater than 2 is equivalent to showing that $a^2+c^2 \geq 2ac$ and $b^2+d^2 \geq 2bd$, which can be done using the AM-GM inequality .

Solution 97. We express the left-hand side as

$$\frac{a^2}{ab+ac}+\frac{b^2}{bc+bd}+\frac{c^2}{cd+ce}+\frac{d^2}{de+da}+\frac{e^2}{ae+be}$$

and apply inequality (1.11) to get

$$\frac{a^2}{ab+ac}+\frac{b^2}{bc+bd}+\frac{c^2}{cd+ce}+\frac{d^2}{de+da}+\frac{e^2}{ae+be} \geq \frac{(a+b+c+d+e)^2}{\sum ab}.$$

Since , $(a+b+c+d+e)^2=\sum a^2+2\sum ab$

we have to prove that

$$2\sum a^2+4\sum ab \geq 5\sum ab,$$

which is equivalent to

$$2\sum a^2 \geq \sum ab$$

The last inequality follows from $\sum a^2 \geq \sum ab$.

Solution 98(i) Using Tchebyshev's inequality with the collections ($a \geq b \geq c$) and ($\frac{a^2}{x} \geq \frac{b^2}{y} \geq \frac{c^2}{z}$), we obtain

$$\frac{1}{3}\left(\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z})\right) \geq \frac{(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z})}{3}.\frac{a+b+c}{3}$$

then by Cauchy-Schwarz Inequality , we can deduce that

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}$$

.

Therefore $\left(\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z})\right) \geq \frac{(a+b+c)^2}{x+y+z}.\frac{a+b+c}{3}$

.

(ii) we have $(\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z})^{1/3}(1+1+1)^{1/3}(\mathrm{x}+\mathrm{y}+\mathrm{z})^{1/3} \geq a+b+c$

Raising to the cubic power both sides and then dividing both sides by 3(x+y+z) we obtain the result.

Solution 99. Using Cauchy-Schwarz Inequality , we obtain

$$\frac{(x_1^2+x_2^2+\cdots+x_{\mathrm{n}}{}^2)}{(x^1+x^2+\cdots+x_{\mathrm{n}})}$$

$$=\frac{x_1^2}{(x_1+x_2+\cdots+x_n)}+\frac{x_2^2}{(x_1+x_2+\cdots+x_n)}+\cdots+\frac{x_n{}^2}{(x_1+x_2+\cdots+x_n)}$$

$$\geq \frac{(x_1+x_2+\cdots+x_n)^2}{\left(n(x_1+x_2+\cdots+x_n)\right)}=\frac{(x_1+x_2+\cdots+x_n)}{n}$$

Thus, it is enough to prove that

$$\left(\frac{(x_1+x_2+\cdots+x_n)}{n}\right)^{\frac{kn}{t}} \geq x_1.x_2.\cdots.x_n$$

Since k = max $\{x_1,x_2,...,x_n\}$ ≥ min $\{x_1,x_2,...,x_n\}$ = t, we have that $\frac{kn}{t} \geq n$ and since $\frac{(x_1+x_2+\cdots+x_n)}{n} \geq 1$, because all the x_i are positive integers, it is enough to prove that

$$\left(\frac{(x_1+x_2+\cdots+x_n)}{n}\right)^{\frac{n}{1}} \geq x_1.x_2.\cdots.x_n$$

which is equivalent to the AM-GM inequality.

Because all the intermediate inequalities are valid as equalities when $x_1 = x_2 = \cdots = x_n$, we conclude that equality happens when $x_1 = x_2 = \cdots = x_n$.

Solution 100. Using the substitution $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$, the inequality becomes:

$$\frac{a^3}{(a^3+2)}+\frac{b^3}{(b^3+2)}+\frac{c^3}{(c^3+2)} \geq 1$$

and with the extra condition, $abc = 1$.

In order to prove this last inequality, the extra condition is used as follows:

$$\frac{a^3}{(a^3+2)}+\frac{b^3}{(b^3+2)}+\frac{c^3}{(c^3+2)}$$

$$=\frac{a^3}{(a^3+2abc)}+\frac{b^3}{(b^3+2abc)}+\frac{c^3}{(c^3+2abc)}$$

$$=\frac{a^2}{(a^2+2bc)}+\frac{b^2}{(b^2+2ca)}+\frac{c^2}{(c^2+2ab)}$$

$$\geq \frac{(a+b+c)^2}{(a^2+b^2+c^2+2bc+2ca+2ab)}=1$$

The inequality above follows from Cauchy-Schwarz Inequality.

EUCLIDEAN GEOMETRY

1. Stewart's Theorem

Let D be a point on side BC such that $BD = m$ and $DC = n$ and $AD = d$. Then $a(d^2 + mn) = b^2 m + c^2 n$.

Lemma:

Let A, B, P, Q be four distinct points on a plane. Then $AB \perp PQ$ if and only if $PA^2 - PB^2 = QA^2 - QB^2$.

2. Carnot's Theorem

Let points D, E, and F be located on the sides BC, AC, and respectively AB of ΔABC. The perpendiculars to the sides of the triangle at points D, E, and F are concurrent if and only if

$$BD^2 - DC^2 + CE^2 - EA^2 + AF^2 - FB^2 = 0$$

3. Ceva's Theorem

If points D, E, F are taken on the sides BC, CA, AB of ΔABC so that the lines AD, BE, CF are concurrent at a point P, then

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \qquad \text{(OR)} \qquad BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$$

4. Trigonometric Form of Ceva's Theorem

Let X, Y, Z be the points taken respectively on the sides BC, CA, AB of ΔABC. Then the lines AX, BY, CZ are concurrent if only if

$$\frac{\sin \angle CAX}{\sin \angle XAB} \cdot \frac{\sin \angle ABY}{\sin \angle YBC} \cdot \frac{\sin \angle BCZ}{\sin \angle ZCA} = 1.$$

5. Converse of Ceva's Theorem

If three cevian AX, BY, CZ satisfy $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1$, then they are concurrent

6. Menelaus Theorem

If a transversal cuts the sides BC, CA, AB of a triangle ABC at X, Y, Z respectively then

$$\frac{BX}{XC} \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$$

7. Converse of Menelaus Theorem

If X, Y, Z are three points on each of the sides BC, CA, AB, of ΔABC or on their extensions such that $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1$, then X, Y, Z are collinear.

8. Pappus Theorem

If A, C, E are three points on one straight line. B, D, F on another and if the three lines AB, CD, EF meet respectively DE, FA and BC at L, M, N, then these three points L, M, N are collinear.

9. Intersecting Chords Theorem

If a line L through P intersects a circle ω at two points A and B, the product $PA \cdot PB$ (of signed lengths) is equal to the power of P with respect to the circle.

More over if there are two lines through P one meets circle ω at points A and B, and let another line meets circle ω at points C and D. Then

$$PA \cdot PB = PC \cdot PD.$$

10. Ptolemy's Theorem

In a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the pairs of opposite sides.

11. m-n Theorem

Let D be a point on the side BC of a ΔABC such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$. Prove that

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) $(m+n)\cot\theta = n\cot B - m\cot C$

12. Bramhagupta's Theorem

In any triangle product of any two sides is equal to the product of the perpendicular drawn to the third side with circum-diameter

QUESTIONS

1. Prove that the in-radius of a right-angled triangle with integer sides is an integer.

2. ABCD is a cyclic quadrilateral with AC ⊥ BD and AC meets BD at E.
   Prove that: $EA^2+EB^2+EC^2+ED^2=4R^2$ where R is the radius of the circumscribed circle.

3. The internal bisector of ∠A in a ΔABC with AC > AB, meets the circumcircle Γ of the triangle in D. Join D to the center O of the circle Γ and suppose DO meets AC in E, possibly when extended. Given that BE ⊥ AD, show that AO || BD.

4. Let BE and CF be the altitudes of an acute ΔABC, with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM ⊥ BE and LN ⊥ CF. Prove that FM || EN.

5. Let ABCD be a convex quadrilateral; P, Q, R, S be the midpoints of sides AB, BC, CD, DA respectively such that △AQR and △CSP are equilateral. Prove that ABCD is a rhombus. Determine its angles.

6. Let ABC be an acute-angled triangle; AD be bisector of ∠BAC with D on BC, and BE be the altitude from B on AC. Show that ∠CED > 45°.

7. Let ABC be an acute-angled triangle; let D, F be the midpoints of BC, AB respectively. Let the ⊥ from F to AC and the ⊥ at B to BC meet in N. Prove that ND is equal to the circumradius of ABC.

8. Let ABC be a triangle in which AB = AC and I be its in-center. Suppose BC = AB + AI. Find ∠BAC.

9. Let ABCDEF be a convex hexagon in which the diagonals AD, BE, CF are concurrent at O. Suppose the area of ΔOAF is the geometric mean of those of OAB and OEF, and the area of ΔOBC is the geometric mean of those of OAB and OCD. Prove that the area of ΔOED is the geometric mean of those of OCD and OEF.

10. A circle passes through the vertex C of a rectangle ABCD and touches its sides AB and AD at M and N respectively. If the distance from C to the line segment MN is equal to 5 units, find the area of the rectangle ABCD.

11. In an acute-angled ΔABC, ∠A = 30, H is the orthocentre, and M is the mid - point of BC. On the line HM, take a point T such that HM = MT. Show that AT =2BC.

12. The inscribed circumference in the ΔABC is tangent to BC, CA and AB at D, E and F respectively. Suppose that this circumference meets AD again at its mid-point X; that is, AX = XD. The lines XB and XC meet the inscribed circumference again at Y and Z, respectively. Show that EY = FZ.

13. T_1 is an isosceles triangle with circumcircle K. Let T_2 be another isosceles triangle inscribed

in K whose base is one of the equal sides of T_1 and which overlaps the interior of T_1. Similarly create isosceles triangles T_3 from T_2, T_4 from T_3 and so on. Do the triangles T_n approach an equilateral triangle as $n \longrightarrow \infty$?

14. The incircle of ΔABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of ΔABC.

15. Let ABC be a triangle and let P be interior point such that ∠BPC= 90°,∠BAP = ∠BCP. Let M. N be the mid points of AC, BC respectively. Suppose, BP = 2 PM. Prove that A, P, N are collinear.

16. Let ABC be an acute angles triangle and let H be its orthocentre. Let h_{max} denote the largest altitude of the ΔABC. Prove that $AH + BH + CH \leq 2h_{max}$.

17. Let ABC be an acute-angled triangle with altitude AK. Let H be its orthocentre and O be its circumcentre. Suppose KOH is an acute-angled triangle and P its circumcentre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the mid points of AB and AC.
18. A square ABCD is inscribed in a circle and a point P is on arc BC then prove that $\frac{PA+PC}{PB+PD} = \frac{PD}{PA}$

19. ABCD is a convex quadrilateral in which $AD=2\sqrt{3}$; ∠A = 60°; ∠D = 120° and AB + CD = 2AD. M is the mid-point of BC. Find DM.

20. In a ΔABC, AB = AC. A circle is internally drawn touching the circum circle of ΔABC, and also touching the sides AB and AC at P and Q, respectively. Prove that the mid-point of PQ is the incentre of ΔABC.

21. A ball of diameter 13 cm is floating so that the top of the ball is 4 cm above the smooth surface of the pond. What is the circumference in centimeters of the circle formed by the contact of the water surface with the ball?

22. Let ABCD be a square, and k be the circle with centre B passing through A and C. Let, I be the semi-circle inside the square with diameter AB. Let, E be a point on I, and the extension of BE meets the circle k at F. Prove that ∠DAF = ∠EAF.

23. OPQ is a quadrant of a circle, and semicircles are drawn on OP and OQ. Show that the shaded areas a and b are equal.

24. A circle AOB, passing through the centre O of another circle, cuts the latter circle at A and B. A straight line APQ is drawn meeting the circle AOB in P and the other circle in Q. Prove that PB = PQ
25. Let, A and B be two points inside a given circle k. Prove that there exist infinitely many circles through A and B which lies entirely in k.

26. If the altitude AD meets the circumcircle of the ΔABC at P and, if H is the orthocentre, show

that HD = PD.

27. ABC is a triangle. O, I and H are its circumcentre, in-centre and ortho- centre. Show that ∠OAI = ∠HAI.

28. Let ABCD be a square. P and Q are any two points on BC and CD respectively.
Such that AP = 4 cm, PQ = 3 cm, AQ = 5 cm. Find the side of the square

29. ABC is a triangle that is inscribed in a circle. The angle bisectors of A, B, C meet the circle at D, E, F. Show that AD is perpendicular to EF.

30. Given a circle and two points A and B inside the circle. If possible, con- struct a right-angled triangle inscribed in the circle, such that one leg of the right- angled triangle contains A and another leg contains B

31. ABC is a triangle, the bisector of ∠A, meets BC in D. Show that AD is less than the geometric mean of AB and AC.

32. Suppose, ABCD is a cyclic quadrilateral. The diagonals AC and BD intersect at P. Let, O be The circumcentre of ΔAPB and H, the orthocentre of ΔCPD. Show that O, P, H are collinear.

33. In Δ ABC, in the usual notation, the area is $\frac{1}{2}bc$ sq. units. AD is the median to BC. Prove that $\angle ABC = \frac{1}{2}\angle ADC$.

34. O is the circumcentre of Δ ABC and M is the mid-point of the median through A. Join OM

and produce it to N so that OM = MN. Show that N lies on the altitude through A.

35. A, B are two fixed points and P is a moving point, such that $\frac{PA}{PB}$ is constant. then prove that the locus of P is a circle.

36. The incircle of ΔABC touch BC at D. Show that the circles inscribed in triangles ABD and CAD touch each other.

37. If H is the orthocentre of DABC and S is the circumcentre and D is a mid-point of BC then prove that AH = 2SD.

38. *In ΔABC, prove that* $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$.

39. *If I is the incentre of a Δ ABC and if AI meets the circumcircle in K prove that KI = KB.*

40. If H is the orthocentre of ΔABC and AH produced meets BC at X and the circumcircle of ΔABC at K then prove that HX = XK.

41. In ΔBC, BM and CN are perpendiculars from B and C respectively on any line passing

through A. If L is the mid-point of BC prove that ML = NL.

42. Inscribe a square in a given triangle, so that, one side of the square may lie along a side of the triangle and the other two vertices lie on the other two sides (one in each) of the triangle. Justify your construction.

43. L is a point on the side QR of DPQR. LM, LN are drawn parallel to PR and QP meeting QP, PR at M and N respectively. MN produced meets QR produced in T. Prove that LT is the geometric mean between RT and QT

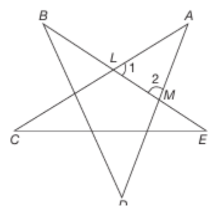
44. L and M are the mid-points of the diagonals BD and AC respectively of the quadrilateral ABCD. Through D draw DE equal and parallel to AB. Show that EC is parallel to LM and is double of it.

45. The side AB of parallelogram is produced both ways to F and G, so that AF = AD and BG = BC. Prove that FD and GC produced intersect at right angles.

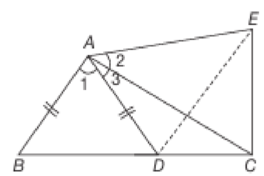
46. PS is the bisector of ∠QPR and PT ⊥ QR show that ∠ TPS $= \frac{1}{2}$(∠Q- ∠R), Where ∠Q < ∠R.

47. Prove that the angle between internal bisector of one base angle and the external bisector of the other base angle of a triangle is equal to one half of the vertical angle.

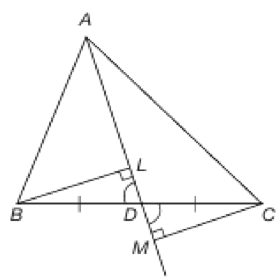
48. The given figure shows a five point star. Find sum of the angle ∠A + ∠B + ∠C + ∠D + ∠E.



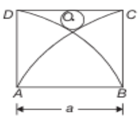
49. In the figure AC = AE , AB = AD , and ∠BAD = ∠ EAC, prove that BC = DE.



50. In the figure AD is a median and BL, CM are perpendiculars drawn from B and C respectively on AD and AD produced. Prove that BL = CM.



51. In the figure ABCD is a square of side ‘a’ units. Find the radius ‘r’ of a smaller circle. Where arc DB and arc AC has centres at A and B respectively



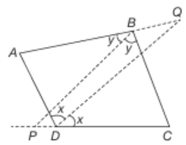
52. Two sides of a triangle are 10 cm and 5 cm in length and the length of the median to the

third side is 6.5 cm. If the area of the triangle is $6\sqrt{P}$ cm^2, find the value of p.

53. In an equilateral ΔABC, a point P is taken in the interior of ΔABC such that $PA^2 = PB^2 + PC^2$,+ Find ∠BPC.

54. Stewart’s Theorem ; Let D be a point on side BC such that BD = m and DC = n and AD = d. Then $a(d^2 + mn) = (b^2 m + c^2 n)$.

55. In the figure bisectors of ∠B and ∠D of quadrilateral ABCD meets CD and AB produced at P and Q respectively. Prove that ∠P+∠Q = $\frac{1}{2}$(∠ABC +∠ADC)



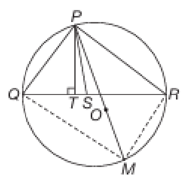
56. ABC is an isosceles triangle in which AB = AC. The bisector of ∠B meets AC at D. Also BC = BD + AD Find the size of ∠A.

57. Show that there is a unique triangle, whose side lengths are consecutive integers and one of whose angles is twice the other.

58. The sides of a triangle are in AP and the greatest angle of the triangle is double the least. Prove that, this triangle is acute angled triangle.

59. ΔABC has incentre I. Let points X, Y be located on the line segments AB, AC respectively, so that, BX · AB = IB2 and CY · CA = IC2. Given that the points X, I, Y. are collinear, find the possible values of ∠A.

60. In the given question if PM is the circum-diameter of DPQR then, prove that PS bisects ∠TPM.



61. In the quadrilateral ABCD, point E is on side DC , AD=AB, $\angle DAB = \angle AEC = \angle BCE = 90°$ and AE=5. Find the area of the quadrilateral ABCD.

62. Given that D is an inner point of the equilateral triangle ABC , such that PA=2, PB=$2\sqrt{3}$, PC=4. Find the length of the sides of triangle ABC.

63. Isosceles triangle has a right angle at point C. point D is inside triangle ABC, such that DA=11, DB=7 , DC=6. legs AC and BC have length $\sqrt{a + b\sqrt{2}}$, where a and b are positive integers. What is $a + b$?

64. Suppose that ABC is an equilateral triangle of side length s, with the property that there is a unique point D inside the triangle such that AD=1, BD=$\sqrt{3}$ and CD=2. Find s.

65. Three concentric circles have radius 3,4,5. An equilateral triangle with one vertex on each circle has side length s. The largest possible area of the triangle can be written as $a + \frac{b}{c}\sqrt{d}$, where a, b, c, d are positive integers, b and c are relatively prime and d is not divisible by the square of any prime. Find $a + b + c + d$

66. ABCD is a square and E is a point inside with $|ED| = 1, |EA| = 2, |EB| = 3$. Find $\angle AED$

67. In triangle ABC, $\angle A = 90°$, AB=AC and M , N are points on side BC. Show that $BM^2 + CN^2 = MN^2$ and $\angle MAN = 45°$

68. As in figure, in triangle ABC , D is the mid-point of BC . $\angle EDF = 90°$,DE intersects AB at E and DF intersect AC at F . Prove that BE+CF>EF ($x + y > z$)

69. Let P be a point inside a given square ABCD. Prove that centroid of triangles ABP, CDP, BCP and DAP forms a square.

70. Two circles C1 and C2 of radii 10 cm and 8 cm respectively are tangent to each other internally at a point A. AD is diameter of C1 and P and M are points on C1 and C2 respectively, such that PM is tangent to C2. As shown in the figure below. If $PM = \sqrt{20}$ and $\angle PAD = x°$, find x.

71. In a scalene triangle ABC with centroid G and circumcircle ω centred at O, the extension of AG meet ω at M and lines AB and CM intersect at P, and line AC and BM intersect at Q. Suppose the circumcentre S of triangle APQ lies on ω and A, O, S are collinear. Prove that $\angle AGO = 90°$

72. In a given acute angled triangle ABC find a point P for which the sum $AP + BP + CP$ of segment lengths is minimal.

73. Let P be a point inside a triangle ABC and let ABD be an equilateral triangle erected externally on side AB. Then $PA + PB + PC \geq CD$

74. Three circles of equal radii have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incentre and the circumcentre of the triangle are collinear with the point O.

75. In a triangle ABC we have AB=AC a circle which is internally tangent with the circumcircle of the triangle is also tangent to the sides AB, AC in the point P and Q respectively. Prove that the mid-point of PQ is the centre of incircle of the triangle ABC.

76. Let A be one point of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centres O_1 and O_2 respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2 , while the other touches C1 at Q1 and C2 at Q2. Let M_1 be the mid-point of P_1Q_1 and M_2 is the midpoint of P_2Q_2 . prove that $\angle O_1AO_2 = \angle M_1AM_2$

77. (Diameter of the incircle lemma) Let the incircle of triangle ABC touch side BC at D, and let DT be a diameter of the incircle. If line AT meets BC at X, then show that BD = CX.

78. Chord AB is given in a circle Ω. Let ω be a circle tangent to chord AB at K and internally tangent to ω at T. Then show that ray TK passes through the midpoint M of arc AB of Ω, not containing T.

79. Triangle ABC has orthocentre H, incentre I and circumcentre O. Let K be the point where the incircle touches BC. If IO is parallel to BC, then prove that AO is parallel to HK

80. In the plane let C be a circle, l a line tangent to the circle C, and M a point on l. Find the locus of all points P with the following property: there exists two points Q, R on l such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

81. If points D, E, F are taken on the sides BC, CA, AB of ΔABC so that the lines AD, BE, CF are concurrent at a point P,

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1 \qquad \text{(OR)} \qquad BD\cdot CE\cdot AF = DC\cdot EA\cdot FB$$

82. In a Δ ABC, ∠A = 2∠B, if and only if, $a^2 = b(b + c)$.

83. Suppose, ABCD is a cyclic quadrilateral. The diagonals AC and BD inter sect at P. Let, O be the circumcenter of D APB and H, the orthocenter of DCPD. Show that O, P, H are collinear.

84. Prove that the diagonals of a (convex) quadrilateral are perpendicular, if and only if the sum of the squares of one pair of opposite sides equals that of the other

85. ABC is a triangle, the bisector of ∠A, meets BC in D. Show that AD is less than the geometric mean of AB and AC

86. let C_1, be any point on side AB of ΔABC. Draw C_1C meeting AB at C_1. The lines through A and B parallel to CC_1 meet BC produced and AC produced at A_1 and B_1 respectively. Prove that $\frac{1}{AA_1} + \frac{1}{BB_1} = \frac{1}{CC_1}$

87. Let O be an arbitrary point situated in the segment AB. Construct equilateral Δ AOC and ΔBOD. Let, E be the point of intersection of AC and BD. Show that CODE is a parallelogram. When will it be a rhombus?

88. ABCDE is a convex pentagon inscribed in a circle of radius 1 units with AE as diameter. It AB = a, BC = b, CD = c, DE = d, then prove that $a^2 + b^2 + c^2 + d^2 + abc + bcd < 4$

89. O is the circumcentre of ΔABC and M is the mid-point of the median through A. Join OM and produce it to N so that OM = MN. Show that N lies on the altitude through A.

90. Prove that the mid-point of the hypotenuse of a right angled triangle is equidistant from all its vertices

91. Prove that the line segment joining the mid-points of the diagonals of a trapezium is parallel to each of the parallel sides and is equal to half the difference of these sides.

92. In the figure BE ⊥ AC. AD is any line from A to BC intersecting BE in H. P, Q and R are respectively the mid-points of AH, AB and BC. Prove that ∠PQR = 90°.

93. In a triangle ABC, points D and E respectively divide the sides BC and CA in the ratio $\frac{BD}{DC}$ = m and $\frac{AE}{EC}$ = n. The segment in a point X. Find the ratio $\frac{AX}{XD}$.

94. On the sides, BC,CA and AB of ΔABC, points D,E,F are taken in such a way that $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = \frac{2}{1}$. Show that the area of the triangle determined by the lines AD,BE,CF is $\frac{1}{7}$ th of the area of ΔABC.

95. L is a point on the side QR of DPQR. LM, LN are drawn parallel to PR and QP meeting QP, PR at M and N respectively. MN produced meets QR produced in T. Prove that LT is the geometric mean between RT and QT

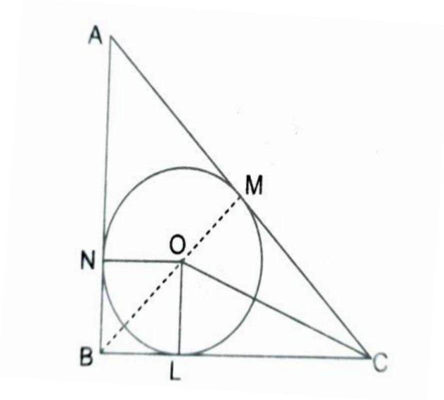
96. Given a parallelogram OBCA, a straight line is constructed such that, It cuts off $\frac{1}{3}$ part of OB and $\frac{1}{4}$ part of OA. Find the fraction of length this line cuts off from the diagonals OC.

97. Let A, B, C be an acute angled triangle in which, D, E, F are points on BC, CA, AB respectively, such that AD ⊥ BC, AE = EC, CF bisects ∠C internally. Sup pose CF meets AD and DE in M and N respectively. If FM = 2, MN = 1, NC = 3, show that the perimeter and area of this triangle are equal numerically.

SOLUTIONS

Solution 1:

Let ABC be a right triangle with $\angle B$ = 90 °. Let O be its incenter and L , M , N the points of contact of the in circle with the a , b , c respectively.



Suppose the in radius is r. now as $\angle ABC$ = 90 °, the quadrilateral NBLO is a square. So NB = BL = r. also, as the two tangents drawn from an external point to a circle are equal, we have AM = AN = AB – NB = c – r and CM = CL = BC – CL = a – r

So, b = AC = AM + CM = c - r + a – r = c + a – 2r,

Therefore, r = $\frac{b-c-a}{2}$

As $\angle B = 90 \implies b^2 = c^2 + a^2$, we have

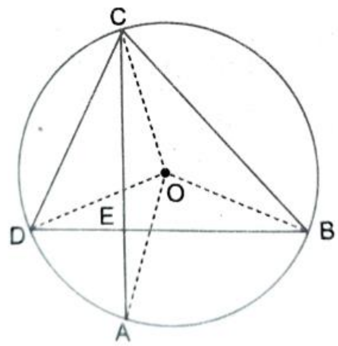
i) If c and a are both odd or even, $c^2 + a^2$ is even $\Longrightarrow b^2$ is even$\Longrightarrow$ b is even $\Longrightarrow$ b – (c + a) is even

ii) If one of c and a is even and other is odd. $C^2 + a^2$ is odd $\Longrightarrow b^2$ is odd $\Longrightarrow$ b is odd $\Longrightarrow$ b – (c + a) is even

So, in any case, if a,b,c are integers, we have, r = $\frac{b-c-a}{2}$ an integer

Solution 2:

let O be the centre of the circle,



$\angle AOB + \angle COD$ = 2 ($\angle ACD + \angle CBD$) = 2 X 90 = 180 ,

SO, $\angle AOB = \theta$,

Then,

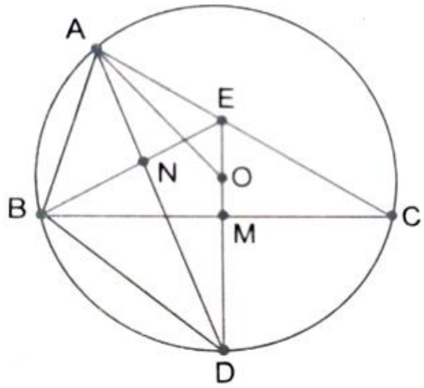
$AB^2 + CD^2 = 2(R^2 - R^2\cos\theta) + 2(R^2 - R^2\cos(\pi - \theta)) = 4R^2$

Similarly, $BC^2 + AD^2 = 4R^2 = AE^2 + EB^2 + EC^2 + ED^2 = \frac{1}{2}\angle(AB^2 + BC^2 + CD^2 + DA^2) = 4R^2$

Solution 3:

We consider here the case when ABC is an acute angled triangle; the case when $\angle A$ is obtuse or one of the $\angle B$ and $\angle C$ is obtuse may be handled similarly.

Let M be the point of intersection of DE and BC; let AD intersect BE in N.



Since, ME is the perpendicular bisector of BC, we have BE = CE. Since AN is the internal bisector of $\angle A$ and $\perp$ BE, it must bisect BE. This in turn implies that DN bisects $\angle BDE$.

But, $\angle BDA = \angle BCA = \angle C$.

Thus $\angle ODA = \angle C$. Since OD = OA, we get $\angle OAD = \angle C$.

It follows that $\angle BDA = \angle C = \angle OAD$.

This implies that OA || BD

Solution 4:

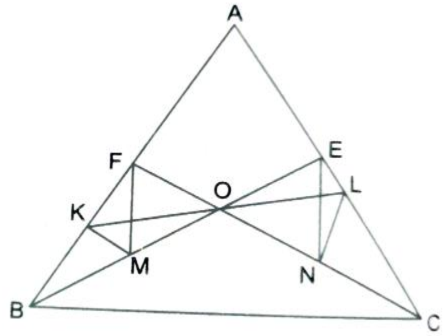
Observe that KMOF and ONLE are cyclic quadrilaterals.

Hence, $\angle FMO = \angle FKO$ and $\angle OEN = \angle OLN$

However,

we see that $\angle OLN = \frac{\pi}{2} - \angle NOL = \frac{\pi}{2} - \angle KOF = \angle OKF$

It follows that $\angle FMO = \angle OEN \Rightarrow$ FM||EN

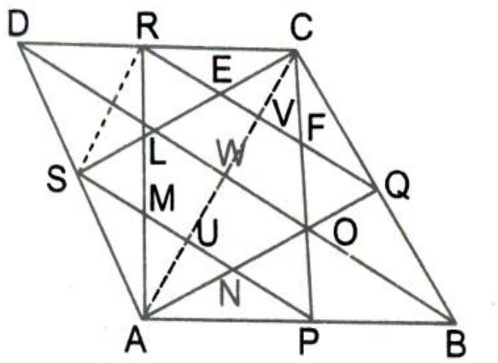


Solution 5:

We have QR = BD/2 = PS. Since AQR and CSP are both equilateral and QR = PS, they must be congruent triangles. This implies that AQ = QR = RA = CS = SP = PC.

Also ∠CEF = 60° = ∠RQA.

Hence, CS || QA. that Now, CS = QA implies that CSQA is a ||gm. In particular SA || CQ and SA = CQ. This shows that AD || BC and AD= BC. Hence, ABCD is a ||gm.



Let the diagonal AC and BD bisect each other at W. Then, DW = DB/2 = QR = CS = AR. Thus, in Δ ADC, the medians AR, DW, CS are all equal. Thus, ADC is equilateral. This implies ABCD is a rhombus. Moreover, the angles are 60 and 120

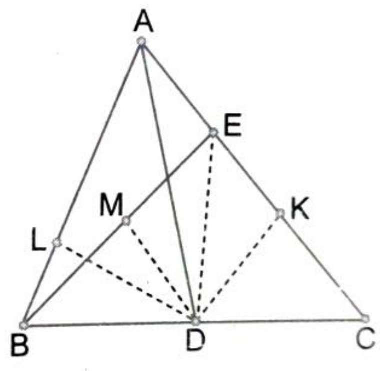
Solution 6:

Draw DL ⊥ AB; DK ⊥ AC; and DM ⊥ BE. Then EM = DK. Since AD bisects angle A, we observe that ∠BAD = ∠KAD.

Thus, in Δ ALD and ΔAKD, we see that ∠LAD =∠KAD; ∠AKD = 90° = ∠ALD and AD is common.

Hence ΔALD and ΔAKD are congruent, giving DL = DK.

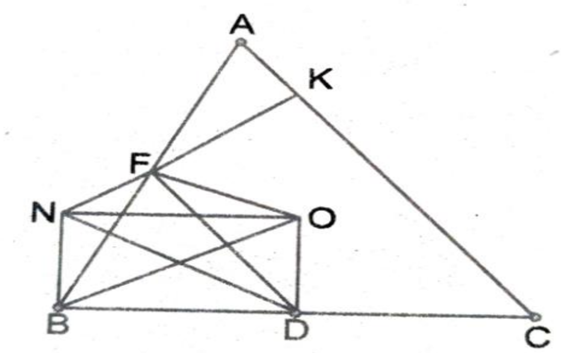
But DL > DM, since BE lies inside the Δ (by acuteness property). Thus EM > DM. This implies that ∠EDM > ∠DEM = 90 -∠EDM. We conclude that ∠EDM > 45. Since ∠CED = ∠EDM. the result follows.



Solution 7:

We can also get the conclusion using trigonometry.

Observe that ∠NFB = ∠AFK = 90° -∠A and ∠BNF = 180° - ∠B since BCKN is a cyclic quadrilateral.



Using the sine rule in the ΔBFN,

$$\frac{NB}{sin\angle NFB} = \frac{BF}{sin\angle BFN}$$

This reduces to NB = $\frac{c}{2}\frac{cosA}{sinC} = RcosA$

But BD = a/2 = R sin A.

Thus $ND^2 = NB^2 + BD^2 = R^2$

This gives ND = R

Solution 8:

We observe that ∠AIB = 90° + (C/2).

Extend CA to D such that AD= AI.

Then, CD = CB by the hypothesis.

Hence ∠CDB = ∠CBD = 90° - (C/2)

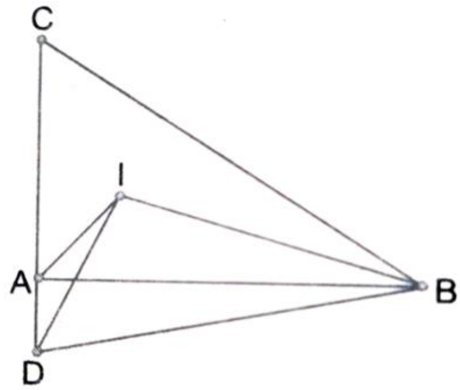
Thus ∠AIB + ∠ADB= 90° + C/2 + 90 – C/2 = 180

Hence, ADBI is a cyclic quadrilateral. This implies that ∠ADI = ∠ABI = B /2

But ADI is isosceles, since AD= AI,

This gives ∠DAI = 180° -2(∠ADI) = 180 - B

Thus, ∠CAI = B and this gives A= 2B.



Since C= B, we obtain 4B = 180° and hence B= 45°: We thus get A = 2B = 90°

Solution 9:

Let OA = a, OB = b, OC = c, OD = d, OE = e, OF = f

[OAB] = x, [OCD] = y, [OEF] = Z, [ODE]=u, [OFA] = v and [OBC] = w.

We are given that $v^2 = zx$, $w^2 = xy$ and we have to prove that $u^2 = yz$.

Since, $\angle AOB = \angle DOE$, we have

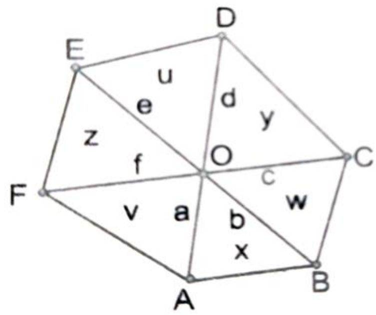
$$\frac{u}{x} = \frac{\frac{1}{2}de\ sin\angle DOE}{\frac{1}{2}ab\ sin\angle AOB} = \frac{de}{ab}$$

Similarly, $\frac{v}{y} = \frac{fa}{cd}, \frac{w}{z} = \frac{bc}{ef}$

Multiplying these three equalities, we get uvw = xyz

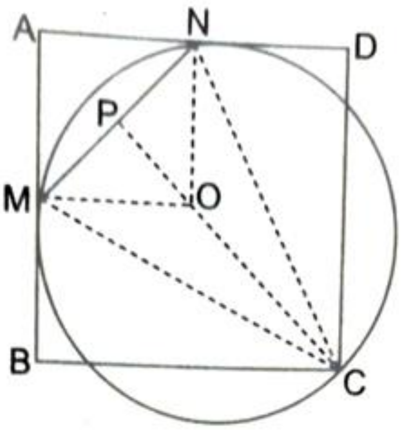
Hence $x^2y^2z^2 = u^2v^2w^2 = u^2(zx)(xy)$

This gives, $u^2 = yz$ as desired



Solution 10:

Let O be the centre of the circle and P be the foot of perpendicular from C to MN.



Then, OM ⊥ AB, ON ⊥ AD and OM = ON = the radius of the circle. So, AMON is a square.

∠MCN = 1/2∠MON = 45°

∠CMP + ∠CNP=135° = ∠CMP + ∠CMB = ∠CNP+ ∠CND.

Hence, ∠CNP = ∠CMB and ∠CMP = ∠CND.

Thus, we see that the right ΔCNP and ΔCMB are similar, and ∠CMP and ∠CND are similar.

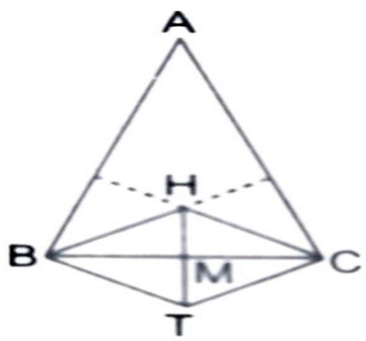
So,$\frac{CN}{CM} = \frac{CP}{CB}, \frac{CM}{CN} = \frac{CP}{CD}$

Multiplying, 1 = $\frac{CP^2}{CB.CD}$

Hence, the area of the rectangle is CB.CD= CP² = 5^2 = 25

Solution 11:

We can assume that the circumcentre of ΔABC is at the origin. If R is the circumradius. BC= 2R sin A = R.



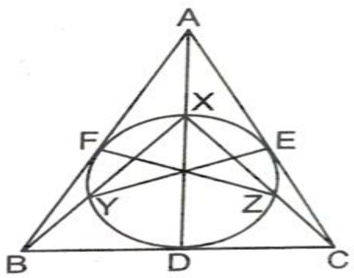
Also, if Z_1, Z_2, Z_3 are complex numbers representing A, B, C respectively than $Z_1 + Z_2 + Z_3$ represents H and M is $\frac{z2+z3}{2}$

If t represents T, then $\frac{t+Z1+Z2+Z3}{2} = \frac{Z2+Z3}{2}$

⟹ t = -Z_1 ⇒ AT = 2T = 2BC

Solution 12:

Since ∠BFY = ∠BXF and ∠FBY = ∠XBF, we have ΔBFY and ΔBXF are similar, so that FY: FX= BF: BX are ...i)



Similarly, we get DY: DX = BD: BX ...(ii)

As BF = BD, we have from Eqs. (i) and (ii) that

FY: FX = DY: DX

Since AX = DX, we get FY: FX= DY:AX ………. (iii)

Since, X, F, Y, D are concyclic we have ∠FYD= ∠AXF …. (iv)

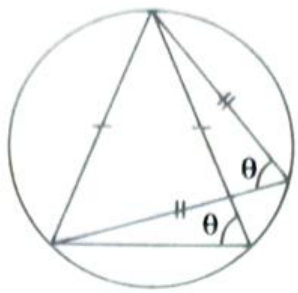
Thus, we get from Eqs. (iii) and (iv) that ΔFYD ~ ΔFXA.

Hence, ∠YFD= ∠XFA = ∠XDF So that FY || XD.

Similarly, we have EZ || XD. Thus, FY|| EZ.

Therefore, FYZE is an isosceles trapezoid and then EY= FZ.

Solution 13:



Note that the base angle of T_n is equal to the angle opposite the base of T_{n+1}(as the figure + indicates). Therefore if θ is the base angle for T_n, then the base angle for the next triangle (T_{n+1}) is $\frac{180-\theta}{2} = 90 - \frac{\theta}{2}$

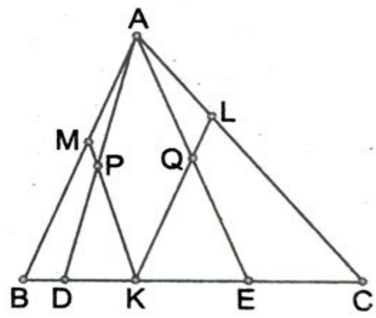
Suppose now that θ is the base angle for T_1.

Then, the base angle for T_n =

$90 - \frac{90}{2} + \frac{90}{4} - \frac{90}{8} + \ldots + (-1)^{n-2}\frac{90}{2^{n-2}} + (-1)^{n-1}\ \frac{\theta}{2^{n-1}}$

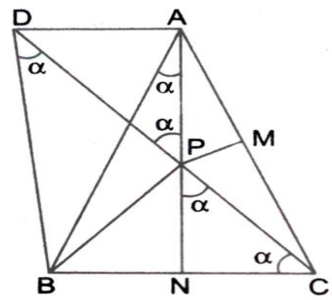
Note that the limit as n ⟶ ∞ of the above is $\frac{90}{1+\frac{1}{2}}$ = 60° by the formula for the sum of an infinite geometric series. Since each T_n is isosceles, the angles of T_n do approach 60° as n⟶ ∞

Solution 14:



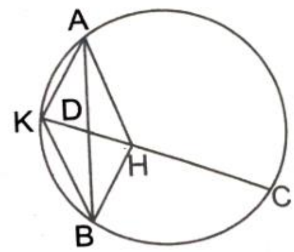
Let AP, AQ produced meet BC in D, E respectively. Since MK || AE, we have ∠AEK = ∠MKB.
Since both BK = BM, being tangents to circle the from B, ∠MKB = ∠BMK. This with the fact that MK || AE gives us ∠AEK = ∠MAE. This shows that MAEK is an isosceles trapezoid. We concude that MA = KE. Similarly, we can prove that AL = DK. But AM = AL We get that DK = KE. Since KP || AE, we get DP = PA and similarly EQ= QA. This implies that PQ || DE and hence bisects AB, AC when produced. [The same argument holds even if one or both of P and Q lie outside triangle ABC].

Solution 15:



Extend CP to D. Now, CP= PD
Let ∠BCP = ∠BAP = α, BC = BD (: BP is perpendicular bisector of CD)
Therefore, BCD is an isosceles triangle.
Thus,∠BDP = α
Then, ∠BDP = ∠BAP = α
Hence, B, P, A, D all lie on a circle.
∠DAB = ∠DPB = 90° (P is mid - point of CD and M is mid - point of CA)
PM || DA where DA = 2PM = BP
Thus, DBPA is an isosceles trapezium
And DB|| PA
Hence, we get ∠DPA = ∠BAP= ∠BCP= ∠NPC (because ∠BPC = 90° and N is mid point of CB.
.: NP = NC = NB for right angled triangle BPC.
Hence, A, P and N are collinear

Solution 16:

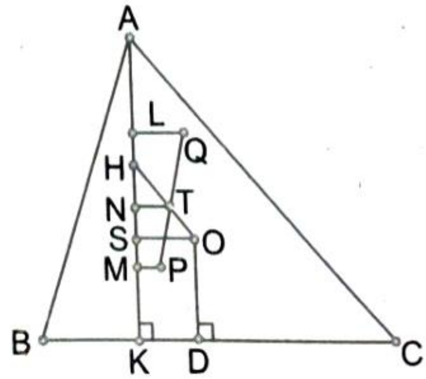


Let ∠C be the smallest angle. So that CA ≥ AB and CB ≥ AB
Here, the altitude through C is the longest one. Let the altitude through C meets AB in D. Let H be the orthocentre of triangle ABC. Let CD extended meet the circumcircle of ABC in K.
We have CD = h_{max}
Using CD= CH + HD
Which reduced to AH + BH ≤ CD + HD
But AH = AK
BH = BK (because triangle DEK ≅triangle DBH)
DH = DK
Apply Ptolemy's theorem quadrilateral BCAK.

We get, AB· CK= AC . BK + BC. AK
≥ AB. BK + AB. AK
⟹ CK ≥ AK + BK

Solution 17:

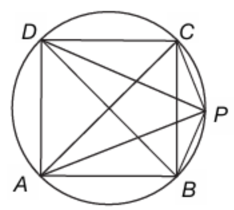
Let D be the mid-point of BC; M that of HK; and T that of OH.
Then, PM is perpendicular to HK and PT is perpendicular to OH.
Since, Q is the reflection of P in HO, we observe that P, T, Q are colinear and PT =TQ.
Let QL, TN and OS be the perpendiculars drawn respectively from Q, T and O on to the altitude AK, (See the figure).
We have LN = NM, since T is the mid - point of OP: HN = NS, since T is the mid - point of OH; and HM = MK, as P is the circumcentre of triangle KHO, we obtain,



LH + HN = LN = NM = NS + SM
which gives LH= SM. We know that,
AH = 2OD
Thus, AL= AH - LH=2OD - LH =2SK - SM
= SK + (SK - SM) = SK+ MK
= SK + HM = SK + HS + SM
= SK + HS + LH = SK + LS = LK
This shows that L is the mid -point of AK and hence lies on the line joining the mid points of the AB and AC.
We observe that the line joining mid points of AB and AC is also perpendicular to AK.
Since QL is perpendicular to AK, we conclude that Q also lies on the line joining the mid points of AB and AC.

Solution 18:

Since in a square ABCD, AB = BC = CD = DA = a (Say) and AC = BD = a√2
In cyclic quadrilateral APCD, by Ptolemy's theorem
PA · CD + AD · PC = PD · AC
⇒ (PA + PC)a = PD · a√2
⇒ PA + PC = PD√2 ..... (1)
In cyclic quadrilateral ABPD, by using Ptolemy's theorem
PD · AB + PB · AD = PA · BD
⇒ (PD + PB)a = PA · a√2
⇒ PB + PD = PA · √2 ...... (2)
From Eq. (1)/ Eq. (2) we get, $\frac{PA+PC}{PB+PD} = \frac{PD}{PA}$



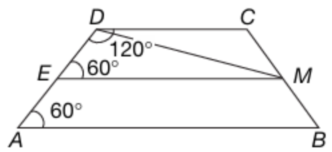
Solution 19:

Since ∠A + ∠D = 180°, AB || CD. Draw ME || BA to meet AD at E. As M is the mid- point of BC, E is the mid-point of AD. Also, EM = $\frac{1}{2}(AB + CD) = AD$ ----- Given

Therefore, EM = AD

From triangle EDM, using cosine rule,

$DM^2 = DE^2 + EM^2 – 2.DE.EM.\cos 60$



i.e. $DM^2 = (\frac{1}{2}DA)^2 + (AD)^2 - 2(\frac{AD}{2})(AD)(\frac{1}{2})$

i.e. $DM^2 = \frac{1}{4}DA^2 + AD^2 - \frac{1}{2}AD^2 = \frac{3}{4}AD^2$

Therefore, $DM^2 = \frac{3}{4}(2\sqrt{3})^2 = 9$

Which implies: DM = 3

Solution 20:

Let, ∠ ABC = ∠ ACB = β °.

AT is the angle bisector of ∠A. I is the mid-point of PQ. Now, AP = AQ as the smaller circle touches AB and AC at P and Q, respectively. The centre of the circle PQT lies on the angle bisector of ∠ A, namely, AT, since PQ is the chord of contact of the circle PQT. PQ ⊥ AT and the mid-point I of PQ lies on AT.

Now, to prove that I is the incentre of ΔABC, it is enough to prove that BI is the angle bisector of ∠B and CI is the angle bisector of ∠C, respectively.

By symmetry, ∠PTI = ∠QTI = α

Now, ∠ ABT – 90° (∵ AT is diameter of ⊙ABC)

∴ ∠PBT = 90°

Also, ∠PIT = 90°

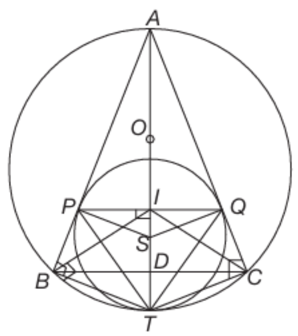
∴ PBTI is cyclic.

∴ ∠PBI = ∠PTI = α (Angle in the same segment)

∠IBD = ∠ABD − ∠ABI = β − α

∠TBC = ∠TAC = 90° − β

∴ ∠IBT = ∠IBD + ∠DBT = β − α + 90° − β = 90° − α



Since, PBTI is cyclic,

∠IPT = ∠IBT = 90° − α (1)

∠BPT = 180° − ∠TPA = 180° − ∠API − ∠IPT

= 180° − β − 90° + α = 90° + α − β (2)

But, APT is a tangent to circle PQT.

∴ ∠BPT = ∠PQT − ∠IQT

From Eqs. (1) and (2), we get

90° + α − β = 90° − α

2α = β

∴ ∠IBD = β − ∠PBI = 2α − α = α

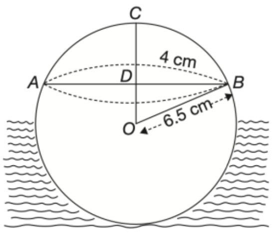
∴ ∠IBD = ∠PBI

∴ BI is the angle bisector of ∠B.

Hence, the result.

Solution 21

We should find the circumference of the circle on the diameter .



CD=4 cm.

OC=OB=13/2 =6.5 cm

So OD = 6.5 cm - 4 cm =2.5 cm.

So , the

Circumference of the circle is $2\pi \times 6\ cm\ = 12\pi\ cm.$

Solution 22:

(i) BA = BF (Radius of the circle k.)
(ii) ∠AEB = 90°
(iii) ∠EAF = 90° − ∠AFE (Angle in the semi-circle.)

= 90° −∠AFB

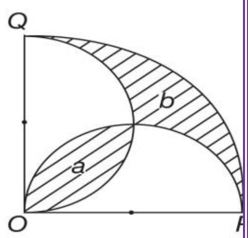
= 90° – ∠BAF (BA =BF by step (i))

= ∠BAD −∠BAF

= ∠FAD or ∠DAF.

Solution 23 :

Area of the quadrant = areas of the two semicircles + b − a [Since the sum of the areas of the two semicircles include the area shaded 'a' twice)



The area of quadrant = $\frac{1}{4}\pi r^2$

ie. $\frac{1}{4}\pi r^2 = \frac{1}{2}\pi(\frac{r}{2})^2 + \frac{1}{2}\pi(\frac{r}{2})^2 + b - a$

$\Rightarrow \frac{1}{4}\pi r^2 = \frac{1}{4}\pi r^2 + b - a$

$\Rightarrow 0 = b - a$

$\Rightarrow a = b.$

Solution 24:

$\angle PQB\ =\ \angle AQB$

$= \frac{1}{2}\angle AOB\ = \frac{1}{2}\angle APB$

$= \frac{1}{2}(\angle PQB\ +\ \angle PBQ) \Rightarrow \angle PQB\ - \frac{1}{2}\angle PQB = \frac{1}{2}\angle PBQ\ \Rightarrow \angle PQB\ = \angle PBQ$

$\Rightarrow PQ\ = PB$

Solution 25:

Join A and B to the centre (O) of the circle k.
If P is a point on OA, any circle with centre P and radius PA lies entirely inside k, since A is an interior point of k.
Similarly, if Q is a point on OB and the circle with its centre Q and radius QB lies entirely inside k.

Since, OA is less than the radius of the circle k, and the circle with O as centre and radius OA lies inside circle k.

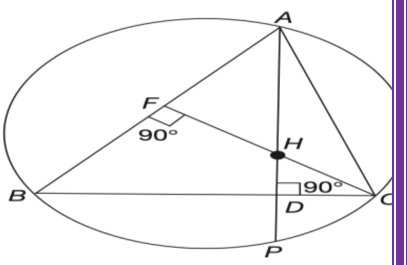
(It is the concentric circle with k) and the circle with centre P and radius PA is a circle touching the concentric circle of k with radius OA internally, and hence, this circle lies entirely inside k. Similarly, for the point Q on OB, the following explanation can be given.

Let the perpendicular bisector of AB meet OA at C (or, this perpendicular bisector may meet OB).

Now, the set of centres of the set of circles passing through A and B are the points on this perpendicular bisector.

Taking any point P on line segment DC as centre and radius PA = PB, an infinite number of circles can be constructed. All those would lie entirely on k. This is because there are an infinite number of points as P on line segment DC.

Solution 26:



∠CPD = ∠CPA
$= \angle CBA = \angle CBF = 90° - \angle FCB = 90° - \angle HCD = \angle DHC = \angle CHD \quad \therefore CP = CH$
∴ CD is the perpendicular bisector of $PH \ \therefore DH = DP \ or \ HD = PD. \quad (\because \angle CDH = 90°)$

Solution:27

Let AI meet the circumcircle at Q. OA = OQ (radii of the circumcircle)

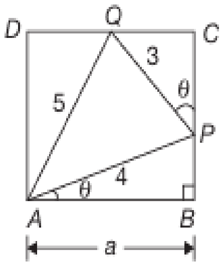
∠OAI = ∠OQI
O is the circumcentre and AQ bisects ∠BAC ∴ arc BQ = arc QC
∴ OQ is perpendicular to chord of arc BC

∴ OQ || AH (both being perpendicular to the same line BC).

∴ ∠HAI = ∠HAQ = ∠AQO = ∠OAQ = ∠OAI

∴ AI bisect ∠HAO.

Solution 28:



Since $5^2 = 3^2 + 4^2$

$i.e., \angle 2 + \angle 3 = 180°$

$\angle 1 + \angle 4 < 180°$ By converse of Baudhayana (Or Pythagoras) theorem

$\angle APQ = 90°$

Let $\angle PAB = \theta$

$\Rightarrow \angle APB = 90° - \theta$

$\angle QPC = \theta.$

Let $AB = a$

In $\Delta APB, \cos\theta = \frac{a}{4} \quad \Rightarrow \quad a = 4\cos\theta$

Also in ΔAPB, $\sin\theta = \frac{PB}{4} \Rightarrow PB = 4\sin\theta$

In ΔPCQ, $\cos\theta = \frac{PC}{3} \Rightarrow PC = 3\cos\theta$

Since $ABCD$ is a square

$AB = BC$

$AB = BP + PC$

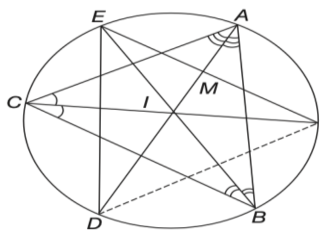
$\therefore 4\cos\theta = 4\sin\theta + 3\cos\theta$

$\therefore \cos\theta = 4\sin\theta$

$\Rightarrow \quad \tan\theta = \frac{1}{4} \quad \Rightarrow \quad \cos\theta = \frac{4}{\sqrt{17}}$

$\therefore AB = a = 4\cos\theta = 4 \times \frac{4}{\sqrt{17}} = \frac{16\sqrt{17}}{17}$ cm.

<u>Solution 29:</u>



Let AD intersect EF at M.

Consider the $\Delta DIMF$

$\angle MFI \;=\; \angle EFC$

$\;=\; \angle EBC$

$= B/2 \, \angle MIF \;=\; 180° - \angle MIC = 180° - [180\,° - \frac{A}{2} - \frac{C}{2}]$ $\quad (In \;\; \Delta AI$

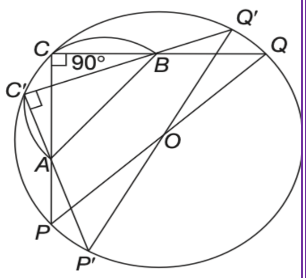
$= 180° - [180\,° - \frac{A}{2} - \frac{C}{2}]$

$= \frac{1}{2}\,(180° \;-\; B)$

$$= 180° - (\frac{B}{2} + 90° - \frac{B}{2})] = 90°$$

i.e., AD is perpendicular to EF. Similarly, we can prove that BE and CF are perpendiculars to FD and ED.

Solution 30:



On AB as diameter, draw a semi-circle to cut the given circle at, say, C and C′. Join CA and CB. Extend them to meet the circle at P, Q.
Then, ΔPCQ is the required triangle. Since, ∠ACB = ∠PCQ = 90°, PQ will be the diameter.

Similarly, if the other point C′ is joined to A and B and extended to meet the given circle at P′, Q′, then $\Delta P'C'Q'$ is the Δ satisfying the given condition.

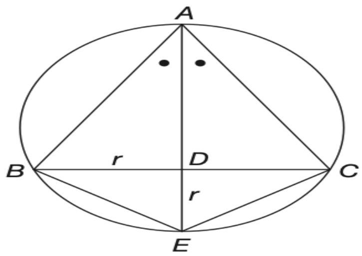
The semi-circle on AB, as diameter, may cut the circle at two points or touch the circle, or the full circle itself may be in the interior of the given circle. Accordingly, there are two right angled triangles, or one right angled triangle , or no right angled triangle satisfying the hypothesis.

SOLUTION 31:

Draw the circumcircle of ΔABC and let the bisector AD of ∠A meet the circumcircle again at E.

$$\Delta\ ABD\ is\ similar\ to\ \Delta AEC$$

$\therefore\ \frac{AD}{AC} = \frac{AB}{AE}$



$\Rightarrow\ AD \times AC = AD.AE\ \ > AD^2 \qquad (AE > AD)$

$\Rightarrow\ \ AD\ < AB \times\ AC$

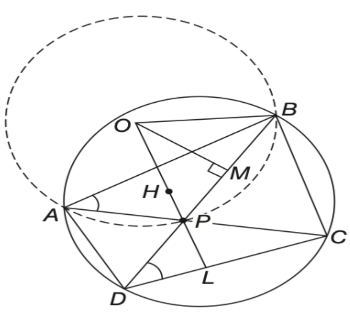
which was to be proved.

Solution 32:

Given, ABCD is a cyclic quadrilateral.

'O' is the circumcentre of ΔAPB.

To explain, if M is the mid-point of PB, then OM is perpendicular to PB in Fig. 3.12, H is the orthocentre of ΔCPD.



Let, OP produced meet DC in L.

To prove: O, P and H, are collinear.
To prove that H lies on OP or OP produced. A Or, in other words, OP produced is perpendicular to DC.

Proof: Since quadrilateral ABCD is cyclic,

$\angle CDB = \angle CAB = \angle PAB = \frac{1}{2}\angle POB$ (Since, O is the circumcentre of Δ PAB)

∠POM (= ∠BOM) as OM is the perpendicular bisector of PB.

$$In\ \Delta LDP\ and\ \Delta MOP, \angle LDP = \angle POM$$

$\angle DPL = \angle OPM\ (Vertically\ opp.)$

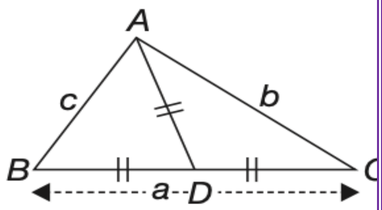
$\therefore \angle PLD = \angle PMO = 90°$ and hence the result.

Solution 33 :

$\Delta = \frac{1}{2}bc\ sinA = \frac{1}{2}bc$

$$\Rightarrow sinA = 1$$

$$\Rightarrow \angle A = 90°$$



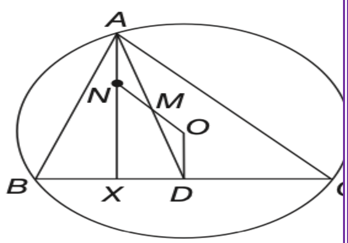
Since AD is the median and $\angle A = 90°$, D, the mid-point of BC is the centre of the circumcircle of ΔABC.

So $AD = BD = DC$

$$\angle ABC = \frac{1}{2}\angle ADC.(Angle\ subtended\ by\ AC\ at\ the\ circumfere$$
$$= \frac{1}{2}(angle\ subtended\ by\ AC\ at\ the\ centre.)$$

Solution 34:

Let AD be the median through A, and M be the mid-point of AD. Join OD. Since, D is the mid-point of BC and O is the circumcentre, OD is perpendicular to BC.



In Δ DMO and Δ AMN,

$DM = AM$ (M is the mid point of AD)

$OM = NM$ (given)

$\angle DMO = \angle AMN$ (vertically opposite angles)

So, the triangles are congruent,

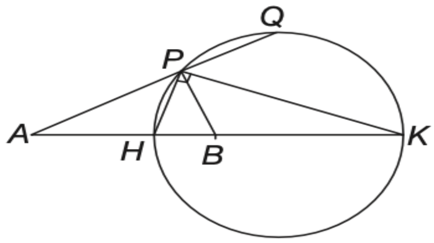
∠MDO = ∠MAN

So, AN || OD . ($\angle MDO$ and $\angle MAN$ are alternate interior angles and are equal)

But, *OD* is perpendicular to *BC* and hence, *AN* produced is perpendicular to *BC* ,

i.e., N lies on the perpendicular through A to BC, i.e., N lies on the altitude through A).

<u>Solution 35:</u>



Produce *AP* to *Q*.

Divide *AB*, internally and externally in the ratio $\frac{PA}{PB} = \lambda$ at H and K, respectively .

$\frac{AH}{HB} = \lambda \;\Rightarrow\; \frac{PA}{PB} = \frac{AK}{BK}$.

So, *PH* and *PK* are the internal and external bisectors of $\angle APB$

hence, $\angle HPK = 90°$.

So, *P* lies on a circle on *HK* as diameter.

<u>Solution 36:</u>

To proof: $AD' = AD_0$

We know that, BD= s-b

$\Rightarrow AD_0 = \frac{c+s-b+AD}{2} - (s-b) = \frac{c+b-s+AD}{2}$

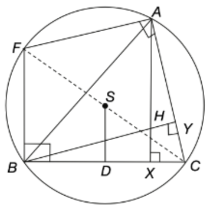
and $AD' = \frac{c+s-c+AD}{2} - (s-c) = \frac{c+b-s+AD}{2}$

$\Rightarrow AD' = AD_0$

Hence we can say D0 and D' are the same points.

<u>Solution 37:</u>

Join *CS* and produce it to cut the circumcircle at *F*. Join *FB* and *FA*. Since *CF* is a diameter



∴ ∠*FBC* = ∠*FAC* = 90°
Since *FB* ⊥ *BC* and *AX* ⊥ *BC*

∴ *FB* | | *AX* | | *AH*
also *FA* ⊥ *AC*, *BY* ⊥ *AC*
∴ *FA* || *BY* || *BH*
∴ In quadrilateral *AFBH*
AF || *HB* and *FB* || *AH*
∴ *AFBH* is a parallelogram
∴ *AH* = *FB*
also in D *CFB*, *S* and *D* are the mid-points of *CF* and *CB* respectively

∴ By mid-point theorem.

SD||*FB* and $SD = \frac{1}{2}FB \Rightarrow SD = 1/2\;AH$ [∴*AH*=*FB*]

$\Rightarrow AH = 2SD.$

<u>*Solution 38:*</u>

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{(s-a)}{\Delta}+\frac{(s-b)}{\Delta}+\frac{(s-c)}{\Delta}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{(s-a)+(s-b)+(s-c)}{\Delta}$$

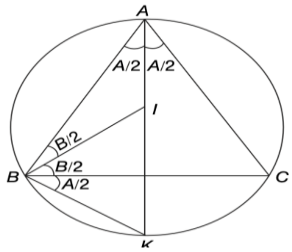
$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{3s-(a+b+c)}{\Delta}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{3s-2s}{\Delta}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{s}{\Delta}$$

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}.$$

<u>*Solution:39*</u>



I is the incentre of ΔABC ,

$$\angle IAB = (1/2)\angle A$$

$$\angle IBA = (1/2)\angle B$$

$\angle KBC = \angle KAC = (1/2)\angle A$ *(1)*

$$\therefore \angle IBK = \angle IBC + \angle CBK = 12\,(\angle A + \angle B)$$

Also In ΔABI by exterior ∠ property

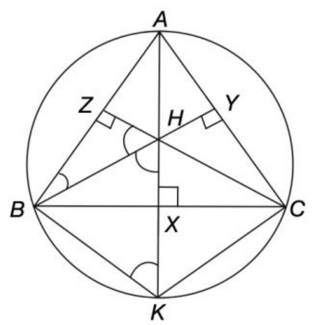
$\angle BIK = \angle IAB + \angle IBA = A/2 + B/2$ *(2)*

In DIBK,

$\angle IBK = \angle BIK = 1/2(\angle A + \angle B)$ *(From Eqs. (1) and (2))*

$$\therefore KI = KB$$

<u>Solution 40:</u>



In *ΔBXH* and *ΔBYC*

$\angle BXH = \angle BYC = 90°$

$\angle XBH = \angle YBC$

∴ By AA similarly

ΔBXH ~ ΔBYC

∴ ∠BHX = ∠BCY = ∠C

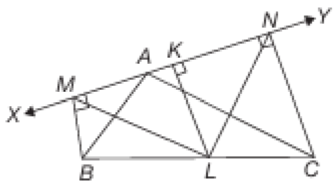
Also $\angle ACB = \angle AKB = \angle C$

In ΔBXH and ΔBXK

$$\angle BHX = \angle BKX = \angle C$$

$BXH = \angle BXK = 90°$
$BX = BX$
$\therefore$ *By AAS Congruence* $\Delta BXH \cong \Delta BXK \therefore HX = KX.$

Solution 41:



Given: ΔABC, XAY is any line passes through A. BM ⊥ XY and CN ⊥ XY. And BL = CL, L is mid-point of BC. To prove: LM = LN Construction: Draw LK ⊥ XAY

Proof: Since perpendiculars drawn on the same line are parallel to each other BM || LK || CN Also by proportional intercept property

BL /LC = MK /KN

1= MK /KN ∵ (BL= LC)

⇒ MK = KN

In ΔMKL and ΔNKL

MK = NK

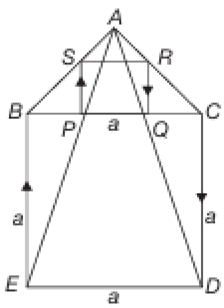
∠MKL = ∠NKL = 90°

KL = KL (By SAS congruence),

ΔMKL ≅ ΔDNK

⇒ LM = LN.

Solution 42:



Let ABC be the triangle in which a square is to be inscribed as desired.

Construct a square BCDE on the opposite side of ∠A. Join AE and AD to cut

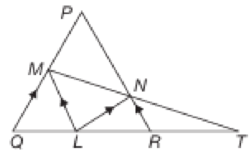
BC at P and Q respectively.

Erect perpendiculars at P and Q to cut AB at S and AC at R, join SR. Then PQRS is the square inscribed in ΔABC as desired.

Solution43 :

In ΔMLT, NR || ML

$\frac{TR}{TL} = \frac{TN}{TM}$ (BY BPT)



In ΔTQM,

$\frac{TL}{TQ} = \frac{TN}{TM}$ (BY BPT)

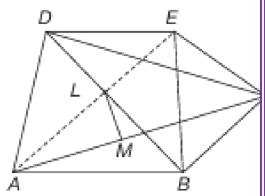
By equating Eqs. (1) and (2) we get,

TL^2 = TR.TQ

That is, is the geometric mean between TR and TQ.

Solution 44

Given quadrilateral ABCD, L and M are the mid-points of diagonals BD and AC respectively.



To prove: LM || EC and LM = $\frac{1}{2}$EC

Proof: Since DE = AB, DE || AB and in a quadrilateral if one pair of opposite side is equal and parallel then it is a parallelogram

∴ABED is a parallelogram.

Its diagonals bisect each other so L is also the midpoint of AE

In · ΔAEC, L and M are the midpoint of AE and AC respectively

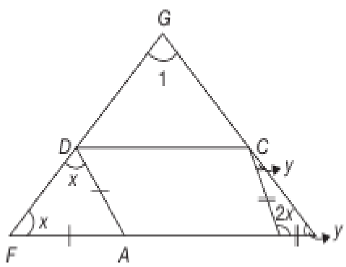
By midpoint theorem LM||EC ⇒ and LM = $\frac{1}{2}$ EC

Solution 45.

Given: ABCD is a parallelogram AB is produced both ways AF = AD and BG = BC.

To prove: FD and GC produced cut at right angles

Proof: Since in ΔAFD, AF = AD ∴∠AFD =∠ADF = x (Say)



∠ DAB =2x (Exterior angle theorem)

AD||CB ⇒∠CBG = ∠ DAB = x

In ΔBCG, BC = BG, ∠ BCG = ∠BGC= y ,

And, 2x+y+z = 108^0

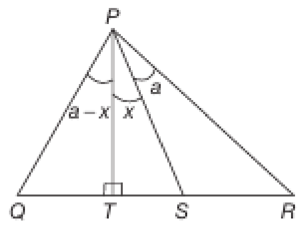
Y + z = 90^0

Therefore, ∠1 = 90^0

<u>Solution 46 :</u>

Let ∠QPS = ∠SPR = a and ∠TPS = x

∴∠QPT = a – x



In ΔPTR, by using exterior angle property,

∠ QTP = 90^0 = a +x + ∠R.

In ΔPTQ, by using exterior angle property,

∠ PTR = 90^0 = a –x + ∠Q.

a +x + ∠R. = a –x + ∠Q.

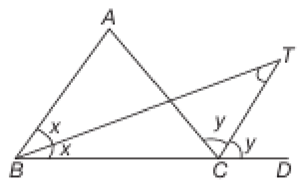
2 x = ∠Q- ∠R

that x = $\frac{1}{2}$ (∠Q- ∠R)

<u>Solution 47</u>

Proof: In ΔABC, by using exterior angle property of a triangle

∠ACD = ∠ABC + ∠ A



2 y = 2 x + ∠A … (1)

In ΔTBC, by using exterior angle property,

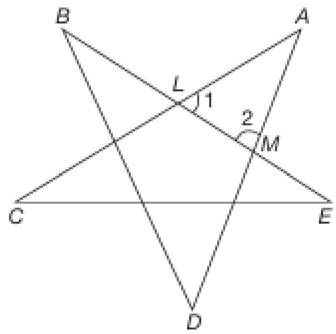
∠ y = ∠x + ∠ T … (2)

From 1 and 2 we get,

∠BTC = $\frac{1}{2}$ ∠A

<u>Solution 48 :</u>

Let BE intersects AC and AD at L and M respectively



Now, in ΔMBD, by using exterior angle property ∠2 =∠B +∠ D

Similarly, in ΔLCE C E ,∠1=∠C +∠ E

In ΔALM ∠A +∠1+∠2 =180^0

⇒∠A +∠C +∠E +∠B +∠ D = 1800

Or ∠A +∠C +∠E +∠B +∠ D = 180^0

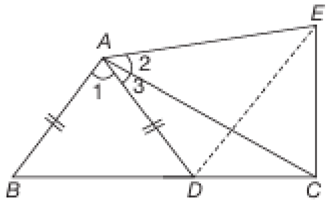
<u>Solution 49 :</u>

Construction: Join DE

Proof: In ΔABC and ΔADE

AB = AD (Given) ….1

Also ∠1 = ∠2



⇒∠ 1 +∠ 3 =∠2 +∠3

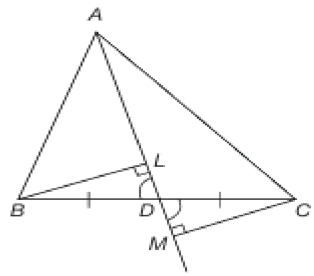
⇒∠ BAC = ∠ DAE …2

Also AC = AE (Given) ….3

Using equation 1 , and 3 we get

Δ ABC ≅ ΔADE

⇒ BC = DE , (by CPCT)



Solution 50 :

Proof: In ΔBDL and ΔCDM,

∠ BLD =∠CMD =90 °

∠ BLD =∠CDM (VOA—Vertically Opposite Angle)

BD = CD

By AAS congruences

ΔBDL ≅ ΔCDM

⇒ BL = CM (CPCT)

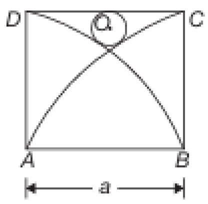
Note: In this figure BLCM will be a parallelogram

Solution51 :

Proof: Since if two circles are touching then the line segment joining their centres passes through their point of contact.

BO = a + r

MO = a - r



And BM = $\frac{a}{2}$ by symmetry

In right angle ΔOMB,

$BO^2 = MO^2 + BM^2$

$(a+r)^2 = (a-r)^2 + (\frac{a}{2})^2$

$(a+r)^2 - (a-r)^2 = \frac{a^2}{4}$

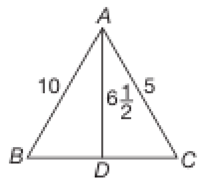
$4ar = \frac{a^2}{4}$

$r = \frac{a}{16}$

Solution 52 :

Let D be the mid-point of BC. By Apollonius theorem

$AB^2 + AC^2 = 2(BD^2 + AD^2)$ (By Apollonius Theorem)



$4AD^2 = 2AB^2 + 2AC^2 - BC^2$

$BC^2 = 2AB^2 + 2AC^2 - 4AD^2$

$= 2(10)^2 + 2(5)^2 - 4\left(\frac{13}{2}\right)^2 = 81$

BC = 9 cm.

$S = \frac{9+10+5}{2} = 12$

Area = $\sqrt{s(s-a)(s-b)(s-c)} = 6\sqrt{p}$ cm^2

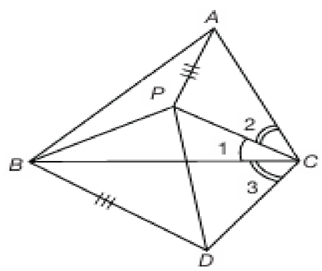
P = 14

Solution 53. :

Construct ∠BCD = ∠ACP and CD = CP

In ΔACP and Δ BCD

CP = CD



By SAS Congruency

ΔACP ≅ ΔBCD,

AP = BD

Also ∠1 + ∠3 = ∠2 + ∠3 = 60°

And PC = CD

ΔPCD is an equilateral D with PC = PD = CD and ∠DPC = 60°

Since $PA^2 = PB^2 + PC^2$

$BD^2 = PB^2 + PD^2$ (As PA = BD, PC = PD)

By converse of Baudhayna (or Pythagoras) theorem

$\angle BPD = 90^0$

$\angle BPC = \angle BPD + \angle DPC$

$= = 90° + 60°$

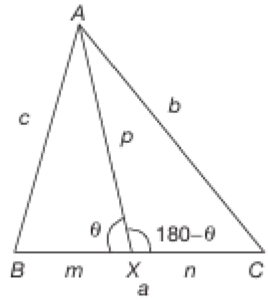
$\angle BPC = 150°$.

<u>Solution 54.</u>

Proof: Let $\angle ADB < \angle \mathbf{ADC}$

$\Rightarrow \angle ADB$ is acute and $\angle ADC$ is obtuse.

In ΔABD, by using acute angle theorem, we



$AB^2 = AD^2 + BD^2 - 2\ BD\ MD.$ 1

$\Rightarrow c^2 = d^2 + m^2 - 2\ mx$

In ΔADC, by using obtuse angle theorem, we get

$AC^2 = AD^2 + DC^2 - 2\ DC.DM$ 2

$\Rightarrow b^2 = d^2 + n^2 - 2nx$

On solving equation 1 and 2 we get

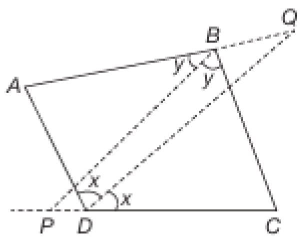
$nc^2 + mb^2 = d^2 (m + n) + mn^2 + m^2 n$

$a(d^2 + mn) = (b^2 m + c^2 n)$.

Solution 55.

Let $\angle ABP = \angle PBC = y = \frac{1}{2} \angle B$

And $\angle ADQ = \angle QDC = x = \frac{1}{2} \angle D$



$\Rightarrow \angle PDQ = 180 - x$ and $\angle PBQ = 180 - y$.

In quadrilateral PDQB, $\angle P + \angle PDQ + \angle Q + QBP = 360^0$

$\Rightarrow \angle P + 180° - x + \angle Q + 180° - y = 360^0$.

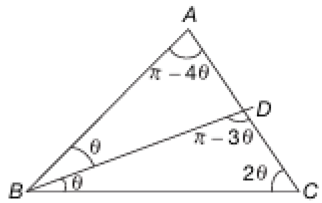
$\Rightarrow \angle P + \angle Q = \frac{1}{2} (\angle ABC + \angle ADC)$

Solution 56. :

Let ∠DBC = θ, so that ∠ACB = 2θ and ∠BDC = π − 3θ, also ∠BAC = π − 4θ.

Now by sine rule, in ΔBDC and ΔABD respectively, we get

$\frac{BC}{sin3\theta} = \frac{BD}{sin2\theta}$ and $\frac{AD}{sin\theta} = \frac{BD}{sin4\theta}$



It is given that BC = BD + AD.

$\frac{BC}{BD} = 1 + \frac{AD}{BD}$

$\frac{sin3\theta}{sin2\theta} = 1 + \frac{sin4\theta}{sin\theta}$

On solving

Sin4θ = Sin5θ

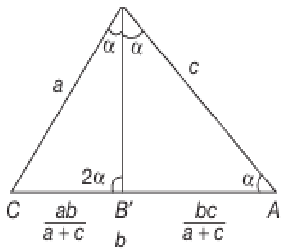
4θ = 5θ or 4θ+ 5θ = 180^0 ,

$4\theta \neq 5\theta$, 9θ = 180^0 ,

$\theta = 20^0$.

Solution 57.

Let $\angle B = 2\alpha$



The bisector of $\angle B$ intersects AC at B', so that, $CB' = \frac{ab}{a+c}$ and $AB' = \frac{bc}{a+c}$

Now $\Delta ABC \sim \Delta BB'C$

$\therefore \frac{BC}{B'C} = \frac{AC}{BC} \Rightarrow BC^2 = AC \cdot B'C$

That is, $a^2 = (b)\left(\frac{ab}{a+c}\right)$ or $a^2 = \frac{ab^2}{a+c}$

i.e., $a(a+c) = b^2$ (1)

According to our assumption of the angles, $b > a$ holds.

$\therefore$ Either $b = (a+1)$ or $b = (a+2)$ (as a, b, c are consecutive)

In the first case, *i.e.*, $b = a + 1 \Rightarrow b^2 = a(a+c)$

$\Rightarrow (a+1)^2 = a(a+c)$, *i.e.*, $a^2 + 2a + 1 = a^2 + ac$

$\Rightarrow 2a + 1 = ac \Rightarrow a \mid 1 \Rightarrow a = 1 \Rightarrow c = 3$ and $b = 2$

Which is impossible, thus $b \neq a + 1$.

Then, let $b = a + 2$ then $c = a + 1$, now $(a+2)^2 = a(a+a+1) = 2a^2 + a$

$\Rightarrow a^2 - 3a - 4 = 0$

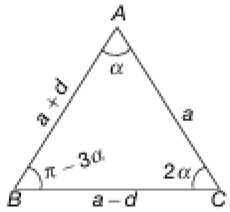
$\therefore a = -1$ or 4, but $a \neq -1$ (reject)

$\therefore a = 4$; thus $b = 6$ and $c = 5$.

$\therefore$ There is only one triangle satisfying the conditions of the problem, *i.e.*, the triangle whose measures are 4, 5, and 6.

Solution 58.

Let the sides be a − d, a, a + d (a > 0, d > 0)



Let α be the smallest angle of the triangle opposite to (a − d); then the greatest angle 2α is opposite to (a + d).

Applying sine rule for ΔABC,

$\frac{a-d}{sin\alpha} = \frac{a}{\sin(\pi-3\alpha)} = \frac{a+d}{sin2\alpha}$

$\frac{a-d}{a+d} = \frac{1}{2cos\alpha}$

$4\cos^2\alpha = (\frac{a-d}{a+d})^2$ … 1

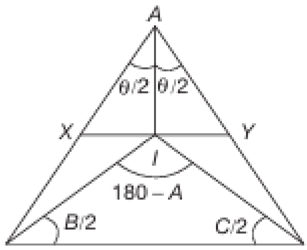
Also $\frac{a-d}{a} = \frac{sin\alpha}{sin3\alpha}$ on solving $4\cos^2\alpha = \frac{2a-d}{a-d}$ … 2

Fron equation 1 and 2

a = 5d

Ratio of the sides is (a − d) : a : (a + d) = 4d : 5d : 6d, i.e., 4 : 5 : 6.

Solution 59. :



Let ABC be the triangle with incentre I. Let X, Y be points on AB, AC respectively such that, $BX \cdot BA = BI^2$ and $CY \cdot CA = CI^2$.

Hence by secant tangent theorem we can conclude that there are circles passing through AIX and AIY respectively, so that, BI is a tangent and BXA is secant in the first circle

and CI is a tangent and CYA is a secant to the second circle.

Thus ∠BIX = ∠BAI and ∠CIY = ∠CAI (Alternate segment theorem)

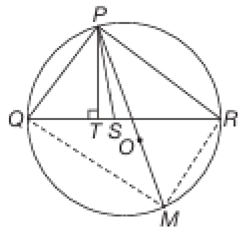
∠BIX = A /2 and ∠CIY = A /2 ∠ BIC = 180 - ∠A (Alternate segment theorem)

Thus $B/2 + (180^0 – A) + C/2 = 180^0$

$\Rightarrow 90^0 – A/2 – A = 0$

$A = 60^0$

Solution 60 :



Construction: Join QM, RM

Proof: Since POM is a diameter, $\angle PRM = 90^0$

$\Rightarrow \angle QRM = 90^0 - \angle R$

$\Rightarrow \angle QPM = \angle QRM = 90^0 - \angle R$

$\Rightarrow \angle TPM = \angle QPM - \angle QTP$

$= (90^0 - \angle R) - (90^0 - \angle Q)$

$= \angle Q - \angle R$

Since $\angle TPS = \frac{1}{2}(\angle Q - \angle R)$

So, $\angle SPM = \frac{1}{2}(\angle Q - \angle R) \Rightarrow$ PS bisects $\angle TPM$.

Solution 61: .

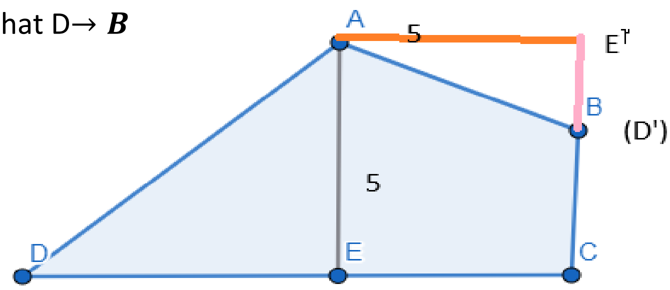
Rotate triangle ADE about A by $\mathbf{90°}$, so that D→ $\boldsymbol{B}$

$ar(\Delta ADE) = ar(\Delta ABE')$

$ar(ABCD) = ar(\Delta ADE) + ar(AECB)$

$ar(ABCD) = ar(\Delta ABE') + ar(AECB)$

$= ar(AECE') = ar(square) = 25$



Solution 62:

Rotate triangle ABD about A at 60° so that $B \rightarrow C$

Now, $\angle DAD' = 60°$

$\Delta DAD'$ is an equilateral triangle ,

So, $DD' = 2$

Now, $\Delta DD'C$ is a right angled triangle

As, $(DD')^2 + (D'C)^2 = DC^2$,So , $\angle DD'C = 90°$

And $\angle AD'D = 60°$

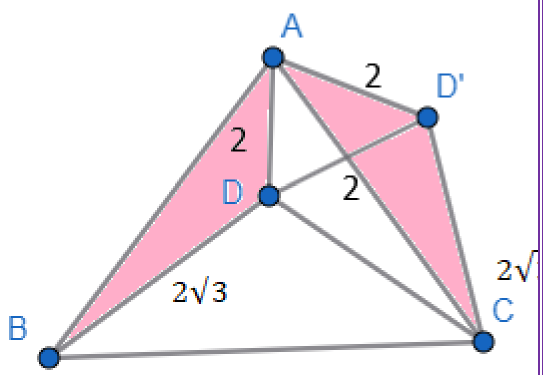
$\angle AD'C = 60° + 90° = 150°$

Using cosine formula in triangle AD'C

$\cos 150° = \frac{2^2+(2\sqrt{3})^2-AC^2}{2\times2\times2\sqrt{3}}$

$AC^2 = 28, AC = 2\sqrt{7}$

Sum of all sides=$6\sqrt{7}$



Solution 63:

Rotate triangle ABC about C at $\mathbf{90°}$, $\boldsymbol{A \rightarrow B}$

$\boldsymbol{\angle D'CD = 90°}$

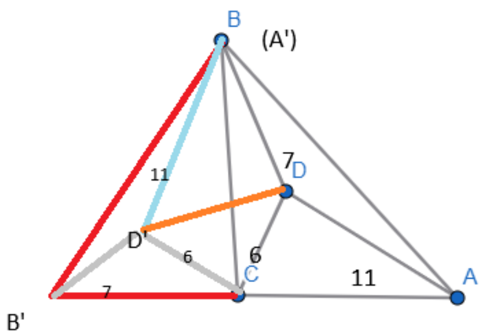
DD'=$\sqrt{\mathbf{6^2 + 6^2}} = \sqrt{\mathbf{72}} = \mathbf{6\sqrt{2}}$

Now, in triangle DD'B, DB=7; D'B=11

and DD'== $\mathbf{6\sqrt{2}}$,

so triangle DD'B is a right-angled triangle

and $\boldsymbol{\angle D'DA' = 90°}$



Now in triangle CDB,

$\angle CDB = 90° + 45° = 135°$

Using cosine formula in triangle CDB,

$$\cos 135° = \frac{7^2+(6)^2-BC^2}{2\times7\times6}$$

$BC^2 = 85 + 42\sqrt{2}$; $BC = \sqrt{85+42\sqrt{2}}$,

by comparing with $\sqrt{a+b\sqrt{2}}$; we get $a = 85$ and $b = 42$

$$a + b = 85 + 42 = 127$$

Solution 64:

Rotate triangle ADB about A at 60°, $B \to C$

$\Delta ADD', \angle DAD' = 60°$,

so $\Delta ADD'$ an equilateral triangle with DD'=1

$\Delta CDD'$ is a right-angled triangle,

as DD'=1, DC=2 and $D'C = \sqrt{3}$

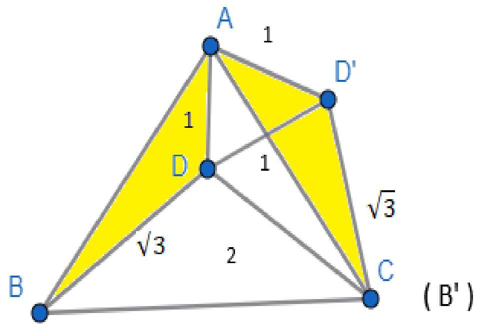
, $\angle DD'C = 90°$ and , $\angle DD'A = 60°$

Using cosine formula in triangle AD'C,

we get AC as

$$AC = \sqrt{1^2 + (\sqrt{3})^2 - 2.1.\sqrt{3}\cos 150°}$$

$$AC = \sqrt{1+3+3} = \sqrt{7}$$



Solution 65:

Rotate triangle AOB about B at 60°

$A \to C$

In $\Delta BOO'$, $\angle OBO' = 60°$

$\Delta BOO'$ is an equilateral triangle, $OO' = 4$

$\Delta COO'$ is a right angled triangle (as sides are 3,4,5)

So, $\angle COO' = 90°$

In $\Delta CBO'$, $\angle BOC = 90° + 60° = 150°$

Using cosine formula in triangle CBO, we get BC as

$BC^2 = 3^2 + 4^2 - 2.3.4\cos(150°)$

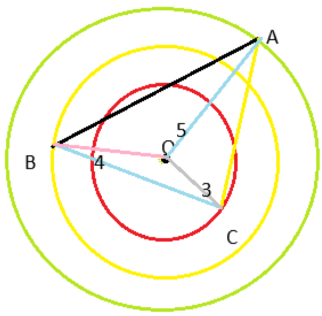
$BC^2 = 25 + 12\sqrt{3}$

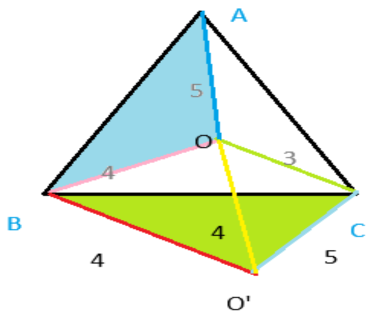
Area of an equilateral triangle=$\frac{\sqrt{3}}{4}(side)^2 = \frac{25}{4}\sqrt{3} + 9$

By comparing it with $a + \frac{b}{c}\sqrt{d}$, we get

$a = 9, b = 25, c = 4\ and\ d = 3$

Now, $a + b + c + d = 41$





Solution 66:

Rotate ΔADE about A at 90°

$D \rightarrow B$

$\Delta AEE'$ is a right-angled isosceles triangle

$EE' = 2\sqrt{2}$

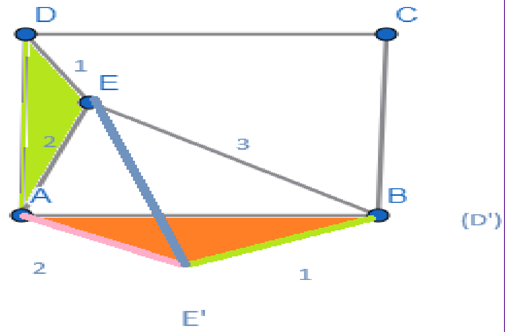
In $\Delta EE'B$, sides are $3, 2\sqrt{2}$ *and* 1,

so it is a right-angled triangle with $\angle EE'B = 90°$

In triangle AE'B, $\angle AE'B = 90° + 45° = 135°$

$\Delta ADE \cong \Delta AE'B$

$so, \angle AE'B = \angle AED = 135°$



Solution 67:

Rotate triangle ABC, about A 90^0 in clockwise

$\angle ACB = 45°$, $\angle ACC' = 45°$, So, $\angle BCC' = 90°$

In triangle NCM', $\angle NCM' = 90°$

By Pythagoras theorem, $NC^2 + M'C^2 = M'N^2$; $BM^2 + CN^2 = MN^2$

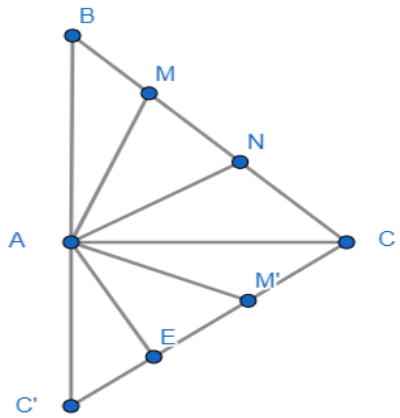
Rotating again 2 times clockwise about A,

then we get a regular octagon

Using angle at a point 360^0

$8\angle's = 360°; \angle's = \frac{360°}{8} = 45°$

So, $\angle MAN = 45°$



Solution 68:

Rotate triangle DFC about D at 180^0

$B \rightarrow C$

BF'=CE=y

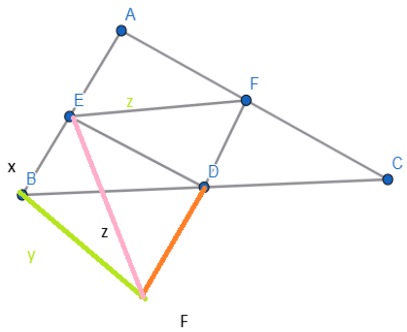
DF=DF'

$\Delta DEF \cong \Delta DEF'$

$EF' = EF = z$

In $\Delta BEF', BE + BF' > EF$

$(x + y > z)$

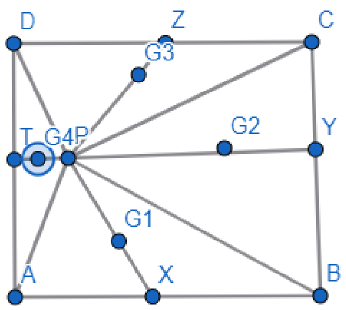


Solution 69:

Let X,Y,Z and T are the mid-points of sides AB,BC, CD and DA of the square.

Here, $\frac{\boldsymbol{PG1}}{\boldsymbol{PX}} = \frac{2}{3} = \frac{\boldsymbol{PG2}}{\boldsymbol{PY}} = \frac{\boldsymbol{PG3}}{\boldsymbol{PZ}} = \frac{\boldsymbol{PG4}}{\boldsymbol{PT}}$

There exists a homothety, $\boldsymbol{H}(\boldsymbol{p}, \frac{3}{2})$, which takes G1 to X,

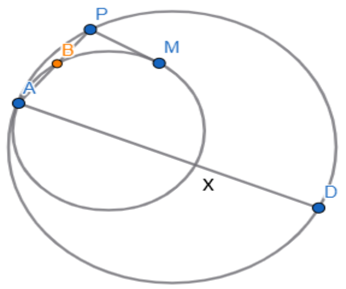


G2 to Y, G3 to X and G4 to T
homothety preserve shapes , XYZT is a square

that implies G1G2G3G4 forms a square.

Solution 70:

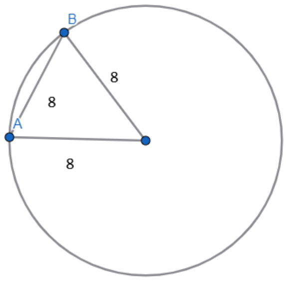
Two circles have internal tangent, so here,
A is the centre of homothety



$$\frac{AB}{AP} = \frac{AX}{AD}$$

$$\frac{AB}{AP} = \frac{2\times 8}{2\times 10} = \frac{4}{5}$$

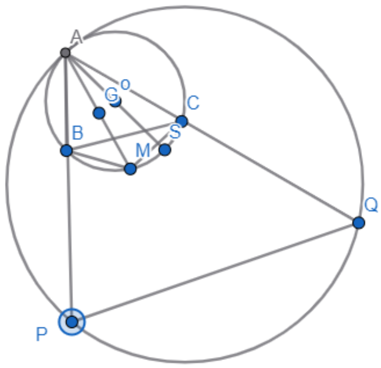
Here, $PB = a, AB = 4a$
Using tangent law, $PM^2 = PA \times PB$
$20 = 5a \times a = 5a^2$
$a = 2$
So, PA=8
In the smaller circle radius=8=PA,
Therefore $x = 60$



Solution 71:

O is the centre of the inner circle
AO=OS
S is the centre of outer circle

By homothety
$$\frac{\boldsymbol{AS}}{\boldsymbol{AO}} = \mathbf{2}$$



the circumcenter S of triangle APQA lies

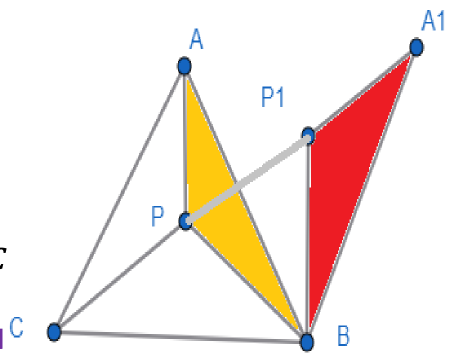
on the circumcircle of △ABC and lies on the line AO,

then: ∠AGO=90^0

solution 72:

P be the arbitrary point inside ABC.
Rotate the triangle ABP clockwise about B by 60^0

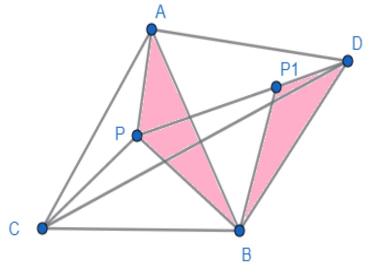
Now we obtain a triangle A1BP1,
since AP=A1P1   BP=PP1=BP1,
It is clear that $\boldsymbol{AP + BP + CP = A1P1 + P1P + PC}$



Thus we see that the sum $AP + BP + CP$ is minimal
when $A1P1 + P1P + PC$ is a straight segment

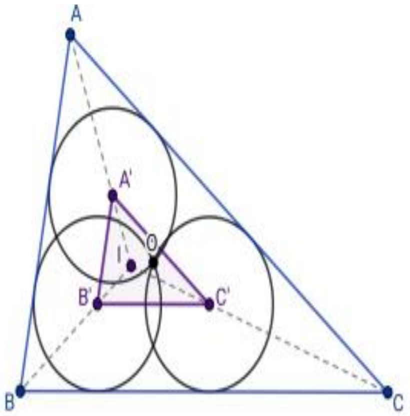
Solution 73:

Rotate the triangle APB through 60^0 about B.
after this rotation A will coincide with D.
Then PB=PP1 and PA=DP1
We know that , sum of any three sides of a quadrilateral Is greater than 4^{th} side.
Hence, PA+PB+PC=P1D+P1B+PC>C



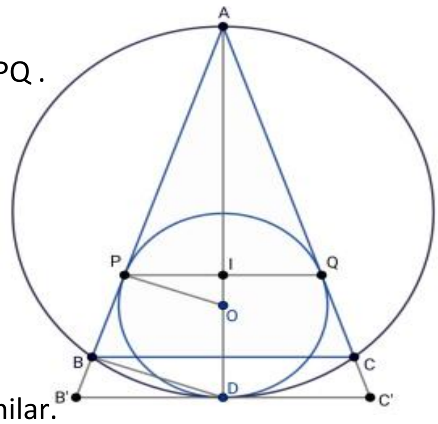
Solution 74:

let A' ,B',C' be the centres of the circles.
Since the radii are the same, so A'B' is parallel to AB,
B'C' parallel to BC, C'A' parallel to CA.
since AA',BB',CC' bisect $\angle A, \angle B, \angle C$ respectively,
they concur at the incentre I of triangle ABC.
Note O is the circumcentre of triangle A'B'C'
as it is equidistant from A',B',C' . then the homothety with centre I sending $\Delta A'B'C'$ to ΔABC will send O to the circumcentre S of ΔABC(not shown in the figure) .
therefore, I,O ,S are collinear.



Solution 75:

let O be the Centre of the circle . let the circle be tangent
to the circumcircle of ΔABC at D. let I be the midpoint of PQ .
then A,I,O,D are collinear by symmetry.
Consider the homothety with Centre A that sends
ΔABC to $\Delta AB'C'$ such that D is on B'C' .
the scaling factor here is $k = \frac{AB'}{AB}$



Now since the right triangles AIP,ADB',ABD and APO are similar.

$$\frac{AI}{AO} = \frac{AI}{AP} . \frac{AP}{AO} = \frac{AD}{AB'} . \frac{AB}{AD} = \frac{AB}{AB'} = \frac{1}{K}$$

This tells us that the homothety sends I to O. then O being the incentre of $\Delta AB'C'$ implies that I is the incentre of ΔABC.

Solution 76:

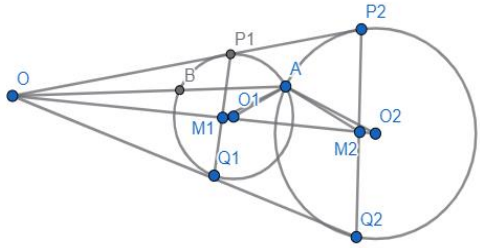
By symmetry ,P_1P_2, O_1O_2, Q_1Q_2 concur at a point O.

consider the homothety with centre at O which sends C_1 to C_2.

Let OA intersect C_1 at B . Then under the Homothety

A is the image of B. since ΔBM_1O_1 is send to ΔAM_2

So $\angle M_1BO_1 = \angle M_2AO_2$



Now $\Delta OP_1O_1 \cong \Delta OM_1P_1$,

which implies $\frac{OO_1}{OP_1} = \frac{OP_1}{OM_1}$

Then, $OO_1.OM_1 = OP_1^2 = OA.OB$, which implies A, M_1, B, O_1 are concyclic.

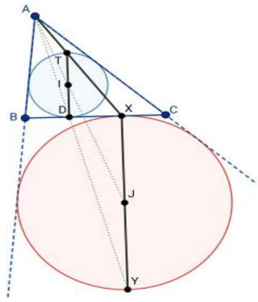
Then, $\angle M_1NO_1 = \angle M_1AO_1$. Hence $\angle M_1AO_1 = \angle M_2AO_2$.

Adding $\angle O_1AM_2$ to both sides, we have

$\angle O_1AO_2 = \angle M_1AM_2$

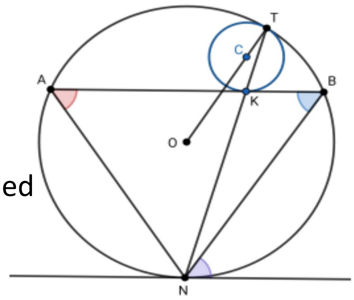
Solution 77:

Assume that AB ≤ AC. Consider the dilation with centre A that carries the incircle to the A-excircle. The line segment DT is the diameter of the incircle that is perpendicular to BC, and therefore its image under the dilation must be the diameter of the excircle that is perpendicular to BC. It follows that T must get mapped to the point of tangency between the excircle and BC. In addition, the image of T must lie on the line AT, and hence T gets mapped to X. Thus, the excircle is tangent to BC at X. From here it is easy to show that BD = CX,



solution 78:

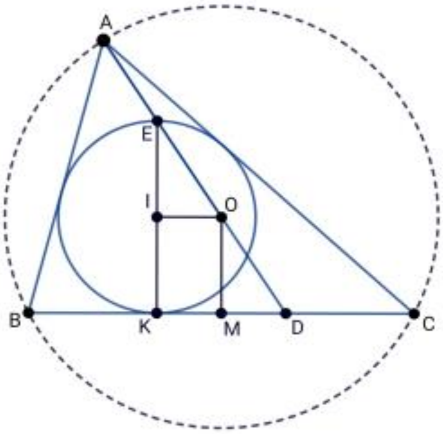
Let us assume that TK meets the arc AB at N and we shall show that AN = NB.
Since Ω and ω are tangent at T,
it follows there is a homothety at T taking ω to Ω.
Clearly, the tangent to ω at K (which is line AB) will be mapped to the tangent to Ω at N.
Hence it follows that the tangent to Ω at N is parallel to AB.
Now AN = NB follows by observing that the three colored angles in the adjacent diagram are equal.

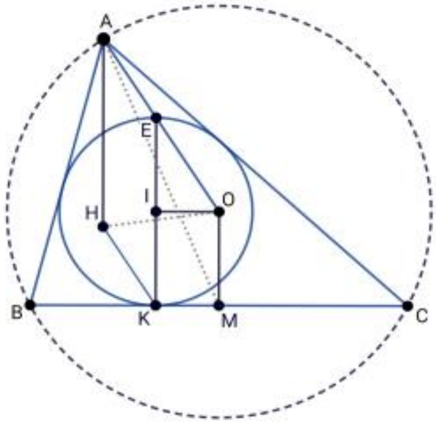


Solution 79:

Let KE be a diameter of the incircle, and let line AE meet BC at D. Let M be the midpoint of BC
By the diameter of the incircle lemma, M is also the midpoint of KD. Since IO is parallel to BC, we can say that KMOI is a rectangle. Since I is the midpoint of KE and

M is the midpoint of KD, we see that1 O must be the midpoint of ED. Thus lines AE and AO coincide



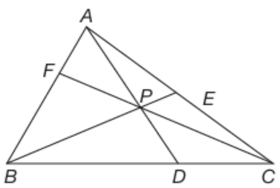


Since AH = 2OM = EK and AH and EK are both perpendicular to BC, it follows that AHKE is a parallelogram. Hence HK is parallel to AE, which coincides with line AO.

Solution 80

Let the line l touch the circle C at D and let T, X be the reflection of D with respect to the centre of C and M respectively. Note that the points T, X are independent of P. But the diameter of the incircle lemma tells us that P must lie on the ray XT beyond T. Conversely, given a point P lying on the ray XT beyond T, let the tangents from P to C meet l at Q and R. By the lemma we must have QD = XR, from which it follows that M is the midpoint of QR. Therefore, the locus is the ray XT beyond T.

Solution 81:



Proof: By Ratio proportion theorem (or area lemma), we have $\frac{[ABD]}{[ADC]} = \frac{BD}{DC}$ (1)

And $$\frac{[BPD]}{[CPD]} = \frac{BD}{DC} \quad (2)$$

$\therefore$ From Eqs. (1) and (2)

$$\frac{BD}{DC} = \frac{[ABD]}{[ADC]} = \frac{[BPD]}{[CPD]} = \frac{[ABD]-[BPD]}{[ADC]-[CPD]} = \frac{[ABP]}{[ACP]}$$

Let $[BPC] = \Delta_1$, $[ACP] = \Delta_2$ and $[ABP] = \Delta_3$

$$\therefore \quad \frac{BD}{DC} = \frac{\Delta_3}{\Delta_2}$$

Similarly $\frac{CE}{EA} = \frac{[BPC]}{[APB]} = \frac{\Delta_1}{\Delta_3}$

$$\frac{AF}{FB} = \frac{[APC]}{[BPC]} = \frac{\Delta_2}{\Delta_1}$$

$$\therefore \quad \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = \frac{\Delta_3}{\Delta_2} \cdot \frac{\Delta_1}{\Delta_3} \cdot \frac{\Delta_2}{\Delta_1} = 1.$$

Solution 82:

Produce CA to D, such that

$AD = AB$

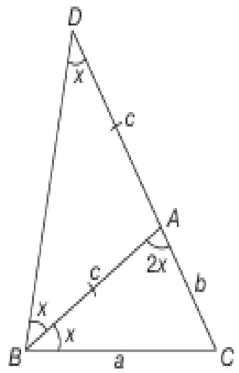
$\therefore \quad \angle ABD = \angle ADB$ and $\angle ABD + \angle ADB = \angle BAC = 2x$

$\therefore \quad \angle ABD = \angle ADB = x.$

$\Delta ABC \sim \Delta BDC$ (AAA similarity)

$$\frac{BC}{DC} = \frac{AC}{BC}$$

i.e., $$\frac{a}{b+c} = \frac{b}{a} \Rightarrow b(b+c) = a^2$$



Solution 83:

'O' is the circumcentre of ΔAPB.

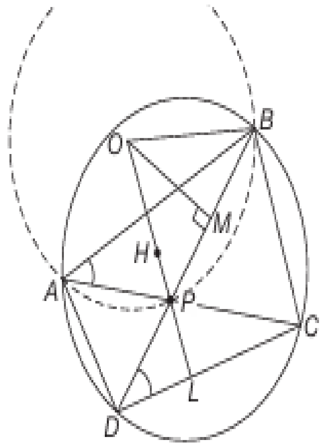
To explain, if M is the mid-point of PB, then OM is perpendicular to PB in the in Fig. 3.12, H is the orthocentre of ΔCPD.

Let, OP produced meet DC in L.

To prove: O, P and H, are collinear.

To prove that H lies on OP or OP produced.

Or, in other words, OP produced is perpendicular to DC.



Proof: Since quadrilateral $ABCD$ is cyclic,

$\angle CDB = \angle CAB = \angle PAB = \frac{1}{2}\angle POB$ (Since, O is the circumcentre of ΔPAB) = $\angle POM$ ($= \angle BOM$) as OM is the perpendicular bisector of PB.

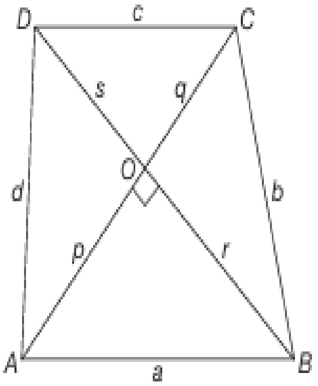
In ΔLDP and MOP,

$\angle LDP = \angle POM$

$\angle DPL = \angle OPM$ (Vertically opp. $\angle^s$)

$\therefore \quad \angle PLD = \angle PMO = 90°$ and hence the result.

Solution 84:



Solution: Let a, b, c and d be the measures of the sides AB, BC, CD, and DA of the quadrilateral. The diagonals intersect at O. Let $OA = p$, $OB = r$, $OC = q$ and $OD = s$.

If AC is not perpendicular to BD, let $\angle AOB$ be obtuse.

Then, by the extension of the Pythagoras theorem,

$$a^2 > p^2 + r^2;\ b^2 < r^2 + q^2$$

$$c^2 > s^2 + q^2;\ d^2 < p^2 + s^2$$

$$a^2 + c^2 > p^2 + r^2 + s^2 + q^2 > b^2 + d^2$$

Thus, $a^2 + c^2 > b^2 + d^2$

which is a contradiction as it is given that $a^2 + c^2 = b^2 + d^2$ and $\angle AOB \not> 90°$

If AC is perpendicular to BD, then

$$a^2 = p^2 + r^2$$

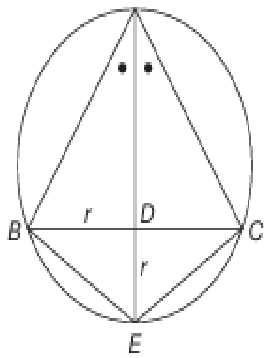
$$c^2 = s^2 + q^2$$

$$a^2 + c^2 = p^2 + q^2 + r^2 + s^2 = (p^2 + s^2) + (q^2 + r^2)$$

$$= d^2 + b^2.$$

Solution 85:

Solution: Draw the circumcircle of ΔABC and let the bisector AD of $\angle A$ meet the circumcircle again at E.



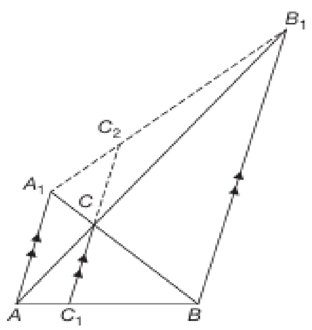
ΔABD is similar to ΔAEC (AA similarity)

$$\therefore\ \frac{AD}{AC} = \frac{AB}{AE}$$

$$\Rightarrow\ AB \times AC = AD \cdot AE > AD^2 \quad (\because AE > AD)$$

$\Rightarrow\ AD < \sqrt{AB \times AC}$ which was to be proved.

Solution 86:



Solution: AA_1, BB_1 and CC_1 are parallel line segments and hence,

$$\frac{CC_1}{A_1A} = \frac{C_1B}{AB}$$

Also $$\frac{CC_1}{B_1B} = \frac{AC_1}{AB}$$

Adding Eqs. (1) and (2), we have

$$\frac{CC_1}{A_1A} + \frac{CC_1}{B_1B} = \frac{C_1B + AC_1}{AB} = \frac{AB}{AB} = 1$$

Dividing Eq. (3) by CC_1, we get

$$\frac{1}{A_1A} + \frac{1}{B_1B} = \frac{1}{CC_1}$$

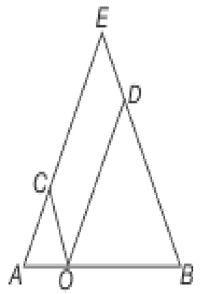
Solution 87:

Solution: In the figure ΔAOC and ΔBOD being equilateral $\angle COD = 180° - (\angle COA + \angle BOD) = 180° - (60° + 60°) = 60°$.

The exterior $\angle ODE$ of $\Delta OBD = 60° + 60° = 120°$. Again, the exterior $\angle OCE$ of $\Delta OCA = 60° + 60° = 120°$.

Therefore, the remaining

$$\angle CED = 360° - (120° + 120° + 60°) = 60°$$



In quadrilateral $OCED$, opposite angles are equal, implying that the opposite sides are parallel. Thus, it is a parallelogram. In this parallelogram, if the adjacent sides $OC = OD$ (*i.e.*, all sides are equal), then it becomes a rhombus. For this, we should have $AO = OC = OD = OB$, *i.e.*, $AO = OB$ or O should be the mid-point of the segment AB (Also note that ΔAEB is also equilateral).

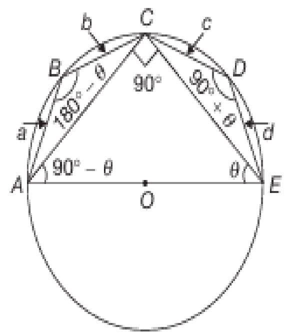
solution 88:

Solution: Since, AE is the diameter $\angle ACE = 90°$ and $AC^2 + CE^2 = AE^2 = 2^2 = 4$. By cosine formula (for ΔABC)

$$AC^2 = a^2 + b^2 - 2ab\cos(180° - \theta)$$
$$= a^2 + b^2 + 2ab\cos\theta$$

Similarly, in ΔCED

$$CE^2 = c^2 + d^2 - 2cd\cos(90° + \theta)$$
$$= c^2 + d^2 + 2cd\sin\theta$$
$$\therefore AC^2 + CE^2 = a^2 + b^2 + c^2 + d^2\ 2ab\cos\theta + 2cd\sin\theta$$



In ΔACE, $\dfrac{AC}{AE} = \sin\theta$

$\Rightarrow \quad AC = 2\sin\theta > b \qquad (AE = 2) \qquad (1)$

and $\dfrac{CE}{AE} = \cos\theta \qquad (AE = 2)$

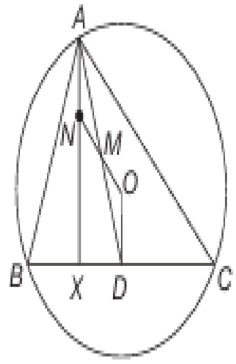
$\Rightarrow \quad CE = 2\cos\theta > c \qquad (2)$

(Because, in ΔABC and ΔCDE, $\angle B$ and $\angle D$ are obtuse angles. Here, AC is the greatest side of ΔABC, and CE is the greatest side of ΔCDE)

$$AC^2 + CE^2 = a^2 + b^2 + c^2 + d^2 + 2ab\cos\theta + 2cd\cos\theta = 4$$
$$\Rightarrow \quad a^2 + b^2 + c^2 + d^2 + ab \cdot 2\cos\theta + cd \cdot 2\sin\theta = 4$$
$$\Rightarrow \quad a^2 + b^2 + c^2 + d^2 + abc + bcd < 4 \qquad \text{(by Eqs. (1) and (2))}$$

Solution 89:

Solution: Let AD be the median through A, and M be the mid-point of AD. Join OD.



Since, D is the mid-point of BC and O is the circumcentre, OD is perpendicular to BC.

In ΔDMO and ΔAMN,

$DM = AM$ (M is the mid-point of AD)

$OM = NM$ (Given)

$\angle DMO = \angle AMN$ (Vertically opposite angles)

So, the triangles are congruent.

$\angle MDO = \angle MAN$ (Corresponding angles of congruent triangles)

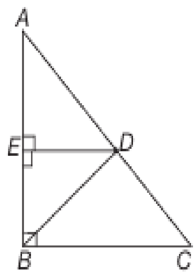
So, $AN \parallel OD$ ($\angle MDO$ and $\angle MAN$ are alternate interior angles and are equal)

But, OD is perpendicular to BC and hence, AN produced is perpendicular to BC, *i.e.*, N lies on the perpendicular through A to BC, *i.e.*, N lies on the altitude through A).

Solution 90:

Solution: Given In ΔABC, $\angle B = 90°$, $AD = DC$

To prove: $BD = \frac{1}{2}AC$



Construction: Draw $DE \parallel CB$

Proof: In ΔABC, D is a mid-point of AC and $DE \parallel CB$

$\therefore$ By converse of mid-point theorem E is a mid-point of AB, *i.e.*, $AE = EB$

also $\angle E = 90°$ $\therefore DE \perp AB$

In ΔAED and ΔBED

$AE = BE$ (Proved above)

$\angle AED = \angle BED = 90°$

$ED = ED$ (Common)

$\therefore$ By SAS congruence $\Delta AED \cong \Delta BED$

$\therefore AD = BD$

but $AD = CD$

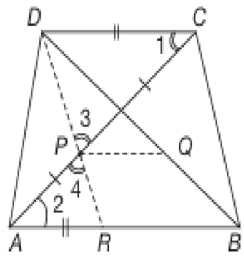
$\therefore \quad BD = AD = CD = \frac{1}{2}AC.$

Solution 91:

Solution: Given In trapezium $ABCD$, $AB \parallel CD$, P and Q are the mid-points of diagonal AC and BD respectively

To prove: $PQ \parallel AB \parallel DC$ and $PQ = \frac{1}{2}(AB - DC)$

Construction: Join DP and produce it to cut AB at R.



Proof: In ΔCPD and ΔAPR

$\angle 1 = \angle 2$ (Alternate interior angles)

$CP = AP$ (As P is the mid-point of AC)

$\angle 3 = \angle 4$ (VOA)

$\therefore$ By ASA congruence $\Delta CPD \cong \Delta APR$

$\therefore$ $CD = AR$ and $DP = RP$

In ΔDRB

P and Q are the mid-points of DR and DB respectively

$PQ \parallel RB$ and $PQ = \frac{1}{2}RB$

$\Rightarrow PQ \parallel AB \parallel DC$ and $PQ = \frac{1}{2}(AB - AR)$ (As $RB = AB - AR$)

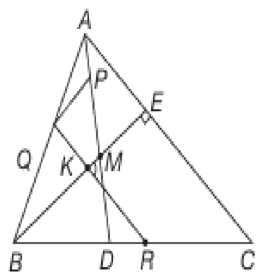
$\Rightarrow PQ = \frac{1}{2}(AB - CD)$. (As $AR = CD$)

Solution 92:

Given: In ΔABC, $BE \perp AC$. Q, R are the mid-points of AB, BC respectively AD is any line which cuts BE at H. P is a mid-point of AH.

To prove: $\angle PQR = 90°$

Construction: Join QR which cuts BE at K



Proof: Since In ΔABC, Q, R are the mid-points of AB, BC respectively.

$\therefore$ By mid-point theorem $QR \parallel AC$,

also, $\angle BEC = 90°$

$\therefore$ $\angle BKR = 90° = \angle HKR$

In ΔABH, Q and P are the mid-points of AB and AH respectively

$\therefore$ By mid-point theorem

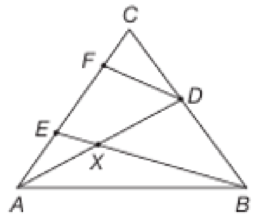
$QP \parallel BH$

$\therefore$ $\angle PQR = \angle HKR = 90°$ (Corresponding angles)

$PQ \perp QR$.

Solution 93:

Given: In ΔABC, $\frac{BD}{DC}=\frac{m}{1}; \frac{AE}{EC}=\frac{n}{1}$ and AD, BE intersect at X.



To find: $\frac{AX}{XD}$.

Construction: Draw $DF \parallel BE$.

Proof: Since $DF \parallel BE$.

In ΔCEB

$\therefore$ By BPT, $\frac{EF}{FC}=\frac{BD}{DC}=\frac{m}{1}$

$$\Rightarrow \frac{EF}{EC}=\frac{m}{m+1}.$$

In ΔADF, $EX \parallel FD$

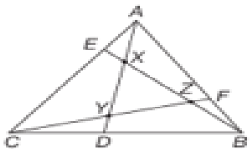
$\therefore$ By BPT

$$\frac{AX}{XD}=\frac{AE}{EF}=\frac{AE}{EC}\cdot\frac{EC}{EF}=\frac{n}{1}\cdot\frac{(m+1)}{m}$$

$$\therefore \quad \frac{AX}{XD}=\frac{n(m+1)}{m}.$$

Note: $\frac{AX}{XD}=\frac{AE}{EC}\cdot\frac{BC}{BD}$ or $\dfrac{\frac{AE}{EC}}{\frac{BD}{BC}}$.

Solution 94:



Given: In ΔABC,

$$\frac{BD}{DC}=\frac{CE}{EA}=\frac{AF}{FB}=\frac{2}{1}$$

To prove: $[XYZ]=\frac{1}{7}[ABC]$

By previous question

$$\frac{AX}{XD}=\frac{\frac{AE}{EC}}{\frac{BD}{BC}}=\frac{\frac{1}{2}}{\frac{2}{3}}=\frac{1}{2}\times\frac{3}{2}=\frac{3}{4}$$

$$\therefore\ \frac{AX}{AD}=\frac{3}{7}$$

Also, $\frac{[ABD]}{[ABC]}=\frac{BD}{BC}=\frac{2}{3}$

$\therefore\ [ABD]=\frac{2}{3}[ABC]$

Now $\frac{[ABX]}{[ABD]}=\frac{AX}{AD}=\frac{3}{7}$

$[ABX]=\frac{3}{7}[ABD]=\frac{3}{7}\times\frac{2}{3}[ABC]$

$\therefore\ [ABX]=\frac{2}{7}[ABC]$

Similarly $[BCZ]=\frac{2}{7}[ABC]$

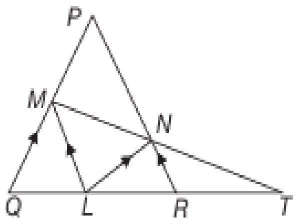
$[ACY]=\frac{2}{7}[ABC]$

Thus $[XYZ]=[ABC]-([ABX]+[BCZ]+[ACY])$

$$=\left(1-\left(\frac{2}{7}+\frac{2}{7}+\frac{2}{7}\right)\right)[ABC]$$

$$=\frac{1}{7}[ABC]$$

Solution 95:



Solution:

In ΔMLT, $NR \parallel ML$

$$\therefore \frac{TR}{TL}=\frac{TN}{TM} \quad \text{(BPT)}$$

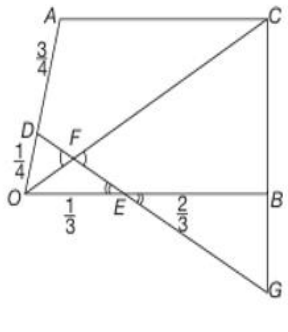
In ΔTQM,

$$\frac{TL}{TQ}=\frac{TN}{TM} \quad \text{(BPT)}$$

By equating Eqs. (1) and (2) we get, $\frac{TR}{TL}=\frac{TL}{TQ} \Rightarrow TL^2=TR\cdot TQ$

That is, TL is the geometric mean between TR and TQ.

Solution 96:



Construction: Extend the line to meet CB extended at G.

$\Delta OFD \sim \Delta CFG$ and $\Delta OED \sim \Delta BEG$

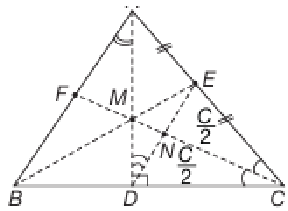
$$\therefore \frac{CF}{OF} = \frac{CG}{OD} = \frac{CB+BG}{OD} = \frac{CB}{OD} + \frac{BG}{OD} = \frac{OA}{OD} + \frac{BE}{OE} = 4+2=6$$

Thus $\frac{OF}{CF} = \frac{1}{6} \quad \Rightarrow \frac{OF}{OC} = \frac{1}{7}$

Thus the line cuts OC at F in the ratio of $OF : FC = 1 : 6$

That is, $\frac{1}{7}$ part of OC.

Solution 97:



$FN = FM + MN = 2 + 1 = 3$ and $NC = 3$

$\therefore$ $FN = NC \Rightarrow N$ is the mid-point of CF.

Also E is the mid-point of $AC \Rightarrow NE \parallel AF$ (By mid-point theorem)

$\therefore$ $DE \parallel AB$

$\therefore$ $BD = DC$ (by converse of mid-point theorem)

Thus AD is both altitude and median to BC

$\therefore$ ΔABC is isosceles $\Rightarrow AB = AC$ (1)

Also AD is the angle bisector of $\angle A$

$\therefore$ $\Delta AMF \sim \Delta DMN$ (AA)

$$\therefore \frac{AM}{MD} = \frac{FM}{MN} = \frac{2}{1}$$

This proves that M is the centroid of ΔABC (as AD is median)

Thus CF is both angle bisector and median to ΔABC

i.e., ΔABC is isosceles $\Rightarrow AC = BC$. (2)

$\therefore$ $AB = AC = BC$ (From Eqs. (1) and (2))

$\therefore$ ΔABC is equilateral.

Let the side of the equilateral triangle be 'a'.

CF, being the altitude,

$$CF = 6 \quad \Rightarrow \quad \frac{\sqrt{3}}{2}a = 6 \quad \Rightarrow \quad a = 4\sqrt{3}$$

$\therefore$ Perimeter $= 3 \times 4\sqrt{3} = 12\sqrt{3}$

$$\text{Area} = \left(\frac{\sqrt{3}}{4}\right)(4\sqrt{3})(4\sqrt{3}) = 12\sqrt{3}$$

Thus area and perimeter are equal numerically.

ALGEBRA

Remainder Theorem: If a polynomial $f(x)$ is divided by $(x-a)$ then the remainder is equal to $f(a)$.

Proof:

$f(x) = (x-a)Q(a) + R \dots (1)$

and so $f(a) = (a-a)Q(a) + R$

if $R = 0$ then $f(x) = (x-a)Q(x)$ and hence $(x-a)$ is a factor $f(x)$.

Further $f(a) = 0$ and thus a is a zero of the polynomial $f(x)$ This leads to the factor theorem.

Factor Theorem: $(x-a)$ is a factor of polynomial f(x) if and only if f(a)=0

Fundamental theorem of Algebra: Every polynomial function of degree ≥ 1 has at least one zero in the complex numbers. In other words, if we have $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ with $n \geq 1$ then there exists at least one such that, $h \in \mathbb{C}$

$$a_n h^n + a_{n-1}h^{n-1} + \dots + a_1 h + a_0 = 0$$

From this it is easy to deduce that a polynomial function of degree 'n' has exactly n zeroes.

Theorem: If $a_1, a_2, \dots \dots \dots, a_n$ and $b_1, b_2, \dots \dots \dots, b_n$ are any real numbers, such that

I. $a_1 \leq \dots \leq a_n, b_1 \leq \dots \leq b_n$, then $n(a_1 b_1 + \dots + a_n b_n) \geq (a_1 + \dots + a_n)(b_1 + \dots + b_n)$

II. $a_1 \geq \dots \geq a_n, b_1 \leq \dots \leq b_n$, then $n(a_1 b_1 + \dots + a_n b_n) \leq (a_1 + \dots + a_n)(b_1 + \dots + b_n)$

Remark: The inequality above can be put in the following symmetric form:

$$\frac{a_1 b_1 + \dots + a_n b_n}{n} \geq \left(\frac{a_1 + \dots + a_n}{n}\right)\left(\frac{b_1 + \dots + b_n}{n}\right)$$

This form suggests the following generalisation which we state without proof.

If $a_1, a_2, \dots \dots \dots, a_n, a_1, b_2, \dots \dots \dots, b_n, k_1, k_2, \dots \dots \dots, k_n,$ are real numbers such that

$$a_1 \leq \dots \leq a_n, b_1 \leq \dots \leq b_n, k_1 \leq \dots \leq k_n \text{ then } \frac{a_1 b_1 k_1 + \dots + a_n b_n k_1}{n} \geq \left(\frac{a_1 + \dots + a_n}{n}\right)\left(\frac{b_1 + \dots + b_n}{n}\right)\left(\frac{k_1 + \dots + k_n}{n}\right)$$

1. **Identities:**

    I. $a + b + c = 0, a^2 + b^2 + c^2 = -2(bc + ca + ab)$

    II. $a + b + c = 0, a^3 + b^3 + c^3 = 3abc$

    III. $a + b + c = 0, a^4 + b^4 + c^4 = 2(b^2c^2 + c^2a^2 + a^2b^2) = \frac{1}{2}(a^2 + b^2 + c^2)^2$

2. **Periodic function**: A function f is said to be periodic, with period k. if $f(x+k) = f(x)$, for all x

3. **Pigeon Hole Principle (PHP)**: If more than n objects are distributed in 'n' boxes, then at least one box has more than one object in it.

4. **Polynomials**:

    a) A function f defined by $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$, where $a_0 \neq 0$, n is a positive integer or zero and a_i $(i = 0,1, \dots, n)$ are fixed complex numbers, is called a polynomial of degree n in x. The numbers $a_0, a_1, a_2, \dots \dots \dots, a_n,$ are called the coefficients of f. If α be a complex number such that f (α)=0, then α is said to be a zero of the polynomial f.

b) If a polynomial f(x) is divided by (x-h) where h is any complex number, the remainder is equal to f(h).
c) If h is a zero of a polynomial f(x), then (x-h) is a factor of f(x) and conversely.
d) Every polynomial equation of degree $n \geq 1$ has exactly n roots.
e) If a polynomial equation wish real coefficients has a complex root $p + iq$ (p, q real numbers, $q \neq 0$) then it also has a complex root $p - iq$
f) If a polynomial equation with **rational** coefficients has an irrational root $p + \sqrt{q}$ (p, q rational, q > 0, q not the square of a rational number), then it also has an irrational root $p - \sqrt{q}$
g) If the rational number $\frac{p}{q}$ (a fraction in its lowest terms so that p, q are integers, prime to each other, $q \neq 0$) is a root of the equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_0, a_{1,}a_{2,} \ldots \ldots \ldots, a_{n,}$ are integers and $a_{n,}$ then p is a **divisor** of $a_{n,}$ and q is a divisor of a_0.
h) A number α is a **common root** of the polynomial equations f(x)=0 and g(x)=0 iff it is a root of h(x)=0 where h(x) is the G.C.D. of f(x) and g(x).
i) A number α is a repeated root of a polynomial equation f(x)=0 iff it is a common root of f(x) and f'(x)

QUESTIONS

1 Let, a, b and c are real and positive parameters. Solve the equation: $\sqrt{a+bx}+\sqrt{b+cx}+\sqrt{c+ax}=\sqrt{b-ax}+\sqrt{c-bx}+\sqrt{a-cx}$

2 Find the general term of the sequence defined by $x_0=3, x_1=4$ and $x_{n+1}={x_{n-1}}^2-nx_n$ for all $n\in\mathbb{N}$

3 Let $x_1, x_2, \dots, x_n$ be a sequence of integers such that

I. $-1\le x_i\le 2$, for $i=1,2,\dots,n$;
II. $x_1+x_2+\ \dots\ +\ x_n=19$;
III. ${x_1}^2+{x_2}^2+\cdots+{x_n}^2=99$.

Determine the maximum and minimum value of ${x_1}^3+{x_2}^3+\cdots+{x_n}^3$

4 The function f, defined by $f(x)=\frac{ax+b}{cx+d}$, where a, b, c and d are non zero real numbers, has the properties $f(19)=19, f(97)=97$ and $f(f(x))=x$, for all values of x, except $-\frac{d}{c}$. Find the range of f.

5 Prove that $\frac{(a-b)^2}{8a}\le\frac{a+b}{2}-\sqrt{ab}\le\frac{(a-b)^2}{8b}$ for all $a\ge b>0$

6 Several (at least two) non zero numbers are written on a board. One may erase any two number write the numbers $a+\frac{b}{2}$ and $b-\frac{a}{2}$ instead. Prove that the set of numbers on the board, after any number of the preceding operations, can not coincide with the initial set.

7 The polynomial $1-x+x^2-x^3+\cdots+x^{16}-x^{17}$ may be written in the form $a_0+a_1y+a_2y^2+\cdots+a_{16}y^{16}+a_{17}y^{17}$, where $y=x+1$ and a_is are constants. Find a_2.

8 Let, a, b and c are distict non zero real numbers such that $a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}$. Prove that $|abc|=1$

9 Find polynomials f(x), g(x) and h(x), if they exist, such that for all x,

$$|f(x)|-|g(x)|+h(x)=\begin{cases}-1, \text{ if } x<-1\\ 3x+2, \text{ if } -1\le x\le 0\\ -2x+2, \text{ if } x>0\end{cases}$$

10 Find all real x for which $\frac{8^x+27^x}{12^x+18^x}=\frac{7}{6}$

11 Evaluate: $\binom{2000}{2}+\binom{2000}{5}+\binom{2000}{8}+\cdots+\binom{2000}{2000}$

12 Let, x, y and z be positive real numbers such that $x^4+y^4+z^4=1$. Determine with proof the minimum value of :

$$\frac{x^3}{1-x^8}+\frac{y^3}{1-y^8}+\frac{x^3}{1-z^8}$$

13 Find the real solution of the equation: $2^x+3^x+6^x=x^2$

14 Find number of distinct real solutions of $x^2+\left(\frac{x}{x+1}\right)^2=3$.

15 Find number of distinct real solutions of $\sqrt{2x-14}-\sqrt{x-6}=1$

16 Find sum of all roots (real & complex) of $(x-1)^{2025}=(x+1)^{2025}$

17 Let matrix $A=\begin{bmatrix} x & -y & z \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$, where $x, y, z \in \mathbb{N}$. If $|adj(adj(adjA))| = 2^{16}.3^{8}$, then find the number of such (x, y, z).

18 Let, $\{a_n\}$ be a sequence such that $a_1 = 2$ and $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$ for all $n \in \mathbb{N}$. Find the explicit formula for a_n.

19 Let, x, y and z be positive real numbers. Prove that $\frac{x}{x+\sqrt{(x+y)(x+z)}} + \frac{y}{y+\sqrt{(y+z)(y+x)}} + \frac{z}{z+\sqrt{(z+x)(z+y)}} \leq 1$

20 Let, $f: \mathbb{N} \longrightarrow \mathbb{N}$ be a function such that $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all n. Evaluate $f(2001)$

21 Find the least positive integer m such that $\binom{2n}{n}^{\frac{1}{n}} < m$ for all positive integers n.

22 Let a, b, c, d, and e be positive integers such that $abcde = a + b + c + d + e$. Find the maximum possible value of max {a, b, c, d, e}.

23 Evaluate: $\frac{3}{1!+2!+3!} + \frac{4}{2!+3!+4!} + \ldots\ldots + \frac{2001}{1999!+2000!+2001!}$

24 Let, $x = \sqrt{a^2+a+1} - \sqrt{a^2-a+1}$, $a \in R$. Find all possible values of x.

25 Find all real numbers x for which $10^x + 11^x + 12^x = 13^x + 14^x$.

26 Let $f: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be a function such that $f(1,1) = 2, f(m+1, n) = F(m, n) + m$ and $f(m, n+1) = f(m, n) - n$ for all $m, n \in \mathbb{N}$. Find all pairs (p, q) such that $f(p, q) = 2001$.

27 Let f is a function defined on $[0,1]$ such that $f(0) = f(1) = 1$ and $|f(a) - f(b)| < |a - b|$ for all $a \neq b$ in the interval $[0,1]$. Prove that $|f(a) - f(b)| < \frac{1}{2}$.

28 Find all pair of integers (x, y) such that $x^3 + y^3 = (x+y)^2$

29 Let $f(x) = \frac{2}{4^x+2}$ for real numbers x. Evaluate $\left(\frac{1}{2001}\right) + \left(\frac{2}{2001}\right) + \cdots + \left(\frac{2000}{2001}\right)$

30 If $a_0, a_1, \ldots, a_{50}$ are the coefficients of the polynomial $(1+x+x^2)^{25}$. Prove that the sum $a_0 + a_2 + \cdots + a_{50}$ is even.

31 If $a_0, a_1, \ldots, a_{50}$ are the coefficients of the polynomial $(1+x+x^2)^{25}$. Prove that the sum $a_0 + a_2 + \cdots + a_{50}$ is even Prove that polynomial $x^{9999} + x^{8888} + x^{7777} + x^{1111} + 1$ is divisible by $x^9 + x^8 + x^7 + \cdots + x + 1$.

32 Prove that the polynomial $f(x) = x^4 + 26x^3 + 52x^2 + 78x + 1989$ cannot be expressed as a product of two polynomials p(x) and q(x) with integral coefficient of degree less than 4.

33 Show that the set of polynomials $P = \{p_k : p_k(x) = x^{5k+4} + x^3 + x^2 + x + 1, k \in \mathbb{N}\}$ has a common non−trivial polynomial divisor.

34 How many zeroes does 6250! end with?

35 Find the last two digits of $(56789)^{41}$.

36 Show that $1^{1997} + 2^{1997} + \cdots + 1996^{1997}$ is divisible by 1997.

37 Find all the ordered pairs of integers (x, z) such that $x^3 = z^3 + 721$.

38 Suppose f is a function on the positive integers, which takes integer values $(\text{i.e. } f: \mathbb{N} \longrightarrow \mathbb{Z})$ with the following properties:

I. $f(2) = 2$
II. $f(m.n) = f(m).f(n)$
III. $f(m) > f(n)$ if $m > n$.

39 Solve for real $x, \frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3}$, where $[x]$ greatest integer less than or equal to x and $\{x\} = x - [x]$.

40 Find all the integral solutions of $y^2 = 1 + x + x^2$.

41 Find sum of all roots of the equation: $|x|^2 - 5|x| + 6 = 0$.

42 If $x = \sqrt{2} + \sqrt{5} + \sqrt{10}$ is a root of $x^4 + ax^3 + bx^2 + cx + d = 0$, where a, b, c, d are integers then find the two-digit prime number that divides $|a + b + c + d|$

43 Find the least value of a^b such that the possible integers $a, b > 1$ satisfy $a^b b^a = a^b + b^a + 5039$.

44 Consider x is a natural number and $R(a, b) = k$ (k is remainder when a divides b). If $R(7, x^2 - 3x + 2) = 5$ and $R(7, x^3 + x - 3) = 1$ then find $R(7, x^5 - 3x^4 + 3x^3 - 6x^2 + 11x + 2)$.

45 If λ is an integer and α, β are the roots of $4x^2 - 16x + \frac{\lambda}{4} = 0$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then how many values can λ take?

46 If $a + b + c = 9$ (where a, b, c are real numbers), then find the minimum value of $a^2 + b^2 + c^2$.

47 If $a^3 + 3a^2 + 9a = 1$, then find the value of $a^3 + \frac{3}{a}$

48 If $x = 3 + 2\sqrt{2}$, then find value of $\frac{x^6 + x^5 - 5x^4 + x^3 + x^2 + 1}{4x^3}$

49 When 752, 1604 and 3095 are divided by two-digit number x, then the remainder in each case is y, what is the value of $(x - y)$.

50 Find the remainder when 24 divides $1!^{1!}! - 2!^{2!}! + 3!^{3!}! - 4!^{4!}! + \cdots + 99!^{99!}!$

51 Find $\phi(340)$ where $\phi(n) = \{a: 1 \leq a \leq n, \gcd(a,n) = 1\}$

52 Find $2^{13^{5^3}} = k(\text{mod } 13)$ or Find remainder when $2^{13^{5^3}}$ divides by 13.

53 If x and y single digit prime numbers and $\phi((xy)^3) = 400$ then find $\phi(xy)$, where $\phi(n) = \{a: 1 \leq a \leq n, \gcd(a,n) = 1\}$.

54 How many positive integer solutions are there for $2x + 3y = 563$.

55 Find number of solutions of $\log_y^x + 5\log_x^y = 6$ where x and y are distinct integers and $2 \leq x, y \leq 2025$.

56 Let x and y be positive integers satisfies $5x^2 + 13y + 50 = x^2 y$. Find value of $\sqrt{x+y}$.

57 If the equations $2x^2 - 4x + 1 = 0$ and $x^2 + ax + b = 0$ have a common root, then find the value of $(1 + ab)$.

58 Let $P(x) = x^6 + x^5 - x^4 + x^2 + x$ and $Q(x) = x^3 + x^2 - x + 1$. If α, β and γ are the roots of $Q(x) = 0$, then find the value of $P(\alpha) + P(\beta) + P(\gamma)$.

59 If $z + \frac{1}{z} = 2\cos 6°$, then find the value of $z^{600} + \frac{1}{z^{600}} + 1$.

60 Find the coefficient of x^{464} in the expansion of $(x+1)(x-2)^2(x+3)^3(x-4)^4 \ldots\ldots (x-30)^{30}$.

61 Find all pairs of integers (a, b) such that the polynomial $ax^{17} + bx^{16} + 1$ is divisible by $x^2 - x - 1$.

62 Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let, $S = p(1) + p(2) + \cdots + p(999)$. What is the largest prime factor of S?

63 Let x_n be a sequence of nonzero real numbers such that $x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}$ for $n = 3, 4, \ldots$.
Establish necessary and sufficient conditions on x1 and x2 for x_n to be an integer for infinitely m

64 Solve the equation $x^3 - 3x = \sqrt{x+2}$.

65 For any sequence of real numbers $A = \{a_1, a_2, a_3 \ldots\}$ define ΔA to be the sequence $\{a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots\}$.
Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_{92} = 0$. Find a_1.

66 Find all real numbers x satisfying the equation $2^x + 3^x - 4^x + 6^x - 9^x = 1$.

67 Determine the number of ordered pairs of integers (m, n) for which $mn \geq 0$ and $m^3 + n^3 + 99mn = 333$.

68 Let a, b, and c be positive real numbers such that $a + b + c \leq 4$ and $ab + bc + ca \geq 4$.
Prove that at least two of the inequalities $|a - b| \leq 2$, $|b - c| \leq 2$ and $|c - a| \leq 2$ are true.

69 Evaluate: $\sum_{k=0}^{n} \frac{1}{(n-k)!\,(n+k)!}$

70 Let, $a_1, a_2, \ldots, a_n$ be real numbers, not all zero. Prove that the equation $\sqrt{1 + a_1 x} + \sqrt{1 + a_2 x} + \cdots \ldots \sqrt{1 + a_n x} = n$ has at most one non-zero real root.

71 Let $\{a_n\}$ be the sequence of real numbers defined by $a_1 = t$ and $a_{n+1} = 4a_n(1 - a_n)$ for $n > 1an + 1 = 4$.
For how many distinct values of t do we have $a_{1998} = 0$?

72 Given eight non − zero real numbers $a_1, a_2, \dots, a_8$ prove that at least one of the following six numbers: $a_1a_3 + a_2a_4$, $a_1a_5 + a_2a_6, a_1a_7 + a_2a_8, a_3a_5 + a_4a_6, a_3a_7 + a_4a_8, a_5a_7 + a_6a_8$ is non − negative.

73 Let a, b and c be positive real numbers such that $abc = 1$. Prove that $\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$.

74 Find all functions $f : R \to R$ such that the equality $f(f(x) + y) = f(x^2 - y) + 4f(x)y$ holds for all pairs of real numbers (x, y).

75 Solve the system of equations: $x + \frac{3x - y}{x^2 + y^2} = 3$ and $y - \frac{x + 3y}{x^2 + y^2} = 0$.

76 Find all positive integers k for which the following statement is true: if F(x) is a polynomial with the condition $0 \leq F(c) \leq k$ for $c = 0, 1, \dots, k + 1$, then $F(0) = F(1) = \cdots F(k + 1)$.

77 Mr. Fat and Mr. Taf play a game with a polynomial of degree at least 4: $x^{2n} + \_\_x^{2n-1} + \_\_x^{2n-2} + \cdots + \_\_x + 1$.
They fill in real numbers to empty spaces in turn. If the resulting polynomial has no real root, Mr otherwise, Mr. Taf wins. If Mr. Fat goes first, who has a winning strategy?

78 Let, $0 < a_1 \leq a_2 \dots\dots a_n$, $0 < b_1 \leq b_2 \dots\dots \leq b_n$ be real numbers such that $\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} b_i$
Suppose that there exists $1 \leq k \leq n$ such that $b_i \leq a_i$ for $1 \leq i \leq k$ and $b_i \geq a_i$ for $i > k$.
Prove that $a_1a_2 \dots\dots a_n \geq b_1b_2 \dots\dots b_n$.

79 Prove that: $16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17$.

80 Does there exist functions $f : R \to R$ and $g : R \to R$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$ for all $x \in R$?

81 The product of two roots of the equation $4x^4 - 24x^3 + 31x^2 + 6x - 8 = 0$ is 1, find all the roots.

82 If α, β and γ are the roots of $x^3 + px + q = 0$, then prove that
$\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \frac{\alpha^3 + \beta^3 + \gamma^3}{3} \times \frac{\alpha^2 + \beta^2 + \gamma^2}{2}$ and $\frac{\alpha^7 + \beta^7 + \gamma^7}{7} = \frac{\alpha^5 + \beta^5 + \gamma^5}{5} \times \frac{\alpha^2 + \beta^2 + \gamma^2}{2}$

83 Find the common roots of $x^4 + 5x^3 - 22x^2 - 50x + 132 = 0$ and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$. Hence solve the equations.

84 Solve the system: $(x + y)(x + y + z) = 18, (y + z)(z + x) = 30, (z + x)(x + y + z) = 2L$ in terms of L.

85 If x_1 and x_2 are non − zero roots of the equation $ax^2 + bx + c = 0$ and $-ax^2 + bx + c = 0$ respectively, prove that
$\frac{a}{2}x^2 + bx + c = 0$ has a root between x_1 and x_2.

86 Find all real values of m such that both roots of the equation $x^2 - 2mx + (m^2 - 1) = 0$ are greater than −2 but less than +4.

87 The roots of the equation $x^5 - 40x^4 + px^3 + qx^2 + rx + s = 0$ are in G. P. The sum of their reciprocal is 10.
Compute the numerical value of |s|.

88 Let, $P(x) = x^4 + ax^3 + bx^2 + cx + d$ where a, b, c, d are constants. If $P(1) = 10, P(2) = 20, P(3) = 30$, then compute $\frac{P(12) + P(-8)}{10}$.

89 Let, $P(x) = 0$ be the polynomial equation of least possible degree with rational coefficients, having $\sqrt[3]{7} + \sqrt[3]{49}$ as a root, Compute the product of all the roots of $P(x) = 0$.

90 The equations $x^3 + 5x^2 + px + q = 0$ and $x^3 + 7x^2 + px + r = 0$ have two roots in common. If the third root of each equation is represented by x_1 and x_2 respectively, compute the ordered pair (x_1, x_2)

91 If a, b, c, x, y, z are all real and $a^2 + b^2 + c^2 = 25, x^2 + y^2 + z^2 = 36$ and $ax + by + cz = 30$, find the value of $\frac{a + b + c}{x + y + z}$.

92 If the integer A its reduced by the sum of its digits, the result is B. If B is increased by the sum of i Compute the largest 3 − digt number A with this property.

93 The roots of $x^4 - kx^3 + kx^2 + lx + m = 0$ are a, b, c, d. If k, l and m are real numbers, compute the minimum value of the sum $a^2 + b^2 + c^2 + d^2$.

94 If $2\left[\frac{x}{6}\right]^2 + 3\left[\frac{x}{6}\right] = 20$, then it must be true that $a \leq x < b$ for some integers a and b. Compute (a, b) where (b − a) as small as possible. Note: [x] represents the greatest integer function.

95 The roots of $x^3 + Px^2 + Qx - 19 = 0$ are each one more than the roots of $x^3 - Ax^2 + Bx - C = 0$. If A, B, C, P and Q are constants, compute A + B + C.

96 Find all ordered pairs of positive integers (x, z) that $x^2 = z^2 + 120$.

SOLUTIONS

1 It is easy to see that x = 0 is a solution. Since the right-hand side is a decreasing function of x and the left-hand side is an increasing function of x, there is at most one solution. Thus x = 0 is the only solution to the equation.

2 We shall prove by induction that $x_n = n + 3$. The claim is evident for n=0,1.
For $k \geq 1$, if $x_{k-1} = k + 2$ and $x_k = k + 3$, then $x_{k+1} = {x_{k-1}}^2 - kx_k = (k + 2)^2 - k(k + 3)$
$= k + 4$

3 Let a, b, and c denote the number of −ls, ls, and 2s in the sequence, respectively. We need not consider the zeros. Then a, b, c are non-negative integers satisfying −a+b+2c= 19 and a+b+4c= 99.
It follows that a = 40 − c and b = 59 − 3c, where $0 \leq c \leq 19$ (Since $b \geq 0$)
So, ${x_1}^3 + {x_2}^3 + \cdots + {x_n}^3 = -a + b + 8c = 19 + 6c$
When c = 0 (a = 40, b = 59), the lower bound (19) is achieved.
When c = 19 (a = 21, b = 2), the upper bound (133) is achieved.

4 For all x, $f(f(x)) = x$, i.e. $\frac{a\left(\frac{ax+b}{cx+d}\right)+b}{c\left(\frac{ax+b}{cx+d}\right)+d} = x \Longrightarrow \frac{(a^2+bc)x+b(a+d)}{c(a+d)x+bc+d^2} = x \Longrightarrow c(a + d)x^2 + (d^2 - a^2)x - b(a + d) = 0$
which implies that c (a + d) = 0. Since $c \neq 0$, we must have a = −d. The conditions f (19) = 19 and f (97) = 97 lead to the equations $19^2c = 2.19a + b$ and $97^2c = 2.97a + b$ Hence $(97^2 - 19^2)c = 2(97 - 19)a$
It follows that a = 58c, which in turn leads to b = −1843c.
$\therefore f(x) = \frac{58x-1843}{x-58} = 58 + \frac{1521}{x-58}$ which never has the value 58. Thus, the range of f is $\mathbb{R} - \{58\}$

5 Note that $$\frac{a+b}{2} - \sqrt{ab} = \frac{\left(\frac{a+b}{2}\right)^2 - ab}{\frac{a+b}{2} + \sqrt{ab}} = \frac{(a-b)^2}{2(a+b) + 4\sqrt{ab}}$$
Thus, the desired inequality is equivalent to $4a \geq a + b + 2\sqrt{ab} \geq 4b$,
which is evident as $a \geq b > 0$(which implies $a \geq \sqrt{ab} \geq b$)

6 Let S be the sum of the squares of the numbers on the board. Note that S increases in the first operation and does not decrease in any successive operation, as
$$\left(a + \frac{b}{2}\right)^2 + \left(b - \frac{a}{2}\right)^2 = \frac{5}{4}(a^2 + b^2) \geq a^2 + b^2 \text{ with equality holds if } a = b = 0$$

7 Let, f(x) denotes the given expression, Then $xf(x) = x - x^2 + x^3 - \cdots - x^{18}$ and $(1 + x)f(x)$
$= 1 - x^{18}$
Hence, $f(x) = f(y - 1) = \frac{1 - (y-1)^{18}}{1 + (y-1)} = \frac{1 - (y-1)^{18}}{y}$
$\therefore a_2$ is equal to the coefficient of y^3 in the expansion of $1 - (y - 1)^{18}$ i.e. $a_2 = \binom{18}{3} = 816$

8 From the given conditions, it follows that $a - b = \frac{b-c}{bc}$, $b - c = \frac{c-a}{ca}$ and $c - a = \frac{a-b}{ab}$
Multiplying the above equations gives $(abc)^2 = 1$, from which the desired result follows.

9 Since x = −1 and x = 0 are the two critical values of the absolute functions, one can suppose that

$$F(x) = a|x+1| + b|x| + cx + d = \begin{cases} (c-a-b)x + d - a, \text{if } x < -1 \\ (a+c-b)x + a + d, \text{if } -1 \le x \le 0 \\ (a+b+c)x + a + d, \text{if } x > 0 \end{cases}$$

which implies that $a = \frac{3}{2}, b = -\frac{5}{2}, c = -1$ and $d = \frac{1}{2}$. Hence $f(x) = \frac{3x+3}{2}, g(x) = \frac{5x}{2}, h(x) = -x + \frac{1}{2}$

10 By setting $2^x = a$ and $3^x = b$, then the equation becomes $\frac{a^3+b^3}{a^2b+b^2a} = \frac{7}{6} \Rightarrow \frac{a^2-ab+b^2}{ab} = \frac{7}{6}$ $\Rightarrow 6a^2 - 13ab + 6b^2 = 0$

$\Rightarrow (2a-3b)(3a-2b) = 0 \therefore 2^{x+1} = 3^{x+1}$ or $2^{x-1} = 3^{x-1}$ which implies that $x = -1$ and $x = 1$. It is easy to check that both $x = -1$ and $x = 1$ satisfy the given equation.

11 Let, $f(x) = (1+x)^{2000} = \sum_{k=0}^{2000} \binom{2000}{k} x^k$

Let, $\omega = \frac{-1+\sqrt{3}i}{2}$ then $w^3 = 1$ and $\omega^2 + \omega + 1 = 0$

Hence, $3\left\{\binom{2000}{2} + \binom{2000}{5} + \binom{2000}{8} + \cdots + \binom{2000}{2000}\right\} = f(1) + \omega f(\omega) + \omega^2 f(\omega^2)$

$= 2^{2000} + \omega(1+\omega)^{2000} + \omega^2(1+\omega^2)^{2000} = 2^{2000} + \omega(-\omega^2)^{2000} + \omega^2(-\omega)^{2000}$

$= 2^{2000} + \omega^2 + \omega = 2^{2000} - 1$

$\therefore \binom{2000}{2} + \binom{2000}{5} + \binom{2000}{8} + \cdots + \binom{2000}{2000} = \frac{2^{2000}-1}{3}$

12 For $0 < u < 1$, let $f(u) = u(1-u^8)$. Let, A be a positive real number.

By AM − GM inequality, $A\left(f(u)\right)^8 = Au^8(1-u^8)^8 \le \left[\frac{Au^8 + 8(1-u^8)}{9}\right]^9$

Setting A = 8 in this inequality, we get $8\left(f(u)\right)^8 \le \left(\frac{8}{9}\right)^9 \Rightarrow f(u) \le \frac{8}{\sqrt[4]{3^9}}$

It follows that: $\frac{x^3}{1-x^8} + \frac{y^3}{1-y^8} + \frac{x^3}{1-z^8} = \frac{x^4}{x(1-x^8)} + \frac{y^4}{y(1-y^8)} + \frac{z^4}{z(1-z^8)}$

$\ge \frac{(x^4+y^4+z^4)\sqrt[4]{3^9}}{8} = \frac{9\sqrt[4]{3}}{8}$

with equality hols iff $x = y = z = \frac{1}{\sqrt[4]{3}}$

13 For $x < 0$, the function $f(x) = 2^x + 3^x + 6^x - x^2$ is an increasing function. So the equation $f(x) = 0$ has unique solution at $x = -1$

for $x \ge 0, 3^x > x^2$. So $f(x) > 0$. Hence there is no non negative solution of the equation $f(x) = 0$. Hence $x = -1$ is the only solution.

14 $x^2 + \left(\frac{x}{x+1}\right)^2 = 3$

$\Rightarrow x^2 + \left(1 - \frac{1}{x+1}\right)^2 = 3$

$\Rightarrow x^2 + 1 - \frac{2}{x+1} + \left(\frac{1}{x+1}\right)^2 = 3$

$\Rightarrow (x+1)^2 - 2x - \frac{2}{x+1} + \left(\frac{1}{x+1}\right)^2 = 3$

$\Rightarrow (x+1)^2 + \left(\frac{1}{x+1}\right)^2 - 2\left(x + \frac{1}{x+1}\right) = 3$

$\Rightarrow (x+1)^2 + \left(\frac{1}{x+1}\right)^2 - 2\left(x + 1 + \frac{1}{x+1}\right) + 2 = 3$

$\Rightarrow \left(x + 1 + \frac{1}{x+1}\right)^2 - 2\left(x + 1 + \frac{1}{x+1}\right) - 3 = 0$

Let, $a = +1 + \frac{1}{x+1}$

$\therefore\ a^2 - 2a - 3 = 0$

$\Rightarrow\ (a-3)(a+1) = 0$

$\Rightarrow a = 3\,, -1$

Case 1:

$x + 1 + \frac{1}{x+1} = 3$

$\Rightarrow (x+1)^2 + 1 = 3(x+1)$

$\Rightarrow (x+1)^2 - 3(x+1) + 1 = 0$

$\Rightarrow x + 1 = \frac{3 \pm \sqrt{9-4}}{2}$

$\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$

Case 2:

$x + 1 + \frac{1}{x+1} = -1$

$\Rightarrow (x+1)^2 + 1 = -1(x+1)$

$\Rightarrow (x+1)^2 + 1(x+1) + 1 = 0$

$\Rightarrow x + 1 = \frac{-1 \pm \sqrt{1-4}}{2}$

$\Rightarrow x = \frac{-3 \pm \sqrt{-3}}{2}$ [it is not real number]

Therefore, total number of solutions are 2.

15 $\sqrt{2x-14}-\sqrt{x-6}=1$

$\Rightarrow \left(\sqrt{2x-14}\right)^2=\left(1+\sqrt{x-6}\right)^2$

$\Rightarrow 2x-14=1+x-6+2\sqrt{x-6}$

$\Rightarrow x-9=2\sqrt{x-6}$

$\Rightarrow (x-9)^2=4(x-6)$

$\Rightarrow x^2-18x+81=4x-24$

$\Rightarrow x^2-22x+105=0$

$\Rightarrow (x-15)(x-7)=0$

$\Rightarrow x=15 \text{ or } x=7$

Checking:

For $x=15$,

$\sqrt{2\times 15-14}-\sqrt{15-6}=\sqrt{16}-\sqrt{9}=4-3=1$ (satisfy)

For $x=7$,

$\sqrt{2\times 7-14}-\sqrt{7-6}=\sqrt{0}-\sqrt{1}=0-1=-1(\neq 1,$ not satisfy)

Therefore, total number of solutions is 1.

16 $(x-1)^{2025}=(x+1)^{2025}$

$\Rightarrow x^{2025}-2025_{C_1}x^{2024}+2025_{C_2}x^{2023}-2025_{C_3}x^{2022}+\cdots-1$

$=x^{2025}+2025_{C_1}x^{2024}+2025_{C_2}x^{2023}+2025_{C_3}x^{2022}+\cdots+1$

$\Rightarrow 2(2025_{C_1}x^{2024}+2025_{C_3}x^{2022}+2025_{C_5}x^{2020}\ldots+1)=0$

$\Rightarrow 2025_{C_1}x^{2024}+2025_{C_3}x^{2022}+2025_{C_5}x^{2020}\ldots+1=0$

$\therefore$ Sum of all roots = $-\frac{\text{coefficient of } x^{2023}}{\text{coefficient of } x^{2024}}=-\frac{0}{\text{coefficient of } x^{2024}}=0$

17 $|A|=\begin{vmatrix} x & -y & z \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{vmatrix}=x+y+z$

$|adjA|=|A|^2,\ |adj(adjA)|=|A|^2.|A|^2=|A|^4,\ |adj(adj(adjA))|=|A|^4.|A|^4=|A|^8$

$\left|adj\left(adj(adjA)\right)\right|=2^{16}.3^8$

$\Rightarrow |A|^8=4^8.3^8$

$\Rightarrow |A|^8=12^8$

$\Rightarrow |A|=\pm 12$

$\Rightarrow x+y+z=\pm 12$

$\Rightarrow x+y+z=12$ (since $x,y,z\in N$)

Total number of possibilities are = $10+9+8+7+6+5+4+3+2+1=\frac{10\times 11}{2}=55$

18 Solving $x=\frac{x}{2}+\frac{1}{x}$ leads to $x=\pm\sqrt{2}$. Note that

$$\frac{a_{n+1}+\sqrt{2}}{a_{n+1}-\sqrt{2}}=\frac{{a_n}^2+2\sqrt{2}a_n+2}{{a_n}^2-2\sqrt{2}a_n+2}=\left(\frac{a_n+2}{a_n-2}\right)^2$$

$$\therefore \frac{a_n+2}{a_n-2}=\left(\frac{a_1+\sqrt{2}}{a_1-\sqrt{2}}\right)^{2^{n-1}}=\left(\sqrt{2}+1\right)^{2^n} \text{ and } a_n=\frac{\sqrt{2}\left[\left(\sqrt{2}+1\right)^{2^n}+1\right]}{\left(\sqrt{2}+1\right)^{2^n}+1}$$

19 Note that $\sqrt{(x+y)(x+z)} \geq \sqrt{xy}+\sqrt{xz} \Longrightarrow x^2+yz$
$\geq 2x\sqrt{yz}$, which is evident by AM − GM inequality. Thus

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} \leq \frac{x}{x+\sqrt{xy}+\sqrt{xz}} = \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$$

Likewise $\frac{y}{y+\sqrt{(y+z)(y+x)}} \leq \frac{\sqrt{y}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$ and $\frac{z}{z+\sqrt{(z+x)(z+y)}} \leq \frac{\sqrt{z}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$

Adding the last thre inequalities leading to the desired result.

20 For integer n, let $n_{(3)} = a_1a_2 \ldots . a_l$ denote the base 3 representation of n.

Using induction, we can prove that $f(n)_{(3)} = \begin{cases} 2a_2 \ldots a_l, \text{if } a_1 = 1 \\ 1a_2 \ldots a_l 0, \text{if } a_1 = 2 \end{cases}$

Since $2001_{(3)} = 2202010$, $f(2001)_{(3)} = 12020100$ or $f(2001) = 1.3^2 + 2.3^4 + 2.3^6 + 1.3^7$
$= 3816$

21 $\binom{2n}{n} < \binom{2n}{0} + \binom{2n}{1} + \binom{2n}{2} + \ldots .. + \binom{2n}{2n} = ((1+1)^{2n} = 4^n$ and for n = 5, $\binom{10}{5} = 252 > 35$

Thus m = 4

22 Suppose that a < b < c < d < e. We need to find the maximum value of e.

Since , e < a + b + c + d + e < $abcde$ < 5e, i.e. 1 < a b c d < 5. Hence (a, b, c, d) = (1, 1, 1, 2), (1, 1, 1, 3), (1, 1, 1, 4), (1, 1, 2, 2),

or (1, 1, 1, 5), which leads to max{e} = 5.

23

Note that $\frac{k+1}{k!+(k+1)!+(k+2)!} = \frac{k+2}{k!(1+(k+1)+(k+2)(k+1))} = \frac{1}{k!(k+2)} = \frac{k+1}{(k+2)!}$

$$= \frac{1}{(k+1)\,!} - \frac{1}{(k+2)\,!}$$

By telescoping sum, the desired value is equal to $= \frac{1}{2} - \frac{1}{2001!}$

24

$\sqrt{a^2+|a|+1} > |a|$ and $x = \frac{2a}{\sqrt{a^2+a+1}-\sqrt{a^2-a+1}}$

we have $|x| < \left|\frac{2a}{2}\right| = 2$

squaring both sides of

$x + \sqrt{a^2-a+1} = \sqrt{a^2+a+1}$

i. e. $2x\sqrt{a^2-a+1} = 2a - x^2$

squaring both sides of above equation we get

$4(x^2-1)a^2 = x^2(x^2-4)$

$a^2 = \frac{x^2(x^2-4)}{4(x^2-1)} > 0$

we must have

$(x^2-4)(x^2-1) > 0, x < -2$ or $x > 2$ or $-1 < x < 1$ since $|x| < 2$ so therefore $-1 < x < 1$

25 It is easy to check that x=2 is a solution. We claim that it is the only one

$$\left(\frac{10}{13}\right)^x + \left(\frac{11}{13}\right)^x + \left(\frac{12}{13}\right)^x = 1 + \left(\frac{14}{13}\right)^x$$

The left hand side is a decreasing function of x and the right hand side is a increasing function of x. Therefore their graphs can have at most one point of intersection.

26 $f(p,q) = f(p-1,q) + p - 1$

$$= f(p-2,q) + p - 2 + p - 1$$
$$= \cdots$$
$$= f(1,q) + \frac{p(p-1)}{2}$$
$$= f(1,q-1) - (q-1) + \frac{p(p-1)}{2}$$
$$= f(1,q-2) - (q-2) - (q-1) + \frac{p(p-1)}{2}$$
$$= \cdots$$
$$= f(1,1) - \frac{q(q-1)}{2} + \frac{p(p-1)}{2}$$
$$= 2001$$
$$\Rightarrow \frac{p(p-1)}{2} - \frac{q(q-1)}{2} = 1999$$

$$\Rightarrow (p-q)(p+q-1) = 2(1999)$$

Note that 1999 is a prime number and that $p - q < p + q - 1$ for $p, q \in \mathbb{N}$ so we have two case
$p - q = 1, p + q - 1 = 3999 \Rightarrow p = 2000, q = 1999$
or $p - q = 2, p + q - 1 = 1999 \Rightarrow p = 1001, q = 999.$
therefore $f(p,q) = (2000,1999), (1001,999)$

27 Two cases

1. $|a-b| \le \frac{1}{2}$, then $|f(a) - f(b)| < |a-b| \le \frac{1}{2}$
2. $|a-b| > \frac{1}{2}$ by symmetry we assume that $a > b$ then

$|f(a) - f(b)| = |f(a) - f(1) + f(0) - f(b)| \le |f(a) - f(1)| + |f(0) - f(b)|$
$< |a-1| + |0-b|$
$= 1 - a + b - 0 \quad = 1 - (a-b) < \frac{1}{2}$ as desired.

28 since $x^3 + y^3 = (x+y)(x^2 - xy + a^2)$ for all pair $(-n,n), n \in \mathbb{Z}$ are solutions
suppose that $x + y \ne 0$ then the equation become

$$x^2 - xy + y^2 = x + y$$
$$x^2 - x(y+1) + y^2 - y = 0$$

here

$$D = (y+1)^2 - 4(y^2 - y)$$
$$D \ge 0 \Rightarrow -3y^2 + 6y + 1 \ge 0$$
$$\frac{3 - 2\sqrt{3}}{3} \le y \le \frac{3 + 2\sqrt{3}}{3}$$

so possible values for y are 0,1,2
Therefore, the integer solutions of the equation are (x, y)=(1,0),(0,1), (1,2),(2,1),(2,2), and (n,-n), for all n ∈Z.

29 Note that f has a half-turn symmetry about point (1/2, 1/2). Indeed
$f(1-x) = \frac{4^x}{4^x+2}$
so $f(x) + f(1-x) = 1$
thus the desired sum is 1000

30 Taking $x = 1$ in the given equation :
$(1 + x + x^2)^{25} = a_0 + a_1x + a_2x^2 + \cdots + a_{50}x^{50}$
We get $3^{25} = a_0 + a_1 + a_2 + \cdots + a_{50}$ Similarly, $x = -1$ gives

$1 = a_0 + a_1 + a_2 + \cdots + a_{50}$ Adding these, we have

$1 + 3^{25} = 2(a_0 + a_2 + a_4 + \cdots + a_{50})$

But $1 + 3^{25} = 3^{25} - 1 + 2 = 2(3^{24} + 3^{23} + 3^{22} + \cdots + 1 + 1)$

There are even number of odd terms in the braces, and hence the sum is even. This implies that $a_0 + a_2 + a_4 + \cdots + a_{50}$ is even.

31 let $P = x^{9999} + x^{8888} + x^{7777+}x^{1111} + 1$

And Q= $x^9 + x^8 + x^7 + \ldots + x + 1$

$P - Q = x^9(x^{9990} - 1) + x^8(x^{8880} - 1) + x^7(x^{7770} - 1) + \cdots + x(x^{1110} - 1)$

$= x^9[(x^{10})^{999} - 1] + x^8[(x^{10})^{888} - 1] + x^7[(x^{10})^{777} - 1] + \cdots + x[(x^{10})^{111} - 1]$ ……(1)

But $(x^{10})^n - 1$ is divisible by $x^{10} - 1$ $for\ all\ n \geq 1$.

$\therefore R.H.S.$ of equation (1) is divisible by $x^{10} - 1$.

$\therefore P - Q$ is divisible by $x^{10} - 1$ and hence divisible by $x^9 + x^8 + \ldots + 1$.

32 If possible, let us express

$x^4 + 26x^3 + 52x^2 + 78x + 1989 \equiv (x^2 + ax + b)(x^2 + cx + d)$

By comparing coefficients of both sides, we get

a + c = 26……….(1)

ac + b + d= 52………..(2)

bc + ad= 78………(3)

bd = 1989 = $13 \times 3^2 \times 17 \ldots..(4)$

Now, we see that 13 is a divisor of 26, 52, 78 and 1989 and 13 is a prime number.

Thus, 13/ b . d

⇒ 13 divides one of b or d but not both.

If 13/b say and 13’/d, them from eq. (3), 13/a.

Now, 13/ac and 13/b, and 13/52.

∴ 13/d, from (2) is a contradiction.

Such a factorization is not prime.

∴ 13’/d, it is a contradiction.

So if 13/d and 13/b, then again from eq. (3), 13/c. [From eq. (1) 13/a also].

Now, $b = 52 - ac - d$.

13/b, 13/52, 13/ac, 13/d but it is again a contradiction.

So there does not exist quadratic polynomials p(x) and q(x) with integral coefficients such that f(x) = $p(x) \times q(x)$.

Similarly, if p(x) is a cubic polynomial and q(x) is a linear one then let

$p(x) = x^3 + ax^2 + bx + c$

$q(x) = (x + d)$

$x^4 + 26x^3 + 52x^2 + 78x + 13 \times 3^2 \times 17 = (x^3 + ax^2 + bx + c)(x + d)$

Again comparing coefficients

$a + d = 26 \ldots\ldots(5)$

$ad + b = 52 \ldots\ldots\ldots..(6)$

$bd + c = 78 \ldots\ldots\ldots.(7)$

$cd = 13 \times 3^2 \times 17 \ldots\ldots\ldots(8)$

As before 13 divides exactly one of c and d.

If 13/d, then 13’/c, then by eq. (7), 13/bd, 13’/c and 13/78 = $bd + c$ is a contradiction.
So let 13/c, 13’/d.
By eq. (7), 13/b,
By eq. (6), 13/b and 13/52
⇒ 13/ad
⇒ 13/a , as 13/d.
By eq. (5), 13/a, 13/d and 13/26 = $a + d$ (a contradiction) and hence there does not exist any polynomials p(x) and q(x) as assumed and hence the result.

33 $P_1(x) = x^9 + x^3 + x^2 + x + 1 = x^9 - x^4 + x^4 + x^3 + x^2 + x + 1 = x^4(x^5 - 1) + (x^4 + x^3 + x^2 + x + 1) = x^4(x - 1)(x^4 + x^3 + x^2 + x + 1) + (x^4 + x^3 + x^2 + x + 1) = (x^4 + x^3 + x^2 + x + 1)[x^4(x - 1) + 1]$Thus, $x^4 + x^3 + x^2 + x + 1$ is a non−trivial polynomial divisor of $P_1(x)$

$P_k(x) = x^{(5k+4)} - x^4 + (x^4 + x^3 + x^2 + x + 1) = x^4[x^{5k} - 1] + (x^4 + x^3 + x^2 + x + 1)$

$(x^5 - 1)$ divides $(x^5)^k - 1, x^4 + x^3 + x^2 + x + 1$ divides $x^5 - 1$ *and hence* $x^{5k} - 1$.Therefore $x^4 + x^3 + x^2 + x + 1$ *divides* $P_k(x)$.

34 We need to find the largest e such that 10^e | 6250!

But as 10 = 2 × 5, this implies that we need to find the largest e such that 5^e|6250! (clearly a larger power of 2| 6250!)

$$e = \sum_{i=1}^{6025} \frac{6250}{5^i} = 1250 + 250 + 50 + 10 + 2 = 1562$$

35 56789 ≡ 89 (mod 100)

= −11 (mod 100)

∴ $(56789)^{41} \equiv (-11)^{41}$ (mod 100)

≡ $(-11)^{40} \times (-11)$ (mod 100)

≡ $11^{40} \times (-11)$ (mod 100)

$11^2 \equiv 21$ (mod 100)

$11^4 \equiv 41$ (mod 100)

$11^6 \equiv 21 \times 41$ (mod 100)

≡ 61 (mod 100)

$11^{10} \equiv 41 \times 61$ (mod 100)

≡ 01 (mod 100)

$11^{40} \equiv (01)^{40}$ (mod 100)

≡ 1 (mod 100)

$(-11)^{41} \equiv 11^{40} \times (-11)$ (mod 100)

≡ 1 × (−11) (mod 100)

≡ −11 (mod 100)

≡ 89 (mod 100)

That is last two digits of $(56789)^{41}$ are 8 and 9 in that order.

36 We shall make groups of the terms of the expression as follows:

$(1^{1997} + 1996^{1997}) + (2^{1997} + 1995^{1997}) + \cdots + (998^{1997} + 999^{1997})$

Here each bracket is of the form
$(a_i^{2n+1} + b_i^{2n+1})$ is divisible by $(a_i + b_i)$
but $(a_i + b_i) = 1997$ for all i
∴ Each bracket and hence, their sum is divisible by 1997.

37 Since $x^3 - z^3 = 721$
$\Rightarrow x^3 - z^3 = (x - y)(x^2 + xz + z^2) = 721$
For integral x, z; $x^2 + xz + z^2 > 0$,
$\because x^3 - z^3 = 721$
$\Rightarrow x^3 - z^3 > 0$
$\Rightarrow x - z > 0$
So $(x - z)(x^2 + xz + z^2) = 721 = 1 \times 721 = 7 \times 103 = 103 \times 7 = 721 \times 1$
Case (i) $x - z = 1 \Rightarrow x = 1 + z$
And $x^2 + xz + z^2 = (1 + z)^2 + (1 + z)z + z^2 = 721$
$\Rightarrow 3z^2 + 3z - 720 = 0$
$\Rightarrow z^2 + z - 240 = 0$
$\Rightarrow (z + 16)(z - 15) = 0$
$\Rightarrow z = -16$ or 15
Solving, we get
$x = -15$ *or* 16
So (−15, −16) and (16, 15) is two of the ordered pairs.
Case (ii) $x - z = 7$ *or* $x = 7 + z$
And $x^2 + xz + z^2 = 103$
$\Rightarrow (7 + z^2) + (7 + z)z + z^2 = 103$
$\Rightarrow 3z^2 + 21z - 54 = 0$
$\Rightarrow z^2 + 7z - 18 = 0$
$\Rightarrow (z + 9)(z - 2) = 0$

$\Rightarrow z = -9$ *or* $z = 2$
So the corresponding values of x are −2 and 9.
So the other ordered pairs are (−2, −9), (9, 2).
Corresponding to x – z = 103 and x – z = 721, the values are imaginary and hence, there are exactly four ordered pairs of integers (−15, −16), (16, 15), (−2, −9) and (9, 2),
satisfying the equation $x^3 = z^3 + 721$.

38 (2) = 2
$f(4) = f(2.2) = f(2).f(2) = 2.2 = 4$
$f(8) = f(2.4) = f(2).f(4) = 2.4 = 8$
Thus, we infer that $f(2^n) = 2^n$.
Let us use M. I. for proving
$f(2^1) = 2$ (by hypothesis) ……..(1)
Assume $f(2^n) = 2^n$ ………..(2)
$f(2^{n+1}) = f(2.2^n) = f(2).f(2^n) = 2.2^n$ ……..(3)

By hypothesis and eq. (1) and (2), we need to find f(n) for all n. Let us see what happens for f(1), f(3) at first.

f(1) < f(2)   (Given)

Now $f(2) = f(1 \times 2) = f(1) \times f(2)$

$\Rightarrow f(1) = 1$

Similarly, f(2) < f(3) < f(4)

2 < f(3) < 4

But the only integer lying between 2 and 4 is 3.

Thus f(3) = 3. So again we guess that $f(n) = n$, for all n.

Let us prove by using the strong principle of mathematical induction.

Let f(n) = n for all n < a, fixed $m \in N$.

Now, we should prove that f(m)= m.

If m is an even integer, then f(m) = $2k$ and k < m.

So, $f(m) = f(2k) = f(2) \times f(k) = 2 \times k = 2k = m$

So, all even m, f(m) = m.

If m is an odd integer, let m = $2k + 1$

And f(2k) < f(2k + 1)< f(2k + 2)

2k < f(2k + 1) < (2k + 1)

(Because the function f(n) = n is true for all even integer n).

But only integer lying between 2k and 2k + 2, is 2k + 1, (since the range of f is integer).

Thus, f(2k + 1) = 2k + 1

i.e., f(m) = m, in the case of odd m also.

Thus, f(n) = n, for all $n \in N$

$\therefore f(1983) = 1983$

39 Since the right hand side is positive, so is the left hand side. Hence x must be positive.

Let $x = n + f$, where $n = [x]$ and $f = [x]$. We consider two cases:

Case 1: $0 \leq f < 1/2$: In this case, we get [2x]= $[2n + 2f] = 2n$, as 2f < 1. Hence the equation becomes

$$\frac{1}{n} + \frac{1}{2n} = f + \frac{1}{3}$$

This forces (1/n)+(1/2n) ≥ 1/3. We conclude that $2n - 9 \leq 0$. Thus n can take values 1, 2, 3, 4.

Among these n = 2,3,4 are all admissible, because for n = 2,3,4 we get f=5/12, 1/6, 1/24 respectively which are all less than ½;

we get three solution in this case

$$x = 2 + \frac{5}{12} = \frac{29}{12}; x = 3 + \frac{1}{6} = \frac{19}{6} \text{ and } x = 4 + \frac{1}{24} = \frac{97}{24}$$

Case 2: $(1/2) \leq f < 1$: Now we get [2x] = $2n + 1$, as $1 \leq 2f < 2$.

The given equation reduces to

$$\frac{1}{n} + \frac{1}{2n+1} = f + \frac{1}{3}$$

We conclude, as in Case 1. 1/n + 1/ (2n+1) ≥ 1/2 + 1/3. This reduces to $10n^2 - 13n - 6 \leq 0$. It follows that $n = 1$. But this is not admissible since n = 1 gives $f = 1$. We do not have any solution in this case.

40 If x > 0, then $x^2 < x^2 + 1 + x < x^2 + 2x + 1 = (x + 1)^2$.

So $x^2 + x + 1$ lies between the two consecutives square integers and hence, cannot be asquare.

$If\ x = 0, y^2 = 1 + 0 + 0 = 1$ is a square number. Thus, the solutions in this case are is (0, 1), (0, −1).

Again if x < −1, then $x^2 > x^2 + x + 1 > x^2 + 2x + 1$, and hence, there exists no solution.

For = −1 , we have. $y^2 = 1 - 1 + (-1)^2 = 1 \quad \therefore y = \pm 1$

Thus, the only integral solutions are (0,1), (0,−1), (−1, +1), (−1, −1)

41 given $|x|^2 - 5|x| + 6 = 0$

If $x > 0$

$x^2 - 5x + 6 = 0$

$\Rightarrow (x-2)(x-3) = 0$

$\Rightarrow x = 2 \text{ or } 3$

If $x < 0$

$x^2 + 5x + 6 = 0$

$\Rightarrow (x+2)(x+3) = 0$

$\Rightarrow x = -2 \text{ or } -3$

Therefore, roots are $-2, -3, 2$ and 3.

Sum of roots = $-2 - 3 + 2 + 3 = 0$

42 $x = \sqrt{2} + \sqrt{5} + \sqrt{10}$

$\Rightarrow x - \sqrt{10} = \sqrt{2} + \sqrt{5}$

$\Rightarrow \left(x - \sqrt{10}\right)^2 = \left(\sqrt{2} + \sqrt{5}\right)^2$

$\Rightarrow x^2 - 2\sqrt{10}x + 10 = 2 + 5 + 2\sqrt{10}$

$\Rightarrow x^2 + 3 = 2\sqrt{10}x + 2\sqrt{10}$

$\Rightarrow (x^2 + 3)^2 = \left\{2\sqrt{10}(x+1)\right\}^2$

$\Rightarrow x^4 + 6x^2 + 9 = 40(x^2 + 2x + 1)$

$\Rightarrow x^4 - 34x^2 - 80x - 31 = 0$

On comparing with equation $x^4 + ax^3 + bx^2 + cx + d = 0$ we get,

$a = 0, b = -34, c = -80, d = -31$

$\therefore |a + b + c + d| = |0 - 34 - 80 - 31| = 145$

145=5×29

The two-digit prime number that divides $|a + b + c + d|$ is 29.

43 $a^b b^a = a^b + b^a + 5039$

$\Rightarrow a^b b^a - a^b - b^a + 1 = 5039 + 1$

$\Rightarrow \left(a^b - 1\right)(b^a - 1) = 63 \times 80$

$\Rightarrow a^b - 1 = 63, b^a - 1 = 80 \text{ or } a^b - 1 = 80, b^a - 1 = 63$

$\Rightarrow a^b = 64 \text{ or } a^b = 81$

The least value of a^b is 64.

44 Given, $R(7, x^2 - 3x + 2) = 5$ and $R(7, x^3 + x - 3) = 1$

Let, $x^2 - 3x + 2 = 7k_1 + 5$ and $x^3 + x - 3 = 7k_2 + 1$

Now, $(x^2 - 3x + 2)(x^3 + x - 3)$

$= x^5 - 3x^4 + 3x^3 - 6x^2 + 11x - 6$

$= x^5 - 3x^4 + 3x^3 - 6x^2 + 11x + 2 - 8$

$\therefore x^5 - 3x^4 + 3x^3 - 6x^2 + 11x + 2 = (x^2 - 3x + 2)(x^3 + x - 3) + 8$

$\Rightarrow x^5 - 3x^4 + 3x^3 - 6x^2 + 11x + 2 = (7k_1 + 5)(7k_2 + 1) + 8$

$\Rightarrow x^5 - 3x^4 + 3x^3 - 6x^2 + 11x + 2 = 49k_1k_2 + 7k_1 + 35k_2 + 5 + 8$

$\Rightarrow x^5 - 3x^4 + 3x^3 - 6x^2 + 11x + 2 = 7(7k_1k_2 + k_1 + 5k_2 + 1) + 6$

Hence, $R(7, x^5 - 3x^4 + 3x^3 - 6x^2 + 11x + 2) = 6$.

45 $4x^2 - 16x + \frac{\lambda}{4} = 0$

$\Rightarrow x = \frac{16 \pm \sqrt{(-16)^2 - 4\times4\times\frac{\lambda}{4}}}{2\times4}$

$\Rightarrow x = \frac{16 \pm \sqrt{256 - 4\lambda}}{8}$

$\Rightarrow x = \frac{8 \pm \sqrt{64 - \lambda}}{4}$

$\therefore \alpha = \frac{8 - \sqrt{64-\lambda}}{4}$ and $\beta = \frac{8 + \sqrt{64-\lambda}}{4}$

Now, $1 < \alpha < 2$

$\Rightarrow 1 < \frac{8 - \sqrt{64-\lambda}}{4} < 2$

$\Rightarrow 4 < 8 - \sqrt{64 - \lambda} < 8$

$\Rightarrow -4 < -\sqrt{64 - \lambda} < 0$

$\Rightarrow 0 < \sqrt{64 - \lambda} < 4$

$\Rightarrow 48 < \lambda < 64$ .....(i)

Again,

$2 < \beta < 3$

$\Rightarrow 2 < \frac{8 + \sqrt{64-\lambda}}{4} < 3$

$\Rightarrow 0 < \sqrt{64 - \lambda} < 4$

$\Rightarrow 48 < \lambda < 64$ ....(ii)

From (i) and (ii) we get, $48 < \lambda < 64$

Therefore, number of values of λ is $63 - 48 = 15$

46 For minimum value $a = b = c$

$\therefore a + b + c = 9$

$\Rightarrow a + a + a = 9$

$\Rightarrow a = 3$

$\therefore a = b = c = 3$

The minimum value of $a^2 + b^2 + c^2 = 3^2 + 3^2 + 3^2 = 27$

47 given, $a^3 + 3a^2 + 9a = 1$

$\therefore a \times (a^3 + 3a^2 + 9a) - 3 \times (a^3 + 3a^2 + 9a) = a \times 1 - 3 \times 1$

$\Rightarrow a^4 + 3a^3 + 9a^2 - 3a^3 - 9a^2 - 27a = a - 3$

$\Rightarrow a^4 - 27a = a - 3$

$\Rightarrow a^4 + 3 = 28a$

$\Rightarrow a^3 + \frac{3}{a} = 28$

48 $x = 3 + 2\sqrt{2}$

$\Rightarrow x - 3 = 2\sqrt{2}$

$\Rightarrow (x-3)^2 = (2\sqrt{2})^2$

$\Rightarrow x^2 - 6x + 9 = 8$

$\Rightarrow x^2 - 6x + 1 = 0$ ...(i)

Again,

$x = 3 + 2\sqrt{2}$

$\Rightarrow \frac{1}{x} = \frac{1}{3 + 2\sqrt{2}} = 3 - 2\sqrt{2}$

$\therefore x + \frac{1}{x} = 6$ ...(ii)

$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 6^2$

$\Rightarrow x^2 + \frac{1}{x^2} = 34$ ... (iii)

Now,

$\frac{x^6+x^5-5x^4+x^3+x^2+1}{4x^3}$

$= \frac{x^6+x^4+x^2+1+x^5-6x^4+x^3}{4x^3}$

$= \frac{x^6+x^4+x^2+1+x^3(x^2-6x+1)}{4x^3}$

$= \frac{x^4(x^2+1)+x^2+1+x^3\times 0}{4x^3}$ [by (i)]

$= \frac{(x^4+1)(x^2+1)}{4x^2.x}$

$= \frac{\left(x^2+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)}{4}$

$= \frac{1}{4} \times 34 \times 6$ [by (ii) and (iii)]

$= 51$

49 Let,

$752 = px + y \ldots .(i)$

$1604 = qx + y \ldots .(ii)$

$3095 = rx + y \ldots .(iii)$

$(ii) - (i) \Rightarrow 852 = (q - p)x \ \ldots (iv)$

$(iii) - (ii) \Rightarrow 1491 = (r - q)x \ldots (v)$

$(v) - (iv) \Rightarrow 639 = (p + r - 2q)x$

Now, 639 = 9×71

$\therefore x = 71$

752=71×10+42

$\therefore y = 42$

$\therefore x - y = 71 - 42 = 29$

50 $2!^{2!}! = 4! = 24$, it is divisible by 24.

$\therefore$ 24 divides $2!^{2!}!, 3!^{3!}!, 4!^{4!}!, \ldots \ldots \ldots, 99!^{99!}!$

$\therefore$ Remainder = $1!^{1!}! = 1$

51 $\phi(340) = \phi(17 \times 20) = \phi(17) \times \phi(20)$

Now, $\phi(17) = n\{1,2,3, \ldots ,16\} = 16$ and $\phi(20) = n\{1,3,7,9,11,13,17,19\} = 8$

$\therefore \phi(340) = \phi(17) \times \phi(20) = 16 \times 8 = 128$

52 By Fermat's little theorem, it is known that if p is a prime number, then for any integer a not divisible by p, $a^{p-1} \equiv 1(\text{mod } p)$.

Since 13 is a prime number, $2^{12} \equiv 1(\text{mod } 13)$

Now, $13^{5^3} = (12 + 1)^{5^3} \equiv 1(\text{mod } 12)$

$\Rightarrow 13^{5^3} = 12k + 1$, where k is an integer.

$\Rightarrow 2^{13^{5^3}} = 2^{12k+1}$

Now, $2^{12} \equiv 1(\text{mod } 13)$

$\Rightarrow (2^{12})^k \equiv 1(\text{mod } 13)$

$\Rightarrow (2^{12})^k.2 \equiv 1 \times 2(\text{mod } 13)$

$\Rightarrow 2^{12k+1} \equiv 2(\text{mod } 13)$

$\Rightarrow 2^{13^{5^3}} \equiv 2(\text{mod } 13)$

∴ Remainder = 2

53 Since x and y single digit prime numbers then $\gcd(x, y) = 1$

$\phi((xy)^3) = 400$

$\Rightarrow \phi(x^3y^3) = 4 \times 100$

$\Rightarrow \phi(x^3)\phi(y^3) = (8-4) \times (125-25)$

$\Rightarrow (x^3 - x^2)(y^3 - y^2) = (2^3 - 2^2)(5^3 - 5^2)$

$\Rightarrow x = 2, y = 5$ or $x = 5, y = 2$

$\Rightarrow xy = 2 \times 5$

$\Rightarrow xy = 10$

$\therefore \phi(xy) = \phi(10) = 4$

54 $2x + 3y = 563$

$$\Rightarrow x = \frac{563 - 3y}{2}$$

If value of y be odd then $(563 - 3y)$ is even i.e divisible by 2.

Now, $563 = 3 \times 187 + 2$

∴ Number of possible values of y $= \frac{187+1}{2} = 94$ (odd numbers from 1 to 187)

∴ No of solutions = 94

55 Let, $t = \log_y^x$

Now, $\log_y^x + 5\log_x^y = 6$

$$\Rightarrow t + \frac{5}{t} = 6$$

$\Rightarrow t^2 - 6t + 5 = 0$

$\Rightarrow (t-1)(t-5) = 0$

$\Rightarrow t = 1,5$

For $t = 1$,

$\log_y^x = 1$

$\Rightarrow x = y$ (Which is not possible since x and y are distinct integers)

For $t = 5$,

$\log_y^x = 5$

$\Rightarrow x = y^5$

Now $2 \le x \le 2025$

$\Rightarrow 2 \le y^5 \le 2025$

$\Rightarrow 2 \le y \le 4$ (Since y is integer)

$\Rightarrow y \in \{2,3,4\}$

No of solutions = 3

56 $5x^2 + 13y + 50 = x^2y$

$\Rightarrow x^2(5-y) + 13y + 50 = 0$

$\Rightarrow x^2(5-y) + 13y - 65 + 115 = 0$

$\Rightarrow x^2(5-y) - 13(5-y) = -115$

$\Rightarrow (x^2 - 13)(5-y) = -115$

$\Rightarrow (x^2 - 13)(y-5) = 5 \times 23$

Case 1:

$\therefore x^2 - 13 = 5$ and $y - 5 = 23$

$\Rightarrow x^2 = 18$ & $y = 28$

Since x is positive integer then $x^2 = 18$ not possible.

Case 2:

$\therefore x^2 - 13 = 23$ and $y - 5 = 5$

$\Rightarrow x = 6 \, \& y = 10$

$\therefore \sqrt{x+y} = \sqrt{6+10} = \sqrt{16} = 4$

57 $2x^2 - 4x + 1 = 0$

$$\Rightarrow x = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 2 \times 1}}{2 \times 2}$$

$$\Rightarrow x = \frac{4 \pm \sqrt{8}}{4}$$

$$\Rightarrow x = \frac{2 \pm \sqrt{2}}{2}$$

If $\frac{2+\sqrt{2}}{2}$ is a root of $x^2 + ax + b = 0$ then $\frac{2-\sqrt{2}}{2}$ is also root of $x^2 + ax + b = 0$

$$\therefore \frac{1}{2} = \frac{a}{-4} = \frac{b}{1}$$

$$\Rightarrow a = -2 \, , b = \frac{1}{2}$$

$$\therefore 1 + ab = 1 + (-2).\frac{1}{2} = 1 - 1 = 0$$

58 Given α, β and γ are the roots of $Q(x) = 0$

$\alpha^3 + \alpha^2 - \alpha + 1 = 0$

$\beta^3 + \beta^2 - \beta + 1 = 0$

$\gamma^3 + \gamma^2 - \gamma + 1 = 0$

And $\sum \alpha = -1 \, , \sum \alpha\beta = -1 \, , \alpha\beta\gamma = -1$

Now,

$P(\alpha) + P(\beta) + P(\gamma)$

$= \sum(\alpha^6 + \alpha^5 - \alpha^4 + \alpha^2 + \alpha)$

$= \sum[\alpha^3(\alpha^3 + \alpha^2 - \alpha + 1) - \alpha^3 + \alpha^2 + \alpha]$

$= \sum(-\alpha^3 + \alpha^2 + \alpha)$

$= \sum(\alpha^2 - \alpha + 1 + \alpha^2 + \alpha) \quad [\text{since } \alpha^3 + \alpha^2 - \alpha + 1 = 0 \,]$

$= \sum(2\alpha^2 + 1)$

$= 2\sum \alpha^2 + \sum 1$

$= 2[(\sum \alpha)^2 - 2\sum \alpha\beta] + 3$

$= 2[(-1)^2 - 2.(-1)] + 3$

$= 9$

59 Given $z + \frac{1}{z} = 2\cos 6°$

$$\Rightarrow \cos 6° = \frac{z + z^{-1}}{2}$$

$$\Rightarrow z = e^{i6°} \quad \left[\text{since } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\right]$$

$$\therefore \; z^{600} + \frac{1}{z^{600}} + 1$$

$= \left(e^{i6°}\right)^{600} + \left(e^{-i6°}\right)^{600} + 1$

$= e^{i3600°} + e^{-i3600°} + 1$

$= 2\cos 3600° + 1$

$= 2 + 1$

$= 3$

60 Degree of the given polynomial is = $1 + 2 + 3 + \cdots \ldots \ldots . +30 = \frac{30 \times 31}{2} = 465$

Zeros of the polynomial are $-1,2,2,-3,-3,-3,4,4,4,4 \ldots \ldots \ldots \ldots$

∴Cofficient of x^{464} is = - (sum of zeros)

$= -(-1+2+2-3-3-3+4+4+4+4-\cdots\ldots.)$

$= -(-1+2\times 2-3\times 3+4\times 4-5\times 5+\cdots\ldots\ldots..+30\times 30)$

$= -(-1^2+2^2-3^2+4^2-5^2+\cdots+30^2)$

$= -[(2^2-1^2)+(4^2-3^2)+\cdots+(30^2-29^2)]$

$= -[(2+1)+(4+3)+\cdots+(30+29)]$

$= -(1+2+3+4+5+\cdots+30)$

$= -\frac{30\times 31}{2}$

$= -465$

61 Let p and q be the roots of x^2-x-1 =0 .By vieta's theorem,p+q=1 and q=-1.Note that p and q must also be the roots of $ax^{17}+bx^{16}+1=0$ .Thus

$ap^{17}+bp^{16}=-1$ and $aq^{17}+bq^{16}=-1$

Multiplying by q^{16} and p^{16} in eq. 1 and 2 respectively and using the fact that pq =-1

$ap+b=-q^{16}$ and $aq+b = -p^{16}$…………………(1)

Thus $a=\frac{p^{16}-q^{16}}{p-q}= (p^8+q^8)(p^4+q^4)(p^2+q^2)(p+q)$

Since

p+q =1

$P^2+q^2 =(p+q)^2-2pq= 1+2 = 3$

$P^4+q^4 = (p^2+q^2)^2-2p^2q^2= 9-2 =7$

$P^8+q^8=(p^4+q^4)-2p^4q^4=49-2 =47$

It follows that a= 1.3.7.47 = 987Type equation here.

Likewise, eliminating a in (1) gives

$-b =\frac{p^{17}-q^{17}}{p-q}$

$=p^{16}+p^{15}q+p^{14}q^2+\ldots\ldots\ldots\ldots.+q^{16}$

$=(p^{16}+q^{16})+pq(p^{14}+q^{14})+p^2q^2(p^{12}+q^{12})+\ldots\ldots\ldots+p^7q^7(p^2+q^2)+p^8q^8$

$=(p^{16}+q^{16})-(p^{14}+q^{14})+\ldots\ldots\ldots-(p^2+q^2)+1$

For n≥ 1,let $k_{2n}=p^{2n}+q^{2n}$ then $k_3=3$ and $k_4=7$ and

$k_{2n+4}=p^{2n+4}+q^{2n+4}$

$=(p^{2n+2}+q^{2n+2})(p^2+q^2)-p^2q^2(p^{2n}+q^{2n})$

$=3k_{2n+2}-k_{2n}$

For n≥ 3.Thus $k_6=18$,$k_8=47$,$k_{10}=123$,$k_{12}=322$,$k_{14}=843$,$k_{16}=2207$

Hence -b=2207-843+322-123+47-18+7-3+1=1597

or (a, b) = (987, -1597)

62 S = p(1) + p(2) + ... + p(999).

What is the largest prime factor of S?

Solution:-

Consider each positive integer less than 1000 to be a three-digit number by prefixing Os to numbers with fewer than three digits. The sum of the products of the digits of all such positive numbers is

(0.0.0+0.0.1+...+9.9.9)-0.0.0

$=(0+1+...+9)^3-0.$

However, p(n) is the product of non-zero digits of n. The sum of these products can be found by replacing 0 by 1 in the above expression, since ignoring 0's is equivalent to thinking of them as 1's in the products. (Note that the final 0 in the above expression becomes a 1 and compensates for the contribution of 000 after it is changed to 111.)

Hence

S=46^3-1=(46-1)(46^2+46+1)=3^3.5.7.103, and the largest prime factor is 103

63 We have $\frac{1}{x_n} = \frac{2x_{n-2}-x_{n-1}}{x_{n-2}x_{n-1}} = \frac{2}{x_{n-1}} - \frac{1}{x_{n-2}}$

Let $y_n = 1/x_n$. Then $Y_n - Y_{n-1} = Y_{n-1} - Y_{n-2}$, i.e., y_n is an arithmetic sequence. If x_n is a nonzero integer when n is in an infinite set S, the y_n's for n ∈ S satisfy $-1 \leq y_n \leq 1$. Since an arithmetic sequence is unbounded unless the common difference is 0, $Y_n - Y_{n-1} = 0$ for all n, which in turn implies that $x_1 = x_2$ = m, a non zero integer.

Clearly, this condition is also sufficient.

64 For x > 2, there is a real number t > 1 such that x = $t^2+\frac{1}{t^2}$.

The equation becomes $(t^2+\frac{1}{t^2})^3-3(t^2+\frac{1}{t^2}.)=\sqrt{t^2+\frac{1}{t^2}+2}$

i.e $t^6+\frac{1}{t^6} = t+\frac{1}{t}$,

$(t^7-1)(t^5-1) = 0$

which has no solutions for t> 1. Hence there are no solutions for x > 2.

65 Suppose that the first term of the sequence Δ**A** is d.

Then Δ**A** ={d,d+1,d+2,...} with the n^{th} term given by d + (n - 1).

Hence A = {a_1, a, +d, a_1 +d+ (d+ 1), a_1 +d+ (d+ 1) + (d+2),...} with the n^{th} term given by

$a_n = a_1 + (n - 1)d + \frac{1}{2}(n - 1)(n - 2)$.

This shows that an is a quadratic polynomial in n with leading coefficient1/2.

Since $a_{19} = a_{92} = 0$, we must have

$a_n = \frac{1}{2}(n - 19)(n - 92)$, so $a_1 = (1 - 19)(1 - 92)/2 = 819$.

66 Setting 2^x = a and 3^x = b, the equation becomes

$1 + a^2 + b^2 - a - b - ab = 0$. Multiplying both sides of the last equation by 2 and completing the squares gives

$(1 - a)^2 + (a - b)^2 + (b - 1)^2 = 0$.

Therefore $1 = 2^x = 3^x$, and x = 0 is the only solution.

67 Note that $(m + n)^3 = m^3 + n^3 + 3mn(m + n)$. If m + n = 33, then

$33^3 = (m + n)^3 = m^3 + n^3 + 3mn(m + n) = m^3 + n^3 + 99mn$.

Hence m + n - 33 is a factor of $m^3 + n^3 + 99mn - 33^3$. We have $m^3 + n^3 + 99mn - 33^3$

$= (m + n - 33)(m^2 + n^2 - mn + 33m + 33n + 33^2) = 1/2\ (m + n - 33)[(m - n)^2 + (m + 33)^2 + (n + 33)^2]$.

Hence there are 35 solutions altogether: (0, 33), (1, 32),…… , (33, 0), and(-33, -33).

68 We have $(a+b+c)^2 \leq 16$,

i.e. i.e. $a^2 + b^2 + c^2 + 2(ab + be + ca) \leq 16$

i.e $a^2 + b^2 + c^2 \leq 8$

i.e. $a^2 + b^2 + c^2 - (ab + be + ca) \leq 4$

i.e $(a - b)^2 + (b - c)^2 + (c - a)^2 \leq 8$, and the desired result follows.

69 Let Sn denote the desired sum. Then

$S_n=\frac{1}{(2n)!}\sum_{k=0}^{n}\frac{(2n)!}{(n-k)!(n+k)!}$

$=\frac{1}{(2n)!}\sum_{k=0}^{n}\binom{2n}{n-k}$

$=\frac{1}{(2n)!}\sum_{k=0}^{n}\binom{2n}{k}$

$=\frac{1}{(2n)!}\cdot\frac{1}{2}[\sum_{k=0}^{n}\binom{2n}{k}+\binom{2n}{n}]$

$=\frac{1}{(2n)!}\cdot\frac{1}{2}[2^{2n}+\binom{2n}{n}]$

$=\frac{2^{2n-1}}{(2n)!}+\frac{1}{2(n!)^2}$

70 Notice that $f_i(x) = \sqrt{1+a_1x}$ is concave.

Hence $f(x) = \sqrt{1+a_1x} + \sqrt{1+a_2x} + \ldots\ldots \sqrt{1+a_nx}$

is concave. Since f'(x) exists, there can be at most one point on the curve y = f (x)with derivative 0. Suppose there is more than one nonzero root. Since x = 0 is also a root, we have three real roots $x_1 < x_2 < X_3$. Applying the Mean-Value theorem to f (x) on intervals $[x_1, x_2]$ and $[x_2,x_3]$,we can find two distinct points on the curve with derivative 0, a contradiction. Therefore, our assumption is wrong and there can be at most one nonzero real root for the equation f (x) = n.

71 Let f (x) = 4x(1 - x). Observe that

$f^{-1}(0) = \{0, 1\}$, $f^{-1}(1) = \{1/2\}$, and $f^{-1}([0, 1]) = [0, 1]$ and $|\{y: f(y) = x\}|=2$ for all $x\in [0,1)$

Let $A_n = \{x \in R: f^n(x)=0\}$;then

$A_{n+1} = \{x \in R: f^{n+1}(x)=0\}$

$= \{x \in R: f^n(f(x))=0\} == \{x \in R: f(x)\in An\}$

We claim that for all $n\geq 1$,A_n C [0, 1], 1 E An, and IAnI = $2^{n-1} + 1$.

For n = 1, we have $A_1= \{x\in R \text{ I } f(x)=0\} = \{0, 1\}$, and the claims hold.

Now suppose $n \geq 1$ and An C [0, 1], $1 \in A_n$, and $|A_n| = 2^{n-1} + 1$. Then

$X \in A_{n+1}$

$= f(x) \in A_n$, C [0, 1] $=x \in [0, 1]$,

so A_{n+1} C [0, 1].

Since f (0) = f (1) = 0, we have $f^{n+1}(1) = 0$ for all $n \geq 1$, so $1 \in A_{n+1}$.

Now we have

$|A_{n+1}| = |\{x: f(x) \in A_n|$

$=\sum_{a\in A_n}|\{x: f(x) = a\}|$

$=|\{x: f(x) = 1\}| + \sum_{a\in A_n}|\{x: f(x) = a\}|$,$a\in [0,1)$

$=1 + \sum_{a\in A_n} 2$, $a\in [0,1)$

$= 1+ 2(|A_n|-1)$

$=1+ 2(2^{n-1}+1-1)$

$= 2^n+1$

Thus the claim holds by induction.

Finally, $a_{1998} = 0$ if and only if $f^{1997}(t) = 0$, so there are $2^{1996} + 1$ such values of t.

72 First, we introduce some basic knowledge of vector operations. Let u = [a, b] and v = [m, n] be two vectors. Define their dot product u.v = am + bn.

It is easy to check that

(i) $v.v = m^2+n^2 = |v|^2$, that is, the dot product of vector with itself is the square of the magnitude of v and $v.v \geq 0$ with equality ifand only if v = [0, 0];

(ii) u.v = v.

(iii) u. (v + w) = u.v + u.w, where w is a vector;

(iv) (cu) v = c(u.v), where c is a scalar.

When vectors u and v are placed tail-by-tail at the origin O, let A and B be the tips of u and v, respectively. Then $\overrightarrow{AB}$ = v - u.

Let <AOB $=\theta$.

Applying the law of cosines to triangle AOB yields

$|v - u|^2 = AB^2$

$= OA^2 + OB^2 - 2OA.OB \cos\theta$,

$\cos\theta = \frac{u.v}{|u||v|}$

Consequently, if $0 \le \theta \le 90°$, u. v ≥ 0. Consider vectors $v_1 = [a_1, a2]$, $v_2 = [a_3, a_4]$, $v_3 = [a_5, a_6]$, and $v_4 = [a_7, a_8]$. Note that the numbers $a_1a_3 + a_2a_4$, $a_1a_5 + a_2a_6$, $a_1a_7 + a_2a_8$, $a_3a_5 + a_4a_6$, $a_3a_7 + a_4a_8$, $a_5a_7 + a_6a_8$ are all the dot products of distinct vectors $v_i$and v_j. Since there are four vectors, when placed tail-by-tail at the origin, atleast two of them form a non-obtuse angle, which in turn implies the desired result.

73 We have $a^5 + b^5 \ge a^2 b^2 (a + b)$,

because $(a^3 - b^3)(a^2 - b^2) \ge 0$,

with equality if and only if a = b. Hence

$\frac{ab}{a^5+b^5+ab} \le \frac{ab}{a^2b^2(a+b)+ab}$

$= \frac{1}{ab(a+b)+1}$

$=\frac{abc}{ab(a+b+c)}$

$=\frac{c}{a+b+c}$

Likewise, $\frac{bc}{b^5+c^5+bc} \le \frac{a}{a+b+c}$ and $\frac{ca}{c^5+a^5+ca} \le \frac{b}{a+b+c}$

Adding the last three inequalities leads to the desired result. Equality holds if and only if a = b = c = 1.

74 Clearly, f (x) = x^2 satisfies the functional equation.

Now assume that there is a nonzero value a such that f (a)$\neq a^2$

Let y = $\frac{x^2-f(x)}{2}$ in the functional equation to find that

$f(\frac{f(x)+x^2}{2}) = f(\frac{f(x)+x^2}{2}) + 2f(x)(x^2-f(x))$

or $0 = 2f(x)(x^2 - f(x))$. Thus, for each x, either f (x) = 0 or f (x) = x^2.

In both cases, f (0) = 0.

Setting x = a, it follows from above that either f (a) = 0 or f (a) = 0 or f (a) = a^2. The latter is false, so f (a) = 0.

Now, let x = 0 and then x = a in the functional equation to find that f (y) = f (-y), f (y) = f (a^2 - y) and so f (y) = f (-y), f(y) = f (a^2 + y); that is, the function is periodic with nonzero period a^2. Let y = a^2 in the original functional equation to obtain $f(f(x)) = f(f(x) + a^2) = f(x^2 - a^2) + 4a^2f(x) = f(x^2) + 4a^2f(x)$.

However, putting y = 0 in the functional equation gives $f(f(x)) = f(x^2)$for all x.

Thus, $4a^2f(X) = 0$ for all x. Since a is nonzero, f (x) = 0 for all x. Therefore, either f(x) = x^2 or f(x) = 0.

75 Multiplying the second equation by i and adding it to the first equation yields

$x+yi +\frac{(3x-y)-(x+3y)i}{x^2+y^2} = 3$

Or $x+yi +\frac{3(x-yi)}{x^2+y^2} - \frac{i(x-yi)}{x^2+y^2} = 3$

Let z = x+ yi, Then $\frac{1}{z}\ \frac{x-yi}{x^2+y^2}$

Thus the last equation becomes $z + \frac{3-i}{z} = 3$,

Or $z^2-3z +(3-i) = 0$

Hence $z = \frac{3\mp(1+2i)}{2}$

that is, (x, y) = (2, 1) or (x, y) = (1,-1).

76 The statement is true if and only if $k \ge 4$. We start by proving that it does hold for each $k \ge 4$. Consider any polynomial F(x) with integer coefficients satisfying theinequality $0 \le F(c) \le k$ for each $c \in \{0, 1, \dots, k + 1\}$.

Note first that F(k + 1) = F(0), since F(k + 1) - F(0) is a multiple of k + 1 not exceeding k in absolute value.

Hence F(x) - F(0) = x(x - k - 1)G(x),
where G(x) is a polynomial with integer coefficients. Consequently,
k ≥ I F(c) - F(O) I = c(k + 1 - c)IG(c)I
for each c ∈ { 1, 2, ... , k}.
The equality c(k + 1 - c) > k holds for each c ∈ {2, 3, ... , k - 1}, as it isequivalent to (c - 1)(k - c) > 0.
Note that the set {2, 3,... , k - 1} is not empty if k ≥ 3, and for any c in this set, (1) implies that $|G(c)|$ < 1.
Since G(c) is an integer, G(c) = 0.
Thus
F(x) - F(0) = x(x - 2)(x - 3)...(x - k + 1)(x - k - 1)H(x),
where H(x) is a polynomial with integer coefficients.
To complete the proof of our claim, it remains to show that H(1) = H(k) = 0.
Note that for c = 1 and c = k, (2) implies that k ≥ $|F(c) - F(0)|$ = (k - 2)! . k .$|H(c)|$.
For k > 4, (k - 2)! > 1.
Hence H(c) = 0.
We established that the statement in the question holds for any k ≥ 4. But the proof also provides information for the smaller values of k as well.
More exactly, if F(x) satisfies the given condition then 0 and k + 1 are roots of F(x) and F(0) for any k ≥1; and if k ≥ 3 then 2 must also bea root of F(x) - F(0).
Taking this into account, it is not hard to find the following counter examples:
F(x) = x(2 - x) for k = 1,
F(x) = x(3 - x) for k = 2,
F(x) = x(4 - x)$(x - 2)^2$ for k = 3.

77 Mr. Taf has a winning strategy.
We say a blank space is odd (even) if it is the coefficient of an odd (even) power of x.
First Mr. Taf will fill in arbitrary real numbers into one of the remaining even spaces, if there are any.
Since there are only n - 1 even spaces, there will be at least one odd space left after 2n - 3 plays, that is, the given polynomial becomes
p(x) = q(x) +_x^s +_x^{2t-1},
where s and 2t - 1 are distinct positive integers and q(x) is a fixed polynomial.
We claim that there is a real number a such that
p(x) = q(x) + ax^s + _x^{2t-1}
will always have a real root regardless of the coefficient of x^{2t-1}.
Then Mr. Taf can simply fill in a in front of x^s and win the game.
Now we prove our claim. Let b be the coefficient of x^{2t-1} in p(x). Note that
$\frac{1}{2^{2t-1}}$ p(2) + p(-1)
= ($\frac{1}{2^{2t-1}}$ q(2) + $2^{s-2t+1}a + b$) + [q(-1) +$(-1)^s$a – b]
= ($\frac{1}{2^{2t-1}}$ q(2) + q(-1)) + a [2^{s-2t+1} + $(-1)^s$]
Since s ≠ 2t – 1, 2^{s-2t+1} + $(-1)^s$ ≠ 0
Thus

$$\frac{\frac{1}{2^{2t-1}}q(2)+q(-1)}{2^{s-2t+1}+(-1)^s}$$

is well defined such that a is independent of b and $\frac{1}{2^{2t-1}}$ p(2) +p(-1) = 0

It follows that either p(-1) = p(2) = 0 or p(-1.) and p(2) have different signs, which implies that there is a real root of p(x) in between -1 and 2. In either case, p(x) has a real root regardless of the coefficient of x^{2t-1} as claimed.

Our proof is thus complete.

78 We define two new sequences. For z = 1, 2, ... , n, let

$a_i' = a_k$ and $b_i' = \frac{b_i a_k}{a_i}$.

Then $a_i' - b_i' = a_k - \frac{b_i a_k}{a_i} = \frac{a_k}{a_i}(a_i - b_i)$

Or

$(a_i' - b_i') - (a_i - b_i) = \frac{(a_k - a_i)(a_i - b_i)}{a_i} \geq 0.$

Therefore

$na_k = a_1' + a_2' + . . . + a_n' \geq b_1' + b_2' + ... + b_n'$

Applying the AM-GM inequality yields

$\left(\frac{b_1 b_2 ... b_n a_k^n}{a_1 a_2 ... a_n}\right)^{\frac{1}{n}} = (b_1' b_2' .... b_n')^{\frac{1}{n}} \leq \frac{b_1' + b_2' + ... + b_n'}{n} \leq a_k$

from which the desired result follows.

79 Note that $2(\sqrt{k+1} - \sqrt{k}) = \frac{2}{(\sqrt{k+1} + \sqrt{k})} < \frac{1}{\sqrt{k}}$

Therefore

$\sum_{k=1}^{80} \frac{1}{\sqrt{k}} > 2\sum_{k=1}^{80}(\sqrt{k+1} - \sqrt{k}) = 16,$

which proves the lower bound. On the other hand,

$2(\sqrt{k+1} - \sqrt{k}) = \frac{2}{(\sqrt{k} + \sqrt{k-1})} > \frac{1}{\sqrt{k}}$

Therefore

$\sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 1 + 2\sum_{k=2}^{80}(\sqrt{k} - \sqrt{k-1}) = 2\sqrt{80} - 1 < 17,$

which proves the upper bound. Our proof is complete.

80 The conditions imply that $f(x^3) = f(g(f(x))) = [f(x)]^2$, whence x ∈ {-1, 0 ,1}

$\Longrightarrow x^3 = x \Longrightarrow f(x) = [f(x)]^2 \Longrightarrow f(x) \in \{0, 1\}$.

Thus, there exist different a, b ∈ {-1, 0, 1} such that f (a) = f (b).

But then $a^3 = g(f(a)) = g(f(b)) = b^3$, a contradiction.

Therefore, the desired functions f and g do not exist.

81 Suppose the roots are $\alpha, \beta, \gamma, \delta$ and $\alpha\beta = 1$

$\text{Now}, \sigma_1 = (\alpha + \beta) + (\gamma + \delta) = -\frac{-24}{4} = 6 \ldots (i)$

$\sigma_2 = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{31}{4} \Longrightarrow (\alpha + \beta)(\gamma + \delta) + \gamma\delta = \frac{31}{4} - 1 = \frac{27}{4} \ldots (ii)$

$\sigma_3 = \gamma\delta(ga + \beta) + \alpha\beta(\gamma + \delta) = -\frac{3}{2} \Longrightarrow \sigma_3 = \gamma\delta(\alpha + \beta) + (\gamma + \delta) = -\frac{3}{2} \ldots (iii)$

$\sigma_4 = \alpha\beta\gamma\delta = -2 \Longrightarrow \gamma\delta = -2 \ldots (iv)$

From Eq. (2) and Eq. (4), we get $(\alpha + \beta)(\gamma + \delta) = \frac{35}{4} \ldots (v)$

From Eq. (3) and Eq. (4), we get $-2(\alpha + \beta) + (\gamma + \delta) = -\frac{3}{2} \ldots (vi)$

From Eq. (1) and Eq. (6), we get $3(\alpha + \beta) = \frac{15}{2} \Longrightarrow (\alpha + \beta) = \frac{5}{2}$

82 (i) Since α, β, γ are the roots of

$$x^3 + px + q = 0, \qquad ...(1)$$

we have,

$$\left.\begin{array}{l}\alpha^3 + p\alpha + q = 0\\ \beta^3 + p\beta + q = 0\\ \gamma^3 + p\gamma + q = 0\end{array}\right\} \qquad ...(2)$$

From (2),

$$\Sigma\alpha^3 + p(\Sigma\alpha) + 3q = 0$$

But $\Sigma\alpha = 0$, from Eq. (1)

$\therefore$ $\Sigma\alpha^3 = -3q$

$$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$= 0^2 - 2\times p \qquad (\because \Sigma\alpha\beta = p)$$

$$= -2p \qquad ...(4)$$

Multiplying (1) by x^2 we get

$$x^5 + px^3 + qx^2 = 0 \qquad ...(5)$$

and α, β, γ are are three roots of Eq. (5). So

$$\left.\begin{array}{l}\alpha^5 + p\alpha^3 + q\alpha^2 = 0\\ \beta^5 + p\beta^3 + q\beta^2 = 0\\ \gamma^5 + p\gamma^3 + q\gamma^2 = 0\end{array}\right\} \qquad ...(6)$$

From Eq. (6), $\Sigma\alpha^5 + p\Sigma\alpha^3 + q\Sigma\alpha^2 = 0$

$$\Sigma\alpha^5 = -(p\Sigma\alpha^3 + q\Sigma\alpha^2)$$

$$= -[p(-3q) + q(-2p)] \qquad ...(7)$$

$$= 3pq + 2pq = 5pq$$

or

$$\frac{1}{5}\Sigma\alpha^5 = pq$$

$$= \left(-\frac{1}{2}\times\Sigma\alpha^2\right)\left(-\frac{1}{3}\Sigma\alpha^3\right)$$

$$= \left[\frac{1}{3}\Sigma\alpha^3\right]\left[\frac{1}{2}\Sigma\alpha^2\right]$$

$$\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \frac{\alpha^3 + \beta^3 + \gamma^3}{3} \times \frac{\alpha^2 + \beta^2 + \gamma^2}{2} \qquad ...(8)$$

Multiplying Eq. (1) by x, we get

$$x^4 + px^2 + qx = 0$$ ...(9)

and hence $\Sigma\alpha^4 + p\Sigma\alpha^2 + q\Sigma\alpha = 0$

$\Rightarrow$ $\Sigma\alpha^4 = -p\Sigma\alpha^2$

$(\because \Sigma\alpha = 0)$

Again multiplying Eq. (1) by x^4, we get

$$x^7 + px^5 + qx^4 = 0$$

...(10)

and hence $\Sigma\alpha^7 + p\Sigma\alpha^5 + q\Sigma g a^4 = 0$

or $\Sigma\alpha^7 = -p\Sigma\alpha^5 - q\Sigma\alpha^4$

$$= -p \times 5pq - q(-p\Sigma\alpha^2)$$

$$= -5p^2q - 2p^2q$$

$$= -7p^2q$$

or $\frac{1}{7}\Sigma\alpha^7 = -p^2q$

$$= pq \times (-p)$$

$$= \left(\frac{1}{5}\Sigma\alpha^5\right) \times \left(\frac{1}{2}\Sigma\alpha^2\right)$$

or $\left(\frac{\alpha^7 + \beta^7 + \gamma^7}{7}\right) = \left(\frac{\alpha^5 + \beta^5 + \gamma^5}{5}\right) \times \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{2}\right)$

83 You can see that $4(x^2 - 5x + 6)$ is H.C.F. of the two equations and hence, the common roots are the roots of

$$x^2 - 5x + 6 = 0 \text{ i.e., } x = 3 \text{ or } x = 2$$

Now, $x^4 + 5x^3 - 22x^2 - 50x + 132 = 0$ ...(1)

and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$ ...(2)

have 2 and 3 as their common roots.

If the other roots of Eq. (1) are α and β, then $\alpha + \beta + 5 = -5$,

$\Rightarrow$ $\alpha + \beta = -10$ from eq. (1)

$$6\alpha\beta = 132$$

$\Rightarrow$ $\alpha\beta = 22$

So, α and β are also roots of the quadratic equation

$$x^2 + 10x + 22 = 0$$

$$\therefore \quad x = \frac{-10 \pm \sqrt{100 - 88}}{2} = \frac{-10 \pm 2\sqrt{3}}{2} = -5 \pm \sqrt{3}$$

So the roots of Eq. (1) are $2, 3, -5 + \sqrt{3}, -5\sqrt{3}.$

For Eq. (2), if α_1 and β_1 be the roots of Eq. 92), then we have

$$\alpha_1 + \beta_1 + 5 = -1$$

$$\alpha_1 + \beta_1 = -6$$

$$6\alpha_1\beta_1 = 24 \text{ or } \alpha_1\beta_1 = 4$$

So α_1 and β_1 are the roots of

$$x^2 + 6x + 4 = 0$$

$$x = \frac{-6 \pm \sqrt{36 - 16}}{2} = -3 \pm \sqrt{5}$$

So the roots of Eq. (2) are $2, 3, -3 + \sqrt{5}, -3 - \sqrt{5}.$

84 Adding the three equations, we get

$$2(x + y + z)^2 = 48 + 2L$$

or

$$x + y + z = \sqrt{24 + L}$$

Dividing the three equations by $(x + y + z) = \sqrt{24 + L},$ we get

$$x + y = \frac{18}{\sqrt{24 + L}},\ y + z = \frac{30}{\sqrt{24 + L}},\ z + x = \frac{24}{\sqrt{24 + L}}$$

and solving we get,

$$x = \frac{\left(\sqrt{24 + L}\right)^2 - 30}{\sqrt{24 + L}} = \frac{L - 6}{\sqrt{24 + L}},$$

$$y = \frac{(24 + L) - 2L}{\sqrt{24 + L}} = \frac{24 - L}{\sqrt{24 + L}},$$

$$z = \frac{24 + L - 18}{\sqrt{24 + L}} = \frac{L + 6}{\sqrt{24 + L}}.$$

85 x_1 *and* x_2 are roots of

$$ax^2 + bx + c = 0 \quad \text{...(1)}$$

and

$$-ax^2 + bx + c = 0 \quad \text{...(2)}$$

respectively.

We have

$$ax_1^2 + bx_1 + c = 0$$

and

$$-ax_2^2 + bx_2 + c = 0$$

Let $$f(x) = \frac{a}{2}x^2 + bx + c.$$

Thus, $$f(x_1) = \frac{a}{2}x_1^2 + bx_1 + c \quad ...(3)$$

$$f(x_2) = \frac{a}{2}x_2^2 + bx_2 + c \quad ...(4)$$

Adding $\frac{1}{2}ax_1^2$in Eq. (3), we get

$$f(x_1) + \frac{1}{2}ax_1^2 = ax_1^2 + bx_1 + c = 0$$

$\Rightarrow$ $$f(x_1) = -\frac{1}{2}ax_1^2 \quad ...(5)$$

Subtracting $\frac{3}{2}ax_2^2$ from Eq. (4), we get

$$f(x_2) - \frac{3}{2}ax_2^2 = -ax_2^2 + bx_2 + c = 0$$

$\Rightarrow$ $$f(x_2) = \frac{3}{2}ax_2^2.$$

Thus $f(x_1)$ *and* $f(x_2)$ have opposite signs and, hence, $f(x)$must have a root between x_1 *and* x_2

86 The roots are $m = \pm 1$ $i.e.(m+1)(m-1)$

$\therefore$ $-2 < (m-1) < (m+1) < 4$ gives

$-1 < m < 3.$

87 Let the roots be $\frac{a}{r^2}, \frac{q}{r}, a, ar, ar^2$

$\therefore$ $$\text{Sum of the root} = a\left(\frac{1}{r^2} + \frac{1}{r} + 1 + r + r^2\right) = 40 \quad ...(1)$$

Sum of be reciprocals $$= \frac{1}{a}\left(r^2 + r + 1 + \frac{1}{r} + \frac{1}{r^2}\right) = 10 \quad ...(2)$$

Dividing (1) by (2), $a^2 = 4 \therefore \; a = \pm 2$ ...(3)

Since s is the –ve of the product of the roots $s = -a^5$ ...(4)

$\therefore$ $s = \pm 32$ or $|s| = 32$ ...(5)

88 We use a trick $Q(x) = P(x) - 10x$ ...(1)

The $Q(1) = Q(2) = Q(3) = 0$ ...(2)

$\therefore$ $Q(x)$ i.e., divisible by $(x-1)(x-2)(x-3)$ ...(3)

Since $Q(x)$ is a 4th degree polynomial

and $Q(x) = (x - 1)(x - 2)(x - 3)(x - r)$

$P(x) = (x - 1)(x - 2)(x - 3)(x - r) + 10x$ ...(4)

∴ $\dfrac{P(12) + P(-8)}{10} = 1984$

89 Let $x = \sqrt[3]{7} + \sqrt[3]{49}$

∴ $x^3 = 7 + 49 + 3 \cdot \sqrt[3]{7} \cdot \sqrt[3]{49}$

i.e., $x^3 = 56 + 21x$

Thus, $P(x) = x^3 - 21 - 56 = 0$ and the product of the root is 56.

90 Common roots must be the roots of $2x^2 + (r - q) = 0$ (Difference of equation)

∴ Their sum is 0.

Then the third root of the first equation must be -5 and of the second equation is –7.

∴ $(x_1, x_2) = (-5, -7).$

91 $\left(\frac{a}{5}\right)^2 + \left(\frac{b}{5}\right)^2 + \left(\frac{c}{5}\right)^2 - 2\left(\frac{ax}{30} + \frac{by}{30} + \frac{cz}{30}\right) + \left(\frac{x}{6}\right)^2 + \left(\frac{z}{6}\right)^2 + \left(\frac{z}{6}\right)^2 = 1 - 2 + 1 = 0$

∴ $\left(\frac{a}{5} - \frac{x}{6}\right)^2 + \left(\frac{b}{5} - \frac{y}{6}\right)^2 + \left(\frac{c}{5} - \frac{z}{6}\right)^2 = 0$

Thus $\dfrac{a}{5} = \dfrac{x}{6}$

∴ $a = kx$

where $k = \dfrac{5}{6}; b = ky$ and $c = kz$.

∴ $\dfrac{a + b + c}{x + y + z} = \dfrac{k(x + y + z)}{x + y + z} = k$

$k = \dfrac{5}{6}$

92 A – (sum of the digits) must be divisible by 9. Then B + (sum of the digits) does not satisfy must be divisible by 9.

Now consider 999 : 999 – 27 = 972(so defined sum of 27)

990 : 990 – 18 = 972(so defined sum of 18)

∴ Answer is 990.

93 Sum of the roots = k; Sum of the roots taken two at a line = –k

Then

$k^2 = (a + b + c + d)^2 = (a^2 + b^2 + c^2 + d^2) + 2(ab + ac + ad + bc + bd + cd)$

$= (a^2 + b^2 + c^2 + d^2) + 2k$

Thus $a^2 + b^2 + c^2 + d^2 = k^2 - 2k$ ...(1)

Thus minimum value of $k^2 - 2k = 1$.

94 Replacing $\left[\frac{x}{6}\right]$ by y and solving, $2y^2 + 3y - 20 = 0 \Longrightarrow y = \frac{5}{2}, -4$

$\therefore$ $-4 \le \frac{x}{6} < -3$

which means $-24 \le x < -18$

$\therefore$ **Ans.** (–24, 18)

95 Now $(a + 1)(b + 1)(c + 1) = 19$

Then $A + B + C = (a + b + c) + (ab + bc + ca) + (abc)$

$= (a + 1)(b + 1)(c + 1) - 1$

$= 19 - 1$

$= 18$

96 $x^2 - z^2 = 120$

$\Rightarrow (x + z)(x - z) = 120 = 1.120 = 2.60 = 3.40 = 4.30 = 5.24 = 6.20 = 8.15 = 10.12$

$\therefore$ $x = 31; z = 29; x = 17, z = 13; x = 13, z = 7; x = 11, z = 1$

$\therefore$ Required ordered pairs are : (31, 29), (17, 13), (13, 7), (11, 1).

IOQM PREVIOUS YEAR'S QUESTIONS WITH SOLUTIONS

IOQM – 2024-25

QUESTIONS

1 The smallest positive integer that does not divide 9! = 1 × 2 × 3 × 4 × 5 × 6 × 7 × 8 × 9 is:

2 The number of four-digit odd numbers having digits 1, 2, 3, 4 each occurring exactly once.

3 The number obtained by taking the last two digits of 5^{2024} in the same order is:

4 Let ABCD be a quadrilateral with <ADC= 70°, <ACD= 70°, <ACB=10° and <BAD =110°. Find the measure of <CAB (in degrees).

5 Let a = $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, let b = $\frac{x}{z}+\frac{y}{x}+\frac{z}{y}$ and let c = $(\frac{x}{y}+\frac{y}{z})(\frac{y}{z}+\frac{z}{x})(\frac{z}{x}+\frac{x}{y})$. The value of I ab-c I.

6 Find the number of triples of real numbers (a, b, c) such that $a^{20} + b^{20} + c^{20} = a^{24} + b^{24} + c^{24} = 1$.

7 Determine the sum of all possible surface area of a cube whose vertices are (1, 2, 0) and (3, 3, 2).

8 Let n be the smallest integer such that the sum of digits of n is divisible by 5 as well as the sum of digits of $(n+1)$ is divisible by 5. What are first two digits of n in the same order?

9 Consider the grid points $X = \{(m,n) |\ 0 \le m,n \le 4\}$. We say a pair of points $\{(a,b),(c,d)\}$ in X is a knight-move pair if $(c = a \pm 2\ and\ d = b \pm 1)$ or $(c = a \pm 1\ and\ d = b \pm 2)$. The number of knight-move pairs in X is:

10 Determine the number of positive integral values of p for which there exists a triangle with sides a, b and c which satisfy $a^2 + (p^2+9)b^2 + 9c^2 - 6ab - 6pbc = 0$.

11 The positive real numbers a, b, c satisfy $\frac{a}{2b+1}+\frac{2b}{3c+1}+\frac{3c}{a+1} = 1$, $\frac{1}{a+1}+\frac{1}{2b+1}+\frac{1}{3c+1} = 2$. What is the value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$?

12 Consider a square $ABCD$ of side length 16. Let E, F be points on CD such that $CE = EF = FD$. Let the line BF and AE meet in M. Then area of ΔMAB is:

13 Three positive integers a, b, c with $a > c$ satisfy the following equations: $ac + b + c = bc + a + 66$, $a + b + c = 32$. Find the value of a.

14 Initially, there are 3^{80} particles at the origin (0, 0). At each step the particles are moved to points above the x-axis as follows: if there are n particles at any point (x, y), then $\left[\frac{n}{3}\right]$ of them are moved to $(x+1, y+1)$, $\left[\frac{n}{3}\right]$ are moved to $(x, y+1)$ and the remaining to (x – 1, y + 1). For example, after the first step, there are 379 particles each at (1, 1), (0, 1) and (–1, 1). After the second step, there are 378 particles each at (–2, 2) and (2, 2), 2 × 378 particles each at (–1, 2) and (1, 2), and 379 particle at (0, 2). After 80 steps, the number of particles at (79, 80)

15 Let X be the set consisting twenty positive integers $n, n+2, \dots . n+38$. The smallest value of n for which any three numbers $a, b, c \in X$, not necessarily distinct, form the sides of an acute angled triangle is:

16 $f: R \to R$ be a function satisfying the relation $4f(3-x) + 3f(x) = x^2$ for any real x, Find the value of $f(27) - f(25)$ to the nearest integer. (R is the set of all real numbers).

17 Consider an isosceles triangle ABC with sides $BC = 30, CA = AB = 20$. Let D be the foot of perpendicular from A to BC and M be the midpoint of AD. Let PQ be a chord of the circumcircle of of the triangle ABC, such that M lies on PQ is parallel to BC. The length of PQ is :

18 Let p, q be two-digit numbers neither of which are divisible by 10. Let r be the four-digit number by putting the digits of p followed by the digits of q (in order). As p, q vary, a computer prints r on the screen if $\gcd(p, q) = 1$ and $p + q$ divides r. Suppose that the largest number that is printed by the computer is N. Determine the number formed by the last two digits of N (in same order).

19 Consider five points in the plane, with no three of them collinear. Every pair of points among them is joined by a line. In how many ways can we color these lines by red or blue, so that no three of the points form a triangle with lines of the same color.

20 On a natural number n you are allowed two operations: (1) multiply n by 2 or (2) subtract 3 from n. For example, starting with 8 you can reach 13 as follows: 8 → 16 → 13. You need two steps and you cannot do in less than two steps. Starting from 11, what is the least number of steps required to reach 121?

21 An integer n is such that $[\frac{n}{9}]$ is a three-digit number with equal digits, and $\left[\frac{n-172}{4}\right]$ is a 4-digit number with the digits 2, 0, 2, 4 in some order. What is the remainder when n is divided by 100?

22 In a triangle ABC $\angle$ BAC = 90^0. Let D be the point on BC such that AB + BD = AC + CD. Suppose BD : DC = 2 : 1. If, $\frac{AC}{AB} = \frac{m+\sqrt{p}}{n}$ where m, n are relatively prime positive integers and p is a prime number, determine the value of m + n + p.

23 Consider the fourteen numbers, $1^4, 2^4, \dots, 14^4$. The smallest natural number n such that they leave distinct remainders when divided by n is:

24 Consider the set F of all polynomials whose coefficients are in the set of {0, 1}. Let q(x) = $x^3 + x + 1$. The number of polynomials p(x) in F of degree 14 such that the product p(x)q(x) is also in F is:

25 A finite set M of positive integers consists of distinct perfect squares and the number 92. The average of the numbers in M is 85. If we remove 92 from M, the average drops to 84. If N^2 is the largest possible square in M, what is the value of N?

26 The sum of [x] for all real numbers x satisfying the equation $16 + 15x + 15x^2 = [x]^3$ is:

27 In a triangle ABC, a point P in the interior of ABC is such that
<BPC - <BAC = <CPA - <CBA = <APB - <ACB. Suppose <BPC=30^o AND AP=12. Let D, E, F be the feet of perpendiculars form P on to BC, CA, AB respectively. If m$\sqrt{n}$ in the area of the triangle DEF where m, n are integers with n prime, then what is the value of the product mn?

28 Find the largest positive integer $n < 30$ such that $\frac{1}{2}(n^8 + 3n^4 - 4)$ is not divisible by square of any prime number.

29 Let $n = 2^{19}3^{12}$. Let M denote the number of positive divisors of n^2 which are less than n but would not divide n. What is the number formed by taking the last two digits of M (in the same order)?

30 Let ABC be a right-angled triangle with $< B = 90^o$. Let the length of the altitude BD be equal to 12. What is the minimum possible length of AC, given that AC and the perimeters of triangle ABC are integers.

ANSWERS

1. 11	2. 12	3. 25	4. 70	5. 01	6. 06
7. 99	8. 49	9. 48	10. 5	11. 12	12. 96
13. 19	14. 80	15. 92	16. 8	17. 25	18. 13
19. 12	20. 10	21. 91	22. 34	23. 31	24. 50
25. 22	26. 33	27.	28. 20	29. 28	30. 25

SOLUTIONS

Q.1. The smallest positive integer that does not divide 9! = 1 × 2 × 3 × 4 × 5 × 6 × 7 × 8 × 9 is:

Sol.

Let n be a smallest integer that does not divide 1 × 2 × 3 × 4 × 5 × 6 × 7 × 8 × 9

Prime factorization of 9! = 2^7 × 3 × 5 × 7

Clearly, we can see that it is divisible by 2, 3, 2 x 2 = 4, 5, 2 x 3 = 6, 7, 2 x 2 x 2 = 8, 3 x 3 = 9, 2 x 5 = 10

Therefore, least positive integer that does not divide 9! is 11 as 11 is a prime number.

Q.2. The number of four-digit odd numbers having digits 1, 2, 3, 4, each occurring exactly once.

Sol.

Let the 4-digit odd no is abcd (_ _ _ _)

Place d can take either 1 or 3 i.e. it can be filled in 2 ways

Place a can be filled in any of the remaining digits i.e. in 4-1 = 3 ways

Place b can be filled in any of the remaining digits i.e. in 4-1 = 2 ways

Place c can be filled in any of the remaining digits i.e. in 4-1 = 1 ways

Total values = 2 × 3 × 2 × 1 = 12 numbers.

Q.3. The number obtained by taking the last two digits of 5^{2024} in same order is:

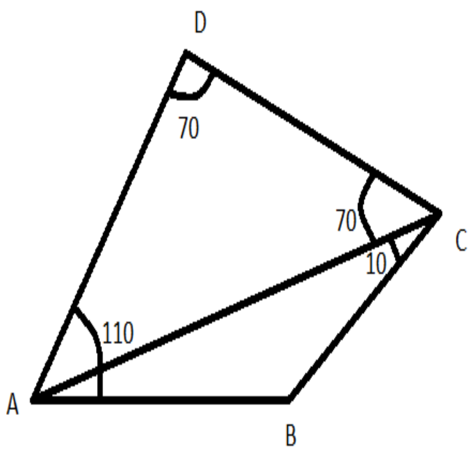
Sol. The number obtained by taking the last two digits of 5^{2024} in the same order is: 25

as Last two digits of any power of 5 is 25.

Q.4. Let ABCD be a quadrilateral with <ADC= 70^o, <ACD= 70^o, <ACB=10° and <BAD =110°.

Find the measure of <CAB (in degrees).

Sol.



In ▲ACD, <ADC= 70 degree, <ACD= 70 degree

So <DAC = 180-70-70=40 degree

Again <BAD =110°

Or, <BAC + <CAD =110°

Or, <BAC = <CAB = 110-40=70^0

Therefore <CAB is 70^0

Q.5. Let a= $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, let b= $\frac{x}{z}+\frac{y}{x}+\frac{z}{y}$ and let c= $(\frac{x}{y}+\frac{y}{z})(\frac{y}{z}+\frac{z}{x})(\frac{z}{x}+\frac{x}{y})$. The value of

I ab-c I.

Sol. c= $(\frac{x}{y}+\frac{y}{z})(\frac{y}{z}+\frac{z}{x})(\frac{z}{x}+\frac{x}{y})$ = $(a-\frac{z}{x})(a-\frac{x}{y})(a-\frac{y}{z})$

$= a^3 - (\frac{x}{y}+\frac{y}{z}+\frac{z}{x})a^2 + (\frac{z}{x}+\frac{y}{x}+\frac{x}{z})a - 1 = a^3 - a^3 + ab - 1 = ab - 1$

Or, ab - c - 1 = 0

Or, ab-c =1

Q.6. Find the number of triples of real numbers (a, b, c) such that $a^{20}+b^{20}+c^{20} = a^{24}+b^{24}+c^{24} = 1$.

Sol.- If $a^{20}+b^{20}+c^{20} = a^{24}+b^{24}+c^{24} = 1$

Then any one of a, b, c is ±1 and others are 0

Since $(a^{24}+b^{24}+c^{24}) - (a^{20}+b^{20}+c^{20}) = 1 - 1 = 0$.

$\Rightarrow a^{24} - a^{20} + b^{24} - b^{20} + c^{24} - c^{20} = 0$

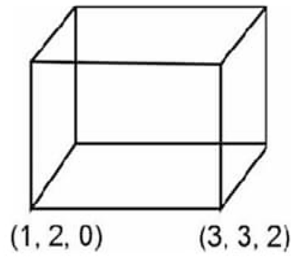
Triples can be (1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1) and (0, 0, -1).

No. of triples =6

Q.7. Determine the sum of all possible surface area of a cube whose vertices are (1,2,0) and (3,2,2).

Sol. Case-1

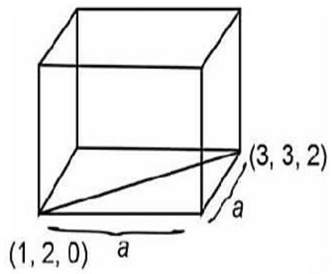
SA= $6 \times \left[\sqrt{4+1+4}\right]^2 = 54\ sq.unit$



Case-2

$2a^2 = 9 \Longrightarrow a = \frac{3}{\sqrt{2}}$

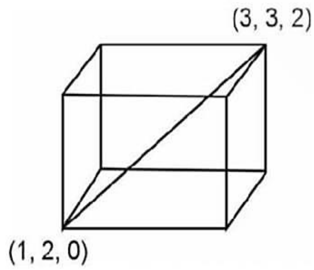
SA= $6 \times \frac{9}{2} = 27\ sq.unit$



Case-3

$\sqrt{3}a = 3 \Longrightarrow a = \sqrt{3}$,

SA=$6 \times 3 = 18\ sq.unit$



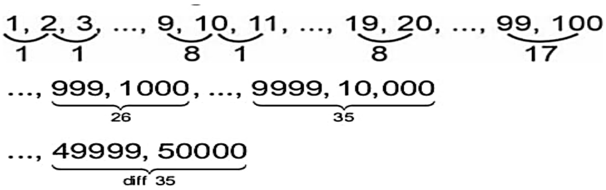
Therefore, possible surface area =18+27+54=99.

Q.8. Let n be the smallest integer such that the sum of digits of n is divisible by 5 as well as the sum of digits of $(n+1)$ is divisible by 5. What are first two digits of n in the same order?

Sol.

Let sum of digits of n-digit number=5λ

Let the sum of digit of n+1 number=$5k$



Therefore, the smallest number = 49999

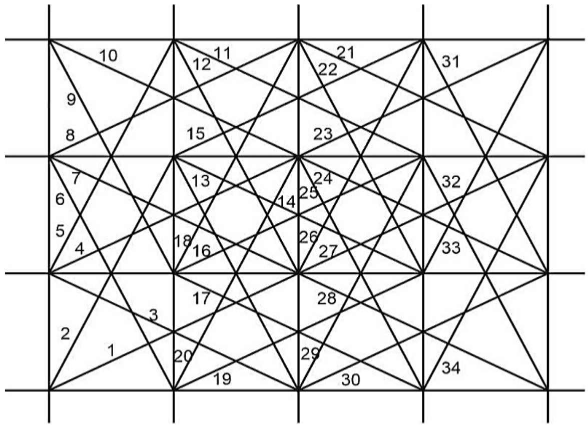
First two digits = 49

Q.9. Consider the grid points $X = \{(m, n)| \ 0 \leq m, n \leq 4\}$. We say a pair of points $\{(a, b), (c, d)\}$ in X is a knight-move pair if $(c = a \pm 2 \ and \ d = b \pm 1)$ or $(c = a \pm 1 \ and \ d = b \pm 2)$. The number of knight-move pairs in X is:

Sol.

Consider the grid as shown above

Each line represents the allowed move from one and points of line to other end point of the line.



We can see the movement allowed in $3 \times 2, 2 \times 2$ rectangles vertical or horizontal.

In each rectangle there are 4 moves allowed and total number of rectangles are 12.

Therefore, total allowed moved=$12 \times 4 = 48$.

Q.10. Determine the number of positive integral values of p for which there exists a triangle with sides a, b and c which satisfy $a^2 + (p^2 + 9)b^2 + 9c^2 - 6ab - 6pbc = 0$.

Sol.

$a^2 + (p^2 + 9)b^2 + 9c^2 - 6ab - 6pbc = 0.$

$\Rightarrow (a - 3b)^2 + (pb - 3c)^2 = 0$

$\Rightarrow a = 3b, pb = 3c \Rightarrow c = \frac{pb}{3}$

If a is the largest side, $b + \frac{pb}{3} > 3b \Rightarrow p > 6 \ \ldots\ldots\ldots (i)$

If c is the largest side, $3b + b > \frac{pb}{3} \Rightarrow p < 12 \ldots\ldots\ldots\ldots (ii)$

From, (i) and (ii) $p = \{7,8,9,10,11\}$

Therefore, number of positive integral value of p is 5

$Q.11$ The positive real numbers a, b, c satisfy $\frac{a}{2b+1}+\frac{2b}{3c+1}+\frac{3c}{a+1}=1$, $\frac{1}{a+1}+\frac{1}{2b+1}+\frac{1}{3c+1}=2$. What is the value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

Sol.

$\frac{a}{2b+1}+\frac{2b}{3c+1}+\frac{3c}{a+1}=1$, $\frac{1}{a+1}+\frac{1}{2b+1}+\frac{1}{3c+1}=2$

Adding the above two equations, we get

$$\frac{3c+1}{a+1}+\frac{a+1}{2b+1}+\frac{2b+1}{3c+1}=3.$$

Now, $AM \geq GM$

$$\Rightarrow \left(\frac{\frac{3c+1}{a+1}+\frac{a+1}{2b+1}+\frac{2b+1}{3c+1}}{3}\right) \geq (1)^{\frac{1}{3}}$$

$$\Rightarrow \frac{3c+1}{a+1}=1 \; and \; \frac{2b+1}{3c+1}=1 \;\; and \;\; \frac{a+1}{2b+1}=1$$

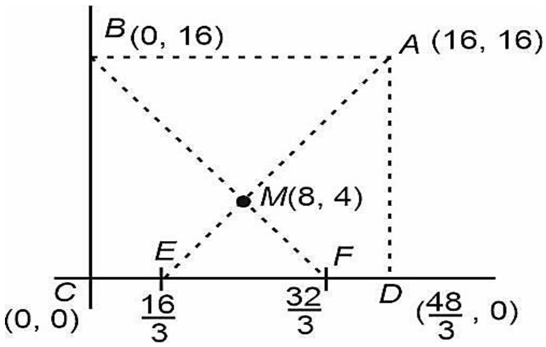
$$\Rightarrow (a+1)=(2b+1)=(3c+1)=k$$

$$\Rightarrow \frac{1}{k}+\frac{1}{k}+\frac{1}{k}=2 \Rightarrow k=\frac{3}{2}$$

$$a=\frac{1}{2}, b=\frac{1}{4}, c=\frac{1}{6} \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2+4+6=12$$

Q.12. Consider a square $ABCD$ of side length 16. Let E, F be points on CD such that $CE = EF = FD$. Let the line BF and AE meet in M. Then area of ΔMAB is:

Sol.



The equation of line AE: $2y - 3x + 16 = 0$

The equation of line BF: $2y + 3x - 32 = 0$

$\Rightarrow M \equiv (8,4)$

The area of triangle AMB= $\left|\frac{1}{2}\begin{vmatrix} 0 & 16 & 1 \\ 16 & 16 & 1 \\ 8 & 4 & 1 \end{vmatrix}\right|$

$$=\frac{1}{2}\left|[-16(8)+1(64-8\times 16]\right|$$

$$=96 \; sq.units$$

Q.13. Three positive integers a, b, c with $a > c$ satisfy the following equations:

$ac + b + c = bc + a + 66$, $a + b + c = 32$. Find the value of a.

Sol.

$ac + b + c = bc + a + 66$ ……(i)

$a + b + c = 32$ ……(ii)

From (i), we get

$c(a - b) + (b - a) + c = 66$

$\Longrightarrow (a - b)(c - 1) + (c - 1) = 66 - 1$

$$\Longrightarrow (c - 1)(a - b + 1) = 65 = 1 \times 65 = 65 \times 1 = 5 \times 13 = 13 \times 5$$

Case-1

$c - 1 = 65, a - b + 1 = 1$

$$c = 66 \ \ but \ a + b + c = 32 \ \Longrightarrow not \ possible$$

Case-2

$c - 1 = 1, a - b + 1 = 65$

$\Longrightarrow c = 2, a - b = 64$

$a + b + c = 32 \Longrightarrow a + b = 30$

$$\Longrightarrow 2a = 94 \Longrightarrow a = 47, b = -17 \ \Longrightarrow not \ possible$$

$Case - 3$

$\mathrm{c} - 1 = 5, a - b + 1 = 13$

$\Longrightarrow c = 6, a - b = 12$

$$a = 19, b = 7$$

Case-4

$\mathrm{c} - 1 = 13, a - b + 1 = 5$

$\Longrightarrow c = 14, a - b = 4$

$a = b = 18$

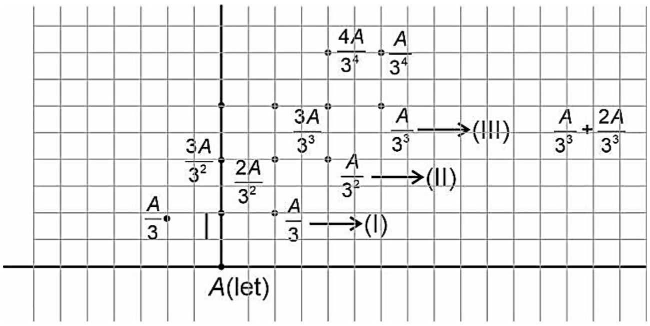
$\Longrightarrow a = 11 \ but \ a < c$

$\Longrightarrow \mathrm{only\ a} = 19$

Q.14. Initially, there are 3^{80} particles at the origin (0, 0). At each step the particles are moved to points above the x-axis as follows: if there are n particles at any point (x, y), then $\left[\frac{n}{3}\right]$ of them are moved to $(x+1, y+1)$, $\left[\frac{n}{3}\right]$ are moved to $(x, y+1)$ and the remaining to (x – 1, y + 1). For example, after the first step, there are 379 particles each at (1, 1), (0, 1) and (–1, 1). After the second step, there are 378 particles each at (–2, 2) and (2, 2), 2 × 378 particles each at (–1, 2) and (1, 2), and 379 particles at (0, 2). After 80 steps, the number of particles at (79, 80)

Sol.

At n^{th} step particle at $(n-1, n)$ is $\frac{nA}{3^n}$



∴ At 80^{th} step particles at (79, 80) is $\frac{80A}{3^{80}}$

$= 80$ (where A = 3^{80})

Q.15. Let X be the set consisting twenty positive integers $n, n+2, \ldots . n+38$. The smallest value of n for which any three numbers $a, b, c \in X$, not necessarily distinct, form the sides of an acute angled triangle is:

Sol.

$X = \{n, n+2, \ldots, n+38\}$

$a, b, c \in X$

For any a, b, c

(i) Triangle should be formed
(ii) Triangle should be acute

$\Rightarrow$ only one angle can be obtuse at max.

(i) Let $a \leq b \leq c$

$\Rightarrow$ for triangle

$a + b > c$ for all possible combination

$\Rightarrow$ even if a, b are smallest $a = b = n$

$\Rightarrow n + n > n + 38 \Rightarrow n > 38 \Rightarrow$ Triangle will form

(ii) Now using cosine formula largest side longest angle

$\Rightarrow cosc = \frac{a^2+b^2-c^2}{2ab} > 0$ for acute Δ

$\Rightarrow a^2 + b^2 - c^2 > 0$ for acute Δ $\forall a, b, c \in X$

$n^2 + n^2 - (n+38)^2 > 0$

$\Longrightarrow n^2 - 76n - 38^2 > 0$

$\Longrightarrow n > 91.74$

$$\Longrightarrow n = 92$$

Q.16. $f: R \to R$ be a function satisfying the relation $4f(3-x) + 3f(x) = x^2$ for any real x, Find the value of $f(27) - f(25)$ to the nearest integer. (R is the set of all real numbers).

Sol.

$4f(3-x) + 3f(x) = x^2\ \forall x \in R$

$\therefore 4f\big(3-(3-x)\big) + 3f(3-x) = (3-x)^2$

$= 4f(x) + 3f(3-x) = (3-x)^2$

$\Longrightarrow 12f(3-x) + 9f(x) = 3x^2$

$\Longrightarrow 12f(3-x) + 16f(x) = 4(x-3)^2$

$\Rightarrow 7f(x) = 4(x-3)^2 = 3x^2$

$= \{4(24)^2 - 3.27^2\} - \{4(22)^2 - 3.25^2\}$

$= 4(24^2 - 22^2) + 3(25^2 - 27^2)$

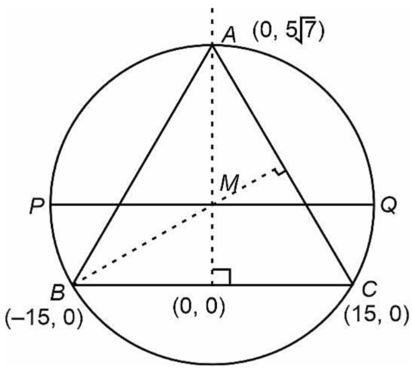
$= 4.46.2 + 3.52.(-2)$

$= 8 \times 46 \times 6 \times 52$

$$\Longrightarrow f(27) - f(25) = \frac{1}{7}(46 \times 8 - 6 \times 52) = \frac{56}{7} = 8$$

Q.17. Consider an isosceles triangle ABC with sides $BC = 30, CA = AB = 20$. Let D be the foot of perpendicular from A to BC and M be the midpoint of AD. Let PQ be a chord of the circumcircle of of the triangle ABC, such that M lies on PQ is parallel to BC. The length of PQ is :

Sol.



The equation of PQ: $y = \frac{5\sqrt{7}}{2}$

The perpendicular bisector of AC:

$$\left(y - \frac{5\sqrt{7}}{2}\right) = \frac{-1}{\left(\frac{5\sqrt{7}-0}{0-15}\right)}\left(x - \frac{15}{2}\right)$$

$$\Longrightarrow \left(y - \frac{5\sqrt{7}}{2}\right) = \frac{15}{5\sqrt{7}}\left(x - \frac{15}{2}\right)$$

Centre ≡ intersection of perpendicular bisector

$$\left(0,-\frac{5}{\sqrt{7}}\right)=\left(0,\frac{-5\sqrt{7}}{7}\right)$$

$\Longrightarrow$ Equation of circumcircle

$$(x-0)^2+\left(y+\frac{5\sqrt{7}}{7}\right)^2=\left(5\sqrt{7}+\frac{5\sqrt{7}}{7}\right)^2$$

$$\Longrightarrow x^2+\left(y+\frac{5}{\sqrt{7}}\right)^2=25\times 7\times\frac{64}{49}=\frac{25\times 64}{7}$$

Intersecting with PQ : $y=\frac{5\sqrt{7}}{2}$

$$x^2=\frac{25\times 64}{7}-\left(\frac{5\sqrt{7}}{2}+\frac{5}{\sqrt{7}}\right)^2=\frac{25\times 64}{7}-\frac{1}{4\times 7}(45^2)$$

$$x^2=\frac{25\times 64\times 4-45^2}{28}=\frac{5^2}{2^2}\left(\frac{64\times 4-81}{7}\right)=\frac{5^2}{2^2}\times 25$$

$$\Longrightarrow x=\pm\frac{25}{2}\Longrightarrow distance=\left[\frac{25}{2}-\frac{25}{2}\right)=25$$

$Q.$18. Let p,q be two-digit numbers neither of which are divisible by 10. Let r be the four-digit number by putting the digits of p followed by the digits of q (in order). As p,q vary, a computer prints r on the screen if $\gcd(p,q)=1$ and $p+q$ divides r. Suppose that the largest number that is printed by the computer is N. Determine the number formed by the last two digits of N (in same order).

Sol.

$r=100p+q$

$p+q|\,r=100p+q$

$\Longrightarrow p+q|\,99r$

But $\gcd(p+q,r)=1$

$\Longrightarrow p+q|\,99$

$\therefore p+q=33\ or\ 99$

For N to be maximum

$p+q=99$

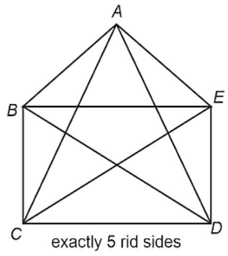
Where $p=76, q=13$

Answer 13

Q.19. Consider five points in the plane, with no three of them collinear. Every pair of points among them is joined by a line. In how many ways can we color these lines by red or blue, so that no three of the points form a triangle with lines of the same color.

Sol.

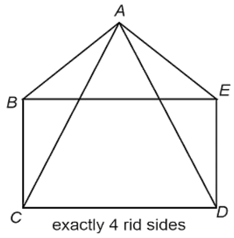
Case-I



exactly 5 rid sides

2 ways (corresponding to each color)
As color of other sides got fixed

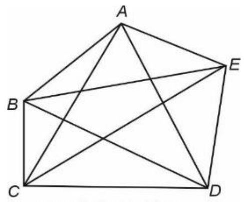
Case-II



exactly 4 rid sides

Not possible, therefore 0 ways as color of other sides got fixed and there will be one triangle which will have all sides rud on block

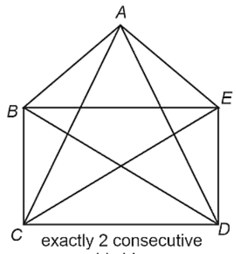
Case-III



exactly 3 rid sides consecutive

Therefore, not possible,0 ways (same as above reason)

Case-IV



exactly 2 consecutive rid sides

Only one way of coloring as color of other sides got fixed

Therefore 2 ways (corresponding to each color for shown figure)

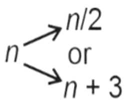
Therefore, total number of ways=2+$^5c_1 \times 2 = 12$ ways

$^5c_1$ ways of choosing 2 will active rides.

Q.20. On a natural number n you are allowed two operations: (1) multiply n by 2 or (2) subtract 3 from n. For example, starting with 8 you can reach 13 as follows: 8 → 16 → 13. You need two steps and you cannot do in less than two steps. Starting from 11, what is the least number of steps required to reach 121?

Sol.

11 → ……. Some steps → 121 minimum steps Notice that for each step the number is a natural number ⇒ Reverse the process Start from 121 and now condition will be



Now, 121 cannot be divide by 2 as result will not be natural number and for minimum step it is better to keep dividing till, we get on odd number

$\Rightarrow$ 1 21$\longrightarrow$ (121+3=124) $\longrightarrow$($\frac{124}{2}=62) \longrightarrow (\frac{62}{2}=31) \longrightarrow (31+3=34) \longrightarrow (\frac{34}{2}) \longrightarrow (17+3=20) \longrightarrow (\frac{20}{2}=10) \longrightarrow (\frac{10}{2}=5) \longrightarrow (5+3=8) \longrightarrow (8+3)$at it doesn't make sense to go half as 1 move to reach 11 .

Q.21. An integer n is such that $[\frac{n}{9}]$ is a three-digit number with equal digits, and n is a 4-digit number with the digits 2, 0, 2, 4 in some order. What is the remainder when n is divided by 100?

Sol. Let $\left[\frac{n-172}{4}\right]=k, k \in I \left[\frac{n-172}{4}\right] \in [k, k+1)$

$$\frac{n-172}{4} \in [4k, 4k+4)$$

n$\in [4k+172, 4k+176)$

$$\frac{n}{9} \in [, \frac{4k+176}{9}]$$

Now, $[\frac{n}{9}] \in \{111,222,333, \ldots \ldots 999\} \Longrightarrow \frac{4k+176}{9} > \overline{aaa} > \frac{4k+172}{9}, for\ a \in \{1,2,3 \ldots \ldots .9\}$

For a=9

K $\in (2203.75, 2204.75) \Longrightarrow k=2204$

$$\left[\frac{n-172}{4}\right]=2204 \ldots \ldots \ldots \ldots (i)$$

And $[\frac{n}{9}]$=999 ....(ii)

From (i)

$\frac{n-172}{4} \in [2204,2205) \Longrightarrow$n$\in [8988,8992]$

From (ii)

$$\frac{n}{9} \in [999,1000)$$

$$\Longrightarrow n \in [8991,9000)$$

$$\Longrightarrow n=8991 \Longrightarrow n=91.$$

Q.22. In a triangle ABC $\angle$ BAC = 90^0. Let D be the point on BC such that AB + BD = AC + CD. Suppose BD : DC = 2 : 1. If, $\frac{AC}{AB} = \frac{m+\sqrt{p}}{n}$ where m, n are relatively prime positive integers and p is a prime number, determine the value of m + n + p.

Sol.

AB+ BD = AC + CD

y+2x = AC + x

$\Rightarrow$ AC =x + y

$\Rightarrow (x+y)^2 + y^2 = (3x)^2$

$\Rightarrow x^2 + y^2 + 2xy + y^2 = 9x^2$

$\Rightarrow 8x^2 - 2y^2 - 2xy = 0$

$\Rightarrow 4x^2 - y^2 - xy = 0$

Dividing by y^2 and taking a= x/y

We get $4a^2 - a - 1 = 0$

Or, $a = \frac{1 \pm \sqrt{1+4*4}}{2*4} = \frac{1 \pm \sqrt{17}}{8}$

Or, $\frac{AC}{AB} = \frac{x}{y} + 1 = a+1$

Or, $\frac{AC}{AB} = \frac{1 \pm \sqrt{17}}{8} + 1 = \frac{1+\sqrt{17}}{8} + 1$ (Since ratio can not be negative)

Or, $\frac{AC}{AB} = \frac{1+\sqrt{17}}{8} + 1 = \frac{9+\sqrt{17}}{8}$

Combining the above with $\frac{AC}{AB} = \frac{m+\sqrt{p}}{n}$ we get m= 9, p = 17, n = 8

Therefore m + n + p = 34

Q.23. Consider the fourteen numbers, $1^4, 2^4, \dots, 14^4$. The smallest natural number n such that they leave distinct remainders when divided by n is:

Sol. $1^4, 2^4, \dots, 14^4$
$x^4 \equiv a \pmod{n}$
$y^4 \equiv b \pmod{n}$ such that $a \neq b$ for $x \neq y$ and x, y $\{1, 2, \dots\ 14\}$
$(x^4 - y^4) \equiv (a-b) \pmod{n}$
$\Rightarrow (x - y)(x + y)(x^2 + y^2) \equiv (a-b) \pmod{n}$
$\Rightarrow n \nmid (x - y)(x + y)(x^2 + y^2)$
We have to find minimum n with condition (i)
Clearly, n > 27 as $(x + y) \in \{3, \dots\ 27\}$

Now n = 28, x = 6, y = 8 works
n = 29, x = 5, y = 2 works
n = 30, x = 8, y = 2 works
for x = 31, there are no such x, y,
$31 | \underline{(x - y)(x + y)(x^2 + y^2)}$
Must be prime factor
$31 \nmid (x^2 + y^2)$ and $31 \nmid (x - y)(x + y)$
$\Rightarrow$ 31 will be the answer

Q.24. Consider the set F of all polynomials whose coefficients are in the set of {0, 1}. Let q(x) = $x^3 + x + 1$. The number of polynomials p(x) in F of degree 14 such that the product p(x)q(x) is also in F is:

Sol.
$p(x)\, q(x) = (x^{14} + \ldots)(x^3 + x + 1)$
$p(x) = x^{14} \rightarrow$ 1 case
$p(x) = x^{14} + x^2$
$\Rightarrow \alpha = 10, 9, 8, \ldots, 1, 0 \rightarrow$ 11 case
$p(x) = x^{14} + x^{\alpha} + x^{\beta}$

α=10,	β=6, 5, 4, 3, 2, 1, 0	25 cases
α=9,	β=5, 4, 3, 2, 1, 0	
α=8,	β=4, 3, 2, 1, 0	
α=7,	β=3, 2, 1, 0	
α=6,	β=2, 1, 0	
α=5,	β=1, 0	
α=4,	β=0	

$p(x) = x^{14} + x^{\alpha} + x^{\beta} + x^{r}$

α=10,	β=6	r=2, 1, 0	6 cases
α=10,	β=5	r=1, 0	
α=10,	β=4	r=0	

α=9,	β=5	r=1, 0	3 cases
	β=4	r=0	

α=8,	β=4	r=0	1 case

Hence, total case = 1 + 11 + 28 + 6 + 3 + 1 = 50 cases.

25. A finite set M of positive integers consists of distinct perfect squares and the number 92. The average of the numbers in M is 85. If we remove 92 from M, the average drops to 84. If N^2 is the largest possible square in M, what is the value of N?

Sol.

Let M contains “n” positive integers each of which is a perfect square.

$M = \{ a_1^2, a_2^2, \ldots, a_n^2, 92\}$

ATQ $\frac{1}{n+1}\{ a_1^2 + a_2^2 + \ldots + a_n^2 + 92\} = 85$ and $\frac{1}{n}\{ a_1^2 + a_2^2 + \ldots + a_n^2\} = 84$

So, from above we have 85n +85 -92=84n

Or, n=7

Therefore, $a_1^2 + a_2^2 + \ldots + a_7^2 = 84*7=588$

For largest possible square all other numbers must be the lowest square numbers i.e. $1^2, 2^2, 3^2, 4^2, 5^2, 6^2$

i.e. $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + a_7^2 = 588$

Or, $a_7^2 = 588-1-4-9-16-25-36=497$ which is not a perfect square (not possible)

But the nearest perfect square is 484 so more 13 must be subtracted where $13 = 7^2 - 6^2$

Therefore, the seven numbers will be $1^2, 2^2, 3^2, 4^2, 5^2, 7^2$ and 22^2

Required value of N is 22

Q.26. The sum of [x] for all real numbers x satisfying the equation $16 + 15x + 15x^2 = [x]^3$ is:

Sol.

$[x]^3 = 15x^2 + 15x + 16$

$[x] \in (x - 1, x]$

Or, $[x]^3 \in ((x - 1)^3, x^3]$

Or, $16 + 15x + 15x^2 \in ((x - 1)^3, x^3]$

Or, $x^3 \geq 16 + 15x + 15x^2$

Or, $x^3 - 16 - 15x - 15x^2 \geq 0$

Or, $x^3 - 16x^2 + x^2 - 16x + x - 16 \geq 0$

Or, $(x - 1)^3 < 16 + 15x + 15x^2$

Or, $x^3 - 3x^2 + 3x - 1 - 16 - 15x - 15x^2 < 0$

Or, $x^3 - 18x^2 - 12x - 17 < 0$

Or. **Let $f(x) = x^3 - 18x^2 - 12x - 17$**

Since **$x \geq 16$**

$f(16) = -721$

$f(17) = -510$

Or, x^2 (x-16) +x(x-16) +1 (x-16) ≥ 0

Or, (x-16) (x^2+x+1) ≥ 0

Or, x ≥ 16 Or, (x^2+x+1)

Combining the above 2 we get x ∈ [16, 19)

⇒ [x] = 16, 17, 18

(i) [x] = 16 ⇒$15x^2 + 15x + 16 = 16^3$ ⇒ $15x^2 + 15x - 4080 = 0$ ⇒ $x^2 + x - 272 = 0$ ⇒ $x^2 + 17x - 16x - 272 = 0$ ⇒ x= 16 or, -17	(ii) [x] = 17 ⇒$15x^2 + 15x + 16 = 17^3$ ⇒ $15x^2 + 15x - 4897 = 0$ ⇒ $x = \frac{1}{30}\{-15 \pm \sqrt{225+4*15*4897}\}$ ⇒ [x] = 17 satisfies	(iii) [x] = 18 ⇒$15x^2 + 15x + 16 = 18^3$ ⇒ $15x^2 + 15x - 5816 = 0$ Here D = 349185>0 so ⇒ x > 19 and x < −20 ⇒No value in the range

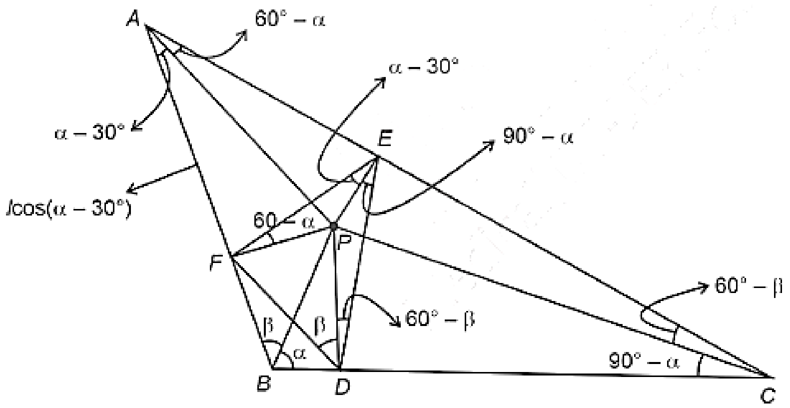
Sum of [x] = 16 + 17 = 33

Q.27. In a triangle ABC, a point P in the interior of ▲ABC is such that

∠BPC - ∠ BAC = ∠CPA - ∠ CBA=∠ APB - ∠ACB. Suppose ∠ BPC = 30° and AP = 12.

Let D, E, F be the feet of perpendiculars form P on to BC, CA, AB respectively. If m√n in the area of the triangle DEF where m, n are integers with n prime, then what is the value of the product mn ?

Sol.



$$EF^2 = l^2\left[\cos^2(\alpha - 30) + \cos^2(60 - \alpha) - \frac{2\sqrt{3}}{2}\cos(\alpha - 30)\cos(\alpha - 60)\right]$$

$=l^2\left[(\frac{\sqrt{3}}{2}\cos\alpha+\frac{1}{2}\sin\alpha)^2+(\frac{1}{2}\cos\alpha+\frac{\sqrt{3}}{2}\sin\alpha)^2-\frac{\sqrt{3}}{2}\left(\cos(2\alpha-90)+\frac{\sqrt{3}}{2}\right)\right]$

$=l^2\left[(\cos\alpha)^2+(\sin\alpha)^2+\frac{\sqrt{3}}{2}\sin\alpha\cos\alpha-\frac{3}{4}\right]$

$=l^2\left[(\frac{\sqrt{3}}{2}\cos\alpha+\frac{1}{2}\sin\alpha)^2+(\frac{1}{2}\cos\alpha+\frac{\sqrt{3}}{2}\sin\alpha)^2-\frac{\sqrt{3}}{2}\left(\cos(2\alpha-90)+\frac{\sqrt{3}}{2}\right)\right]$

$=\frac{l^2}{4}$

Area $=\frac{\sqrt{3}}{4}EF^2=\frac{\sqrt{3}}{4}\frac{144}{4}=9\sqrt{3}$

Q.28. Let $n=2^{19}3^{12}$. Let M denote the number of positive divisors of n^2 which are less than n but would not divide n. What is the number formed by taking the last two digits of M (in the same order)?

Sol.

Let $f(n)=\frac{1}{2}(n^8+3n^4-4)$

$=\frac{1}{2}(n-1)(n+1)(n^2+1)((n+1)^2+1)((n-1)^2+1)$

For $n=2k+1$

$\Rightarrow f(2k+1)=\frac{1}{2}(2k)2(k+1)(4k^2+4k+2)$ ….

Clearly $4|f(2k+1)$

$\Rightarrow$ *Only even cases*

$\Rightarrow n=28, f(28)=\frac{1}{2}\times 27\times 29\times(28^2+1)(27^2+1)(29^2+1)$

Clearly $3^2|f(28)$

When $n=26,\ f(26)=\frac{1}{2}\times 25\times 27\times(25^2+1)(26^2+1)(27^2+1)$

Clearly $5^2|f(26)$

When $n=24,\ f(24)=\frac{1}{2}\times 23\times 25\times(24^2+1)(25^2+1)(23^2+1)$

Again $5^2|f(24)$

When $n=22,\ f(22)=\frac{1}{2}\times 21\times 24\times(22^2+1)(23^2+1)(21^2+1)$

Again $5^2|f(22)$ [*since both* 22^2+1 *and* 23^2+1 *are didisible by* 5^2]

When $n=22,\ f(22)=\frac{1}{2}\times 21\times 23\times(22^2+1)(23^2+1)(21^2+1)$

Which is not divisible by any prime square.

$\Rightarrow n=20$

Q.29. Find the largest positive integer $n<30$ such that $\frac{1}{2}(n^8+3n^4-4)$ is not divisible by square of any prime number.

Sol.

$n^2=2^{38}3^{24}$

Therefore, required number of divisors $\frac{(38+1)(24+1)+1}{2}-20\times 13=228$

Therefore, answer =28

Q.30. Let ABC be a right-angled triangle with $< B = 90^0$. Let the length of the altitude BD be equal to 12. What is the minimum possible length of AC, given that AC and the perimeters of triangle ABC are integers.

Sol.

$ac = 12b$ $[since\ \Delta ABC\ and\ \Delta BDC\ are\ similar] \ldots..(i)$

Let $a + c = l$ $[l\ is\ ineger]$

$\Rightarrow a^2 + c^2 + 2ac = l^2$

$\Rightarrow a^2 + c^2 = l^2 - 24b = b^2$ [AC=b]

a and c are the roots of $x^2 - lx + 12b = 0$

$D \geq 0 \Rightarrow l^2 - 48b \geq 0 \Rightarrow b^2 - 24b \geq 0 \Rightarrow b \geq 24$

$b = k - 12, \Rightarrow k \geq 36$

$l^2 = k^2 - 144 \Rightarrow 144 = (k - l)(k + l)$

$For\ k = 37, l = 35$

Minimum value of =25

IOQM – 2023-24

QUESTIONS

1 Let n be a positive integer such that $1 \le n \le 1000$. Let M_n be the number of integers in the set $X_n=\{\sqrt{(4n+1)}, \sqrt{(4n+2)}, \ldots, \sqrt{(4n+1000)}\}$

Let $a = \max\{M_n : 1 \le n \le 1000\}$ and $b = \min\{M_n : 1 \le n \le 1000\}$. Find $a - b$.

2 Find the number of elements in the set $\{(a,b) \in N : 2 \le a, b \le 2023, \log_a(b) + 6 \log_b(a) = 5\}$.

3 Let α and β be positive integers such that $\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}$. Find the smallest possible value of β.

4 Let x, y be positive integers such that $x^4 = (x-1)(y^3-23)-1$. Find the maximum possible value of $x + y$.

5 In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of triangle ABC is 480, find the area of triangle CYZ.

6 Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X.

7 Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.

8 Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a 2×7 rectangle using some of these tiles.

9 Find the number of triples (a, b, c) of positive integers such that

(a) ab is a prime;

(b) bc is a product of two primes;

(c) abc is not divisible by square of any prime and

(d) $abc \le 30$

10 The sequence $\langle a_n \rangle_{n \ge 0}$ is defined by $a_0 = 1$, $a_1 = -4$ and $a_{n+2} = -4a_{n+1} - 7a_n$, for $n \ge 0$.

Find the number of positive integer divisors of $a_{50}^2 - a_{49}.a_{51}$.

11 A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign).

Find the maximum possible value of $|m - n|$.

12 Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd.

Let pi be the value of P(x) at x = i. Given that $p_1^2+p_2^2+p_3^3=3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

13 The ex-radii of a triangle are $10\frac{1}{2}$, 12 and 14. If the sides of the triangle are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p+q+r}$.

14 Let ABC be a triangle in the xy plane, where B is at the origin (0, 0). Let BC be produced to D such that BC : CD = 1 : 1, CA be produced to E such that CA : AE = 1 : 2 and AB be produced to F such that AB :

BF = 1 : 3. Let G(32, 24) be the centroid of the triangle ABC and K be the centroid of the triangle DEF. Find the length GK.

15 Let ABCD be a unit square. Suppose M and N are points on BC and CD respectively such that the perimeter of triangle MCN is 2. Let O be the circumcenter of triangle MAN and P be the circumcenter of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of m + n.

16 The six sides of a convex hexagon $A_1A_2\ A_3\ A_4\ A_5\ A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_i\ A_j\ A_k$, where $1 \leq i < j < k \leq 6$, has at least one red side, find the sum of the squares of the digits of N.

17 Consider the set S = {(a, b, c, d, e): 0 < a < b < c < d < e < 100}
Where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of S, find the largest integer less than or equal to D.

18 Let P be a convex polygon with 50 vertices. A set F of diagonals of P is said to be minimally friendly if any diagonal d ∈ F intersects at most one other diagonal in F at a point interior to P. Find the largest possible number of elements in a minimally friendly set F.

19 For n ∈ N, let P(n) denote the product of the digits in n and S(n) denote the sum of the digits in n. Consider the set A = {n∈N : P(n) is non-zero, square free and S(n) is a proper divisor of P(n)}. Find the maximum possible number of digits of the numbers in A.

20 For any finite non empty set X of integers, let max(X) denote the largest element of X and |X| denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

max(A) × |B| = 12; and |A| × max(B) = 11

and N can be written as 100a + b where a, b are positive integers less than 100, find a + b.

21 For n∈ N, consider non-negative integer-valued functions f on {1, 2, ..., n} satisfying
f(i) ≥ f(j) for i > j and $\sum_{i=1}^{n}\left(i + f(i)\right) = 2023.$
Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

22 In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If N denotes the number of ways we can choose the pegs such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of N.

23 In the coordinate plane, a point is called a lattice point if both of its coordinates are integers. Let A be the point (12, 84). Find the number of right-angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin (0, 0).

24 A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set {5, 6, 7, 8, 9, 10}.

25 Find the least positive integer *n* such that there are at least 1000 unordered pairs of diagonals in a regular polygon with *n* vertices that intersect at a right angle in the interior of the polygon.

26 In the land of Binary, the unit of currency is called Ben and currency notes are available in denominations 1, 2, 2^2, 2^3, …. Bens. The rules of the Government of Binary stipulate that one cannot use more than two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can give 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 Bens, following the rules of the Government.

27 A quadruple (a, b, c, d) of distinct integers is said to be balanced if $a + c = b + d$.
Let S be any set of quadruples (a, b, c, d) where $1 \le a < b < d < c \le 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in S.

28 On each side of an equilateral triangle with side length n units, where n is an integer $1 \le n \le 100$, consider $n - 1$ point that divide the side into n equal segments. Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up.

Two coins are said to be *adjacent* if the distance between them is 1unit. A move consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves.

29 A positive integer $n > 1$ is called *beautiful* if n can be written in one and only one way as $n = a_1 + a_2 + \ldots + a_k = a_1.a_2....a_k$ for some positive integers $a_1, a_2, ..., a_k$, where $k > 1$ and $a_1 \ge a_2 \ge .... \ge a_k$. (For example, 6 is beautiful since 6 = 3x2x1 = 3 + 2 + 1; and this is unique. But 8 is not beautiful since 8 = 4 + 2 + 1 + 1 = 4x2x1x1 as well as 8 =2+2+2 +1+1=2x2x2x1x1, so uniqueness is lost.)
Then Find the largest beautiful number less than 100.

30 Let d (m) denotes the no. of positive integer divisor of a positive integer m. If r is the no. of integers for n ≤ 2023 for which $\sum_{i=1}^{n} d(i)$ is odd, find the sum of the digits of r.

ANSWERS

1. 22	2. 54	3. 23	4. 07	5. 10	6. 16
7. 24	8. 85	9. 17	10. 51	11. 14	12. 18
13. 58	14. 40	15. 03	16. 94	17. 66	18. 71
19. 92	20. 43	21. 15	22. 77	23. 24	24. 31
25. 30	26. 19	27. 91	28. 67	29. 95	30. 18

SOLUTIONS

Q.1. Let n be a positive integer such that $1 \leq n \leq 1000$. Let M_n be the number of integers in the set $X_n=\{\sqrt{(4n+1)}, \sqrt{(4n+2)}, \ldots\ldots, \sqrt{(4n+1000)}\}$

Let a = max{ M_n:$1 \leq n \leq 1000$} and b = min{ M_n:$1 \leq n \leq 1000$}. Find a – b.

Sol. Let *Mn* be the number of integers in the set

$X_n = \{ \sqrt{4n+1}, \sqrt{4n+2}, \ldots, \sqrt{4n+1000} \}$ where $1 \leq n \leq 1000$

$= \{\sqrt{5}, \sqrt{6}, \ldots, \sqrt{1004}, \ldots, \sqrt{5000}\}$

a = max {M_n : $1\leq n \leq 1000$} = M_1 Cardinality of $X_1 = \{\sqrt{5}, \sqrt{6}, \ldots, \sqrt{1004}\}$

$= \{\pm 3, \pm 4, \pm 31, \ldots, \pm 31\} = 29$

b = min M_n = Cardinality of $X_{1000} = \{\sqrt{4001}, \ldots, \sqrt{5000}\} = \{\pm 64, \ldots, \pm 70\} = 7$

Therefore, a – b = 29-7 = 22

Q.2. Find the number of elements in the set $\{(a,b)\in N : 2 \leq a, b \leq 2023, \log_a(b) + 6 \log_b(a) = 5\}$

Sol. $\log_a(b) + 6 \log_b(a) = 5$

$\Rightarrow \log_a(b) + \frac{6}{\log_b(a)} = 5$

let $\log_a b = t$

$\Rightarrow t + \frac{6}{t} = 5$

$\Rightarrow t^2+6-5t=0 \Rightarrow t = 2$ or 3

case 1 $\log_a b = 2 \Rightarrow b=a^2$

$a^2 \leq 2023$

$\Rightarrow a \leq 44.97$

$\Rightarrow a \in \{2, \ldots., 44\}$

$\Rightarrow$ no. of ways = 44 – 2 + 1 =43

case II $\log_a b = 3 \Rightarrow b=a^3$

$a^3 \leq 2023$

$a \leq 12.64$

$a \in \{2, \ldots,12\}$

no. of ways (12 – 2 + 1) = 11

$\Rightarrow$ total no. of ways = 43 + 11 = 54

Q.3. Let α and β be positive integers such that $\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}$. Find the smallest possible value of β.

Sol. $\frac{\mathbf{16}}{\mathbf{37}} < \frac{\alpha}{\beta} < \frac{\mathbf{7}}{\mathbf{16}}$

$\Rightarrow \quad \frac{\mathbf{37}}{\mathbf{16}} < \frac{\beta}{\alpha} < \frac{\mathbf{16}}{\mathbf{7}}$

$\Rightarrow \quad \frac{\mathbf{37}}{\mathbf{16}}\alpha < \beta < \frac{\mathbf{7}}{\mathbf{16}}\alpha$

$\Rightarrow \quad \frac{\mathbf{2}}{\mathbf{7}}\alpha < \beta - 2\alpha < \frac{\mathbf{5}}{\mathbf{16}}\alpha$, $\beta - 2\alpha \in I$ and lie between $(\frac{\mathbf{2}}{\mathbf{7}}\alpha, \frac{\mathbf{5}}{\mathbf{16}}\alpha) \equiv (\frac{\mathbf{32}}{\mathbf{112}}\alpha, \frac{\mathbf{35}}{\mathbf{112}}\alpha)$

$\Rightarrow \quad \beta - 2\alpha \in (0.28\alpha, 0.31\alpha)$

If $\alpha = 10 \Rightarrow (2.8, 3.1)$

$\Rightarrow \quad \beta - 2\alpha = 3$

$\Rightarrow \quad \beta = 23$

Q.4. Let x, y be positive integers such that $x^4 = (x-1)(y^3 - 23) - 1$. Find the maximum possible value of x + y.

Sol. $\sqrt{(4n+r)} = k \in I$

$\Rightarrow 4n + r = k^2$

$\frac{x^4+1}{x-1} = y^3 - 23$ where x, y$\in$I

Since, $x \neq 1 \Rightarrow x \geq 2$

$y^3 - 23 \in I$

$\Rightarrow \frac{x^4+1}{x-1} \in I$

Let, x – 1 = p

$\Rightarrow \frac{(1+p)^4+1}{p}$

$\Rightarrow \frac{(a_4p^4 + a_3p^3 + \ldots\ldots\ldots a_1p + 2)}{p}$

$\Rightarrow \frac{2}{P} \Rightarrow$ p divides 2

$\Rightarrow p = \{-2, -1, 1, 2\}$

$\Rightarrow x \in \{-1, 0, 2, 3\}$

but $x \geq 2 \Rightarrow x = 2$ or 3

If x = 2 $\quad 2^4 = 1 \times (y^3 - 23) - 1$

$\Rightarrow (17 + 23) = y^3$

$\Rightarrow y \notin I$

If x = 3 $\Rightarrow (3^4 + 1) = 2(y^3 - 23)$

$\Rightarrow y = 4$

$\Rightarrow x + y = 7$

Q.5. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of triangle ABC is 480, find the area of triangle CYZ.

Sol.

Let BE = 6x

CF = 6y

$\frac{area\ (GYZ)}{area\ (GBC)} = (1/4)^2$

Area (GBC) = 16(area of GYZ)

area of GBC = $\frac{1}{3}$of area (ABC) = $\frac{480}{3}$ = 160

area of GYZ = 10

Q.6. Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X.

Sol. Let number of sides of polygon be P.

$\therefore \frac{(P-2).180}{P} = n^\circ$

then P (180 – n) = 360

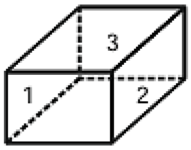
$\therefore P = \frac{360}{180-n}$

But n = 2k then $P = \frac{180}{90-k}$

Hence possible values of (90 – k) are 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45 and 60

$\therefore$ 16 such polygons are possible.

Q.7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.



Sol. Fix 1 and 2 anywhere in 1 way

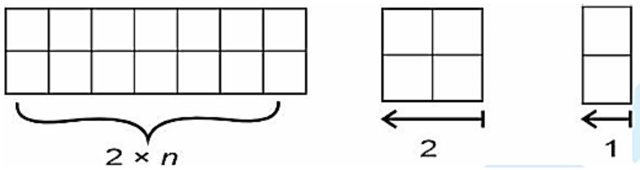
Fix 3 anywhere in 1 way

Now remaining 3 faces can be filled in $\frac{3!}{2}$ ways and coloring them in $2 \times 2 \times 2$ ways

$\therefore$ Total distinct dice $= \frac{3!}{2} \times 8 = 24$ ways

Q.8. Given a 2 × 2 tile and seven dominoes (2 × 1 tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a 2 × 7 rectangle using some of these tiles.

Sol.



$a_n = 2a_{n-2} + a_{n-1}$

$a_1 = 1, a_2 = 3$

$a_3 = 2a_1 + a_2 = 2(1) + 3 = 5$

$a_4 = 2a_2 + a_3 = 2(3) + 5 = 11$

$a_5 = 2a_3 + a_4 = 2(5) + 11 = 21$

$a_6 = 2(11) + 21 = 43$

$a_7 = 2(21) + 43 = 42 + 43 = 85$

Q.9. Find the number of triples (a, b, c) of positive integers such that

(a) ab is a prime;

(b) bc is a product of two primes;

(c) abc is not divisible by square of any prime and

(d) $abc \leq 30$

Sol.

ab is prime mean one of them is 1.

$abc \not\in \{4, 9, 25\}$

Case I : $a = 1 \Rightarrow b$ is prime and bc prime product

$\Rightarrow$ c is prime

$\Rightarrow$ abc $\leq$ 30

$\Rightarrow$ bc $\leq$ 30 & bc $\notin$ {4, 9, 25}

$\Rightarrow$ (b, c) are different prime

$\Rightarrow$ b = {2, 3, 5, 7}

c = {3, 5, 7} and b $\neq$ c. Then total number of cases = 14.

Case II : When b = 1 then possible values of (a, b, c) are (2, 1, 15), (3, 1, 10) and (5, 1, 6).

Total number of ways = 14 + 3 = 17

Q.10. The sequence $\langle a_n \rangle_{n \geq 0}$ is defined by $a_0 = 1$, $a_1 = -4$ and $a_{n+2} = -4a_{n+1} - 7a_n$, for $n \geq 0$.

Find the number of positive integer divisors of $a_{50}^2 - a_{49}.a_{51}$.

Sol. The sequence $\{a_n\}$, $n \geq 0$ and $a_0 = 1$, $a_1 = -4$

and given that $a_{n+2} = -4a_{n+1} - 7a_n$ ....(1)

Now $a^2_n - a_{n+1}.a_{n-1} = a^2_n - (4a_n - 7a_{n-1})a_{n-1}$

$= a^2_n + 4a_n.a_{n-1} + 7a^2_{n-1}$

$= -a_n(7a_{n-2}) + 7a^2_{n-1}$

$= 7^2(a_{n-1}^2 - a_{n-1}.a_{n-3})$

....

$\therefore (a_n)^2 - a_{n+1}.a_{n-1} = 7^{n-1}(a_1^2 - a_0.a_2) = 7^{n-1}(16 - 1.9) = 7^2$

$\therefore (a_{50})^2 - a_{49}.a_{51} = 7^{50}$

$\therefore$ Number of positive divisors of 7^{50} = 50+1 = 51

Q.11. A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$.

Sol. Since $m^2 = 4n^2 - 5n + 16$

$\therefore 4n^2 - 5n + 16 - m^2 = 0$

$8n = 5 \pm \sqrt{(16m^2 - 231)}$

Let $D^2 = 16m^2 - 231$ and $D > 0$

$\Rightarrow (4m + D)(4m - D) = 231 = 7 \times 11 \times 13$

Case-I:

$4m + D = 231$ and $4m - D = 1$

then $m = 29$ and $D = 115$, $n = 15$

$\therefore$ $|m - n| = 14$

Case-II:

$4m + D = 77$ and $4m - D = 3$

$\therefore$ $m = 10$, $D = 37$ and $n = -4$

$\therefore$ $|m - n| = 14$

Case-III: $4m + D = 33$ and $4m - D = 7$

$\therefore$ $m = 5$, $D = 13$ and $n = -1$

$\therefore$ $|m - n| = 6$

Case-IV: $4m + D = 21$ and $4m - D = 11$

$\therefore$ $m = 4$, $D = 5$ and $n = 0$

Hence $|m - n| = 4$

Maximum value of $|m - n| = 14$

Q.12. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd.

Let pi be the value of P(x) at x = i. Given that $p_1^2+p_2^2+p_3^3=3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

Sol: $p_1^2+p_2^2+p_3^3=3p_1p_2p_3$

$\therefore$ either $p_1+p_2+p_3=0$ or $p_1=p_2=p_3$

Here $p_1 = a + b + c + 1$ .....(i)

$P_2 = 4a + 2b + c + 8$ .....(ii)

$P_3 = 9a + 3b + c + 27$ .....(iii)

Here $p_1+p_2+p_3= 14a + 6b + 3c + 36$

Here $(14a + 6b + 36) + 3c = 0$

Even + odd $\neq$ even (c is odd)

$\therefore$ either $p_1+p_2+p_3\neq 0$

Hence, either $p_1 = p_2 = p_3$ ...(iv)

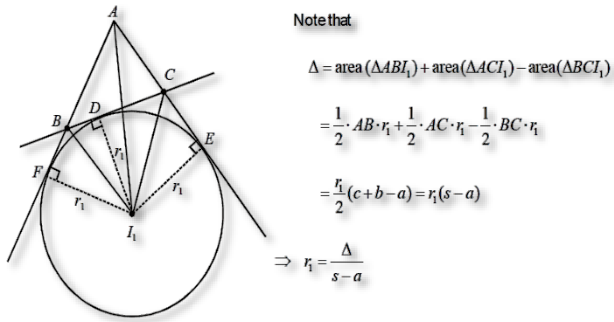
Hence solving equation (i), (ii), (iii) and (iv) we get

$a = -6$, $b = 11$

but $p_2+2p_1-3p_0 = 6a + 4b + 10 = 18$

Q.13. The ex-radii of a triangle are $10\frac{1}{2}$,12 and 14. If the sides of the triangle are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p+q+r}$.

Sol. $r_1 = \frac{\Delta}{S-a} = \frac{21}{2}$



$r_2 = \frac{\Delta}{S-b} = 12$

$r_1 = \frac{\Delta}{S-c} = 14$

On solving above equations,

we get a = 13, b = 14, c = 15

Let $f(x) = x^3 - px^2 + qx - r$

$= (x - 13)(x - 14)(x - 15)$

$f(-1) = -1 - p - q - r$

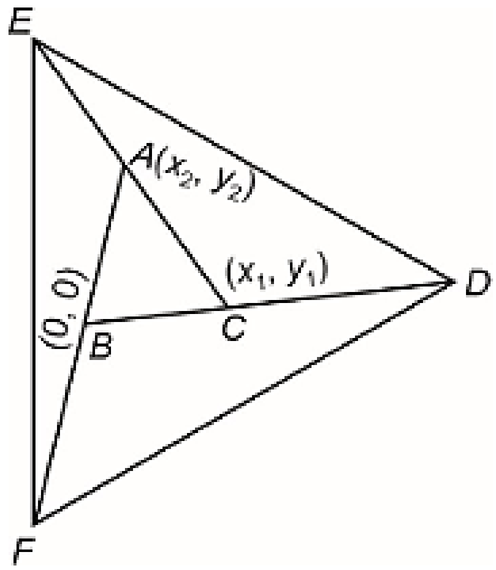
$= (-14)(-15)(-16)$

$\therefore\ p + q + r = 14 \cdot 15 \cdot 16 - 1$

$\therefore\ \sqrt{p+q+r} = \sqrt{3364} \approx 58$

Q.14. Let ABC be a triangle in the xy plane, where B is at the origin (0, 0). Let BC be produced to D such that BC : CD = 1 : 1, CA be produced to E such that CA : AE = 1 : 2 and AB be produced to F such that AB : BF = 1 : 3. Let G(32, 24) be the centroid of the triangle ABC and K be the centroid of the triangle DEF. Find the length GK.

Sol.



Given BC : CD = 1 : 1

Let coordinate of D (x,y)

C(x_1,y_1)=$\left(\frac{1\times x+1\times 0}{1+1},\frac{1\times y+1\times 0}{1+1}\right)$

(x_1,y_1)=$\left(\frac{x}{2},\frac{y}{2}\right)$

hence D = ($2x_1$, $2y_1$)

similarly,

CA : AE = 1 : 2 hence E = ($3x_2 - 2x_1$, $3y_2 - 2y_1$)

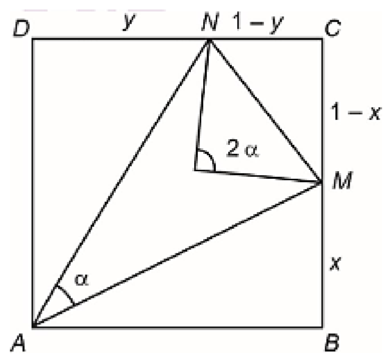
AB : BF = 1 : 3 hence F = ($-3x_2$, $-3y_2$)

$\therefore$ Centroid of ΔDEF = K = (0, 0)

Given Centroid of ΔABC = G = (32, 24)

$\therefore$ GK =$\sqrt{32^2+24^2} = 40$

Q.15. Let ABCD be a unit square. Suppose M and N are points on BC and CD respectively such that the perimeter of triangle MCN is 2. Let O be the circumcenter of triangle MAN and P be the circumcenter of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of m + n.



Sol. OA = circumradius of ΔAMN

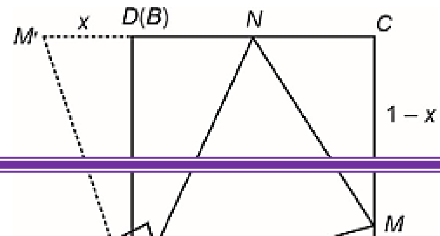
$= \frac{MN}{2\sin\alpha}$

OP = circumradius of ΔOMN

$= \frac{MN}{2\sin 2\alpha}$

$\left(\frac{OP}{OA}\right)^2$=$\left(\frac{1}{2\cos\alpha}\right)^2$

Perimeter of ΔMCN = 2 = (1 – x) + (1 – y) + MN



$\Rightarrow$ MN = x + y

Now rotate ΔABM about A so that AB overlaps with AD (by 90°)

Clearly ΔAMN ≡ ΔAM'N (by SSS)

So, angle M'AN=angle MAN= α

$2\alpha = 90°$

$\therefore$ $\alpha = 45°$

Hence $\left(\frac{OP}{OA}\right)^2 = \frac{1}{2} = \frac{m}{n}$

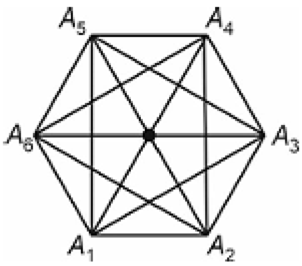
$\therefore$ m + n =3

Q.16. The six sides of a convex hexagon $A_1A_2\ A_3\ A_4\ A_5\ A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_i\ A_j\ A_k$, where $1 \le i < j < k \le 6$, has at least one red side, find the sum of the squares of the digits of N.

Sol. Number of ways such that at least one side of $\Delta A_2A_3A_6$ is red

$= {}^3C_1 \times 2^2 - {}^3C_1$ [illegible]

$=7$



Number of ways such that at least one side of $\Delta A_1A_3A_5$ is red = 7

Number of ways to color diagonals A_1A_4, A_2A_5, A_2A_6 = $2^3 = 8$

$\therefore$ Required number = 8 × 7 × 7 = 392 = N

$\therefore$ Sum of square of digits = $3^2 + 9^2 + 2^2$

= 94

Q.17. Consider the set S = {(a, b, c, d, e): 0 < a < b < c < d < e < 100}

Where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of S, find the largest integer less than or equal to D.

Sol. d = 4 sum = $4 \times {}^{95}C_1 \times {}^3C_3$

d = 5 sum = $5 \times {}^{94}C_1 \times {}^4C_3$

d = 6 sum = $6 \times {}^{93}C_1 \times {}^5C_3$

⋮

d = 98 sum = $98 \times {}^{1}C_1 \times {}^{97}C_3$

Total = $4.{}^{95}C_1 . {}^{3}C_3 + 5 \cdot {}^{94}C_1 \cdot {}^{4}C_3 \ldots \times 98\, {}^{1}C_1 \cdot {}^{97}C_3$

$=4 \sum_{r=1}^{95} \frac{99-r}{4} . r\, {}^{98-r}C_3$

$=4\sum_{r=1}^{95} r\, {}^{98-r}C_3$

$=4\times {}^{100}C_6$

$\therefore$ A.M. = $4 \times {}^{100}C_6 / {}^{99}C_5$

$=\frac{200}{3}$

$\therefore \left[\frac{200}{3}\right] = 66$

Q.18. Let P be a convex polygon with 50 vertices. A set F of diagonals of P is said to be minimally friendly if any diagonal d $\in$ F intersects at most one other diagonal in F at a point interior to P. Find the largest possible number of elements in a minimally friendly set F.

Sol.

Total number of non-intersecting diagonals

$A_1A_3, A_1A_4, A_1A_5, \ldots, A_1A_{49} \rightarrow 47$

Total number of intersecting diagonals at only one point to the non-intersecting diagonals

$A_2A_4, A_4A_6, A_6A_8, \ldots, A_{48}A_{50} \rightarrow 24$

Total = 47 + 24 = 71

Q.19. For n $\in$ N, let P(n) denote the product of the digits in n and S(n) denote the sum of the digits in n. Consider the set A = {n$\in$N : P(n) is non-zero, square free and S(n) is a proper divisor of P(n)}. Find the maximum possible number of digits of the numbers in A.

Sol.

P(n) is non-zero, square free & S(n)/P(n)

$\Rightarrow n = x_1 x_2 x_3 \ldots x_n \Rightarrow x_i \neq 0$

x_i are square free & distinct

$\Rightarrow x_i \in \{1, 5, 6, 7\}$ or $\{1, 2, 3, 5, 7\}$

$P(n) = x_1 x_2 \ldots x_n$

$S(n) = x_1 + x_2 + \ldots + x_n$

Now, S(n) is proper divisor of P(n)

and P(n) max is 2×3×5×7 therefore S(n) maximum is 3×5×7 = 105

⇒ For S(n) = 105

⇒ 2+2+5+7+1+1+….+1(1 is x times) = 105

⇒ x = 88 ⇒ (88 + 4) = 92 digits.

Q.20. For any finite non-empty set X of integers, let max(X) denote the largest element of X and |X| denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

$\max(A) \times |B| = 12$; and $|A| \times \max(B) = 11$

and N can be written as 100a + b where a, b are positive integers less than 100, find a + b.

Sol.

$A = \{a_1, a_2, a_3\ldots a_p\}$

$B = \{b_1, b_2, b_3\ldots b_q\}$

$A_{p.q} = 12$

$p \cdot b_q = 11$

Case-A : $p = 11, b_q = 1$

$A = \{a_1, a_2, a_3\ldots a_{11}\}, B = \{1\}$

$\Rightarrow a_{11} = 12, q = 1$

$\therefore\ {}^{11}C_{10}$ = total ways

Case-B : p = 1, bq = 11

(1) A = {12}, B = {11} → 1 way

(2) $A = \{6\}, B = \{b_1, 11\} \rightarrow {}^{10}C_1$ ways

(3) $A = \{4\}, B = \{b_1, b_2, 11\} \rightarrow {}^{10}C_2$ ways

(4) $A = \{3\}, B = \{b_1, b_2, b_3, 11\} \rightarrow {}^{10}C_3$ ways

(5) $A = \{2\}, B = \{b_1, b_2, b_3, b_4, b_5, 11\} \rightarrow {}^{10}C_5$ ways

(6) $A = \{1\}, B = \{b_1, b_2, \ldots b_{11}, 11\} \rightarrow 0$ ways

∴ Total ways = 11 + 1 + 10 + 45 + 120 + 252

= 439

= 100 × 4 + 39

a + b = 43

Q.21. For n∈N, consider non-negative integer-valued functions f on {1, 2, ..., n} satisfying

f(i) ≥ f(j) for i > j and $\sum_{i=1}^{n}\left(i + f(i)\right) = 2023$.

Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

Sol.

$$\sum_{i=1}^{n}\left(i + f(i)\right) = 2023.$$

$$\sum_{i=1}^{n} f(i) = 2023 - \frac{n(n+1)}{2}$$

For $\sum_{i=1}^{n} f(i)$ be least, n=63

$\therefore$ $\sum_{i=1}^{n} f(i)$ least =7

Total number of possible functions equal to number of possible partitions of 7

i.e. 7 = 1 + 1 + 1 + 1 + 1 + 1 + 1

= 1 + 1 + 2 + 3

……………………..

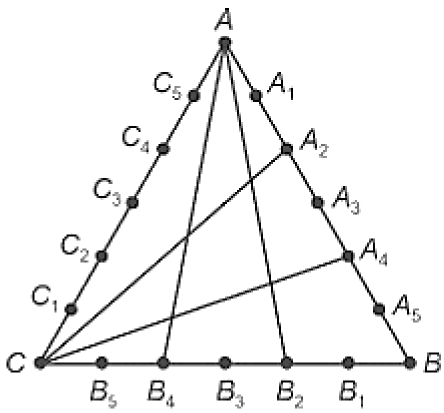
……………………..

Total number of partitions = 15

Number of possible functions = 15

Q.22. In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If N denotes the number of ways we can choose the pegs such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of N.

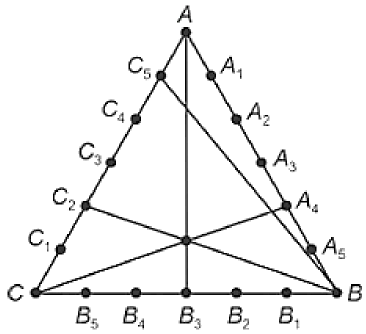
Sol. **case I:**



To divide the triangle into 9 regions. Two pegs must be selected from a side and the other two from a different side.

This can be done in ${}^3C_2 \times {}^5C_2 \times {}^5C_2 = 300$ ways.

case II:



Now we are choosing 3 points on three sides, such that three lines from those points are concurrent.

By using Ceva's theorem in which product of three different ratio leads to 1.

Possible ratio on side AB, BC and CA will be of the form

i.e. $\frac{m}{n} \times \frac{n}{m} \times 1 = 1$

Ratio 1 : 1 can be chosen in 3 ways for all three sides other ratio can be choosen in 4 ways for other two sides

i.e. there are 3 × 4 + 1 = 13 ways

Fourth point can be chosen in ${}^{12}C_1$, ways

Total such possibilities = 12 × 13 = 156 ways

Total ways = 300 + 156 = 456

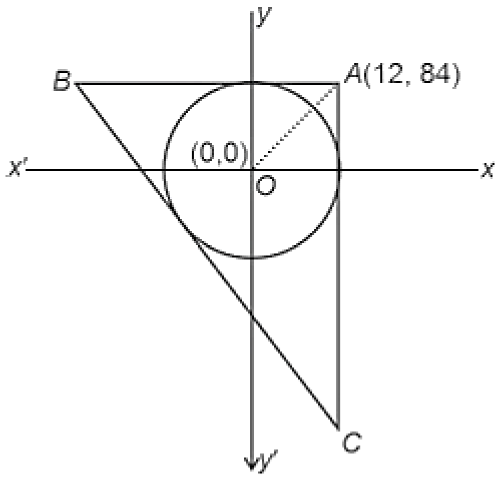
Sum of squares of digit = $4^2 + 5^2 + 6^2 = 16 + 25 + 36 = 77$

Q.23. In the coordinate plane, a point is called a lattice point if both of its coordinates are integers. Let A be the point (12, 84). Find the number of right-angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin (0, 0).

Sol.

We know that if inradius is integer then sides of triangle is also integer.

Here OA = $60\sqrt{2}$



Inradius = 60

Hence AB, BC, CA are integer.

Let AB = $m^2 - n^2$

BC = $m^2 + n^2$

AC = $2 m^2 + n^2$

Here, r = $\frac{\Delta}{s} = \frac{mn.(m^2-n^2)}{3(m+n)}$ = n(m - n) = 60

$\therefore$ n(m – n) = $2^2.3.5$.

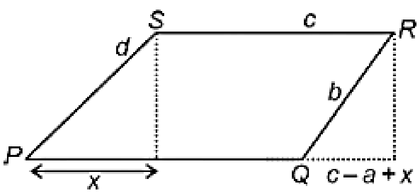
$\therefore$ Possible values of n are 12.

Then 12 such triangles are possible but when B and C interchange their position 12 more triangles are possible.

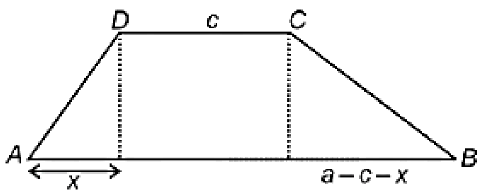
Then total number of distinct triangles = 24.

Q.24. A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set {5, 6, 7, 8, 9, 10}.

Sol. Without losing generality, assume a > c and d > b and sides AB || CD



or



$d^2 - x^2 = b^2 - (c - a + x)^2,$

similarly, $d^2 - x^2 = b^2 - (a - c + x)^2$

$\Rightarrow x = \frac{(a-c)^2+d^2-b^2}{2(a-c)}$

If x $\in$ (0, d), then there will be unique trapezoid

$\Rightarrow \frac{(a-c)^2+d^2-b^2}{2c(c-a)} \in$ (0, d)

$\Rightarrow$ $(a - c)^2 + d^2 - b^2 - 2d (a - c) < 0$

$\Rightarrow (a - c - d)^2 - b^2 < 0$

$\Rightarrow (a - c - d - b)\,(a - c - d + b) < 0$

$\Rightarrow (a - c - d + b) > 0$

$\Rightarrow a + b > c + d$ and $a > c$, $d > b$

Using this inequality, numerate these pairs (a, b, c, d)

Case I : a = 10 $\Rightarrow$ total no. of cases = 16

Case II : a = 9 $\Rightarrow$ total no. of cases = 9

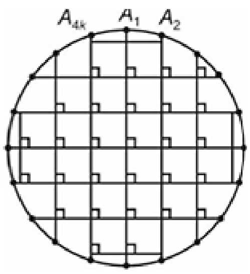
Case III : a = 8 $\Rightarrow$ total no. of cases = 4

Case IV : a = 7 $\Rightarrow$ (7, 9, 5, 10) and (7, 8, 5, 9) $\Rightarrow$ 2 cases

$\Rightarrow$ Total = 31

Q.25. Find the least positive integer *n* such that there are at least 1000 unordered pairs of diagonals in a regular polygon with *n* vertices that intersect at a right angle in the interior of the polygon.

Sol. -



Case 1

Let n = 4k

So (1+3+5+..........+ (2k-1) +......... +3+1)* k

$= k^2 + (k-1)^2\,) \times k \geq 1000$

Or, $k(2k^2-2k+1) \geq 1000$

Or, $k \geq 9$ as $k \in N$ Or, $4k \geq 36$ Or, $n \geq 36$

Case 2 –

Let n = 4k+2

$(1+3+5+...+2k-1).2.(2k+1) \geq 1000$

Or, $(2k+1).2k^2 \geq 1000$

Or, $(2k+1).k^2 \geq 500$

Or, $k \geq 7$

Or, $n \geq 30$

Therefore min (30, 36) = 30

Q.26. In the land of Binary, the unit of currency is called Ben and currency notes are available in denominations 1, 2, 2^2, 2^3, …. Bens. The rules of the Government of Binary stipulate that one cannot use more than two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can give 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 Bens, following the rules of the Government.

Sol.

1,2,2^2……….

$\therefore a+2b+4c+.... = 100$ $\qquad$ $0 \le a \le 2,\ 0 \le b \le 2$

a=0	a=1	a=2
$\Rightarrow b+2c+.... = 50$	$\Rightarrow 2(b+2c+...) = 99$(ODD) $\Rightarrow a=1$ (Not possible)	$\Rightarrow b+2c+..... = 49$ $\Rightarrow c+2d+...... = 24$

Let the no. of solutions be a_{100}

$a_{100} = a_{50} + a_{24}$ $\quad (a_n = a_{n/2} + a_{n/4-1})$

$= (a_{25} + a_{24}) + a_{24}$

$= a_{25} + 2a_{24}$

$= a_{12} + 2(a_{12} + 2a_5)$

$= 3\,a_{12} + 2a_5$

$= 3\,(a_6 + a_2) + 2a_2$

$= 5a_2 + 3a_6$

Let the no. of solutions be a_{100}

For a_3

Let $a+2b+4c=3$

If a=1 then $2b+4c=2$

Or $b+2c=1$

So, b=1, c=0

Hence (1, 1, 0) is the only solution

Therefore $a_3 = 1$

For a_2: $a+2b=2$

Q.27. A quadruple (a, b, c, d) of distinct integers is said to be balanced if $a + c = b + d$. Let S be any set of quadruples (a, b, c, d) where $1 \le a < b < d < c \le 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in S.

Sol.

Given $a+c=b+d$, $1 \le a<b<d<c \le 20$

i.e a, b , c, d lies from 1 to 20

SUM	**Ordered pairs (a,c) & (b,d)**	**No. of possibilities**	**Selecting a,b,c,d**
$a+c=37$	(20,17), (18,19)	2	1way
$a+c=36$	(20,16), (19,17)	2	1way
$a+c=35$	(20,15), (19,16), (18,17)	3	$^3C_2$way
$a+c=34$	(20,14), (19,15), (18, 16)	3	$^3C_2$way
$a+c=33$	(20,13), (19,14),… (18,15), (17,16)	4	$^4C_2$way
$a+c=32$	(20,12), (19,13)…(17,15)	4	$^4C_2$way
$a+c=31$	(20,11),…………., (16,15)	5	$5C_2$way
$a+c=30$	(20,10),……….., (16,14)	5	$5C_2$way
$a+c=29$	(20,9),………….,(15, 14)	6	$6C_2$way
$a+c=28$	(20,8),……………, (15,13)	6	$6C_2$way
$a+c=27$	(20,7),……………., (14,13)	7	
$a+c=26$	(20,6),……………, (14,12)	7	
$a+c=25$	(20,5),……………, (13,12)	8	
$a+c=24$	(20,4),……………, (13,11)	8	
$a+c=23$	(20,3),……………, (12,11)	9	
$a+c=22$	(20,2),……………, (12,10)	9	
$a+c=21$	(20,1),………………,(11, 10)	10	
$a+c=20$	(19,1),(18,2)…………. (11,9)	10	
$a+c=19$	(18,1)….(10,9)	9	
$a+c=18$	(17,1),….(10,8)	8	
$a+c=17$			
:			
:			
$a+c=5$			

Q.28. On each side of an equilateral triangle with side length n units, where n is an integer $1 \le n \le 100$, consider $n-1$ points that divide the side into n equal segments. Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up.

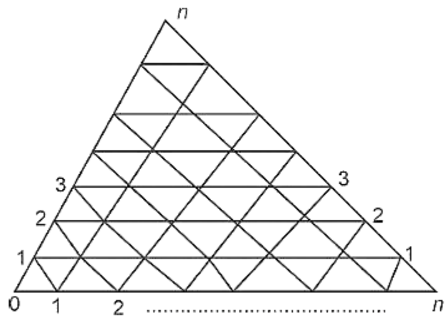
Two coins are said to be *adjacent* if the distance between them is 1unit. A move consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves.

Sol.

The total no. of vertices= (n+1) (n+2)/2 , n is the no. of triangles

Let Head be denoted as 0 and The Tail is denoted by 1

Initial sum of all vertices = 0



After each move three “0“becomes three ‘1‘.

Let after kth move the sum be f(k)

So ATQ f(k+1) = f(k) +3

Finally, number of tails sum = (n+1) (n+2)/2

As initially it was o so (n+1) (n+2)/2 is divisible by 3

Therefore, n can’t be multiple of 3 and so n= 3k+1 Or, 3k+2

Number of such n which are less than 100 are 67.

Q.29. A positive integer $n > 1$ is called *beautiful* if n can be written in one and only one way as $n= a_1+ a_2+ \dots + a_k= a_1.a_2\dots.a_k$ for some positive integers $a_1, a_2, \dots, a_k$, where $k > 1$ and $a_1 \geq a_2 \geq \dots \geq a_k$. (For example 6 is beautiful since 6 = 3x2x1 = 3 + 2 + 1; and this is unique. But 8 is not beautiful since 8 = 4 + 2 + 1 + 1 = 4x2x1x1 as well as

8 =2+2+2 +1+1=2x2x2x1x1, so uniqueness is lost.) Then Find the largest beautiful number less than 100.

Sol.

99=9×11×1×1 ×1×1.....................................×1(79 times ‘1’)
=9+11+1+1+------------------------79 times
=33×3×1×1×1(66times1)
=33+3+1+1+...........................66times
Hence 99 is not beautiful.
98=49 ×2×1×1 ×47 times
=49+2+1+1+......................... 47times
=14×7×1× 1×...... 77 times

=14+7+1+1+77times

Hence 98 is not beautiful.

Also 97 is not beautiful as it cannot be written in these expanded forms.

96 = 2 × 48 × 1 × 1 × ....46 times

=2+48+1+1+.........................46times

=3×32×1×1×.........................61times

=3+32+1+1+......1times Hence 96 is not beautiful.

95=19 ×5×1×1×......................71times

=19+5+1+1+.........................71times

As 95 is uniquely represents hence it is beautiful.

Q.30. Let d (m) denotes the no. of positive integer divisor of a positive integer m. If r is the no. of integers for n ≤ 2023 for which $\sum_{i=1}^{n} d(i)$ is odd, find the sum of the digits of r.

Ans.- Given $_{i=1}{}^{n}$ d(i) is odd

Or, d(1)+d(2)+d(3) +d(4) +........ + d(n) is odd.

d(square number) = odd as every square no has exactly 3 factors except 1

for n=1 $\sum_{i=1}^{n} d(i)$= odd

for n=2 $\sum_{i=1}^{n} d(i)$= 3 (odd)

For n=3

$\sum_{i=1}^{n} d(i)$ = d(1) + d(2) + d(3) = 1+2+2 = odd + even + even= odd

For n=4

$\sum_{i=1}^{n} d(i)$ = even as d(4) is odd {1,2,4}

similarly for n= 5,6,7,8 $\sum_{i=1}^{n} d(i)$ is even

For n=9 $\sum_{i=1}^{n} d(i)$= 1+2+2+3+2+4+2+4+3=23 (odd)

For n=10, $\sum_{i=1}^{n} d(i)$ = odd

For n=11, $\sum_{i=1}^{n} d(i)$ is again odd

For n=12, n=13, n=14, n=15 $\sum_{i=1}^{n} d(i)$ = odd

But for n=16 $\sum_{i=1}^{n} d(i)$ = odd as 16 has odd no. of factors

So, n ϵ [1,3] and n ϵ [9,15]

i.e. n ϵ [1^2, 2^2-1] or n ϵ [3^2, 4^2-1]

Similarly, n ϵ [5^2, 6^2-1] and n ϵ [7^2, 8^2-1] … n ϵ [43^2, 44^2-1]

as 43^2= 1849 44^2 = 1936 45^2 =2025

No. of elements in

$[1,3] = [1^2, 2^2-1] = 2^2 - 1 - 1^2 + 1 = 3$

$[9,15] = [3^2, 4^2-1] = 4^2 - 1 - 3^2 + 1 = 7$

$[5^2, 6^2-1] = 6^2 - 1 - 5^2 + 1 = 11$

$[7^2, 8^2-1] = 8^2 - 1 - 7^2 + 1 = 15$

$[43^2, 44^2-1] = 44^2 - 1 - 43^2 + 1 = 87$

r = 3+7+11+15+…….+87 = 990

The sum of the digits of r =18

IOQM 2022-23
QUESTIONS

1 A triangle ABC with AC=20 is inscribed in a circle ω. A tangent t to ω is drawn through B. The distance of t from A is 25 and that from C is 16. If S denotes the area of the triangle ABC, find the largest integer not exceeding S/20.

2 In a parallelogram ABCD, the point P on a segment AB is taken such that $\frac{AP}{AB} = \frac{61}{2022}$ and a point Q on the segment AD is taken such that $\frac{AQ}{AD} = \frac{61}{2065}$, if PQ intersects AC at T, find $\frac{AC}{AT}$ to the nearest integer.

3 In a trapezoid ABCD, the internal bisector of $\angle A$ intersects the base BC(or its extension) at the point E. Inscribed in the triangle ABE is a circle touching the side AB at M and BE at P. Find the $\angle DAE$ in degrees, if AB:MP=2.

4 Starting with a positive integer M, written on the board, Alice plays the following game: In each move, if x is the number on the board, she replaces it with $3x + 2$. Similarly, starting with a positive integer N, written on the board, Bob plays the following game: In each move if x is the number on the board, he replaces it with $2x + 27$. Given that Alice and Bob reach the same number after playing four moves each, find the smallest value of $M + N$.

5 Let m be the smallest positive integer such that $m^2 + (m+1)^2 + \cdots + (m+10)^2$ is the square of a positive integer n. Find $m + n$.

6 Let a, b be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$.

7 Find the number of ordered pair (a,b) such that $a, b \in \{110,11, \ldots \ldots 29,30\} and\ GCD(a,b) + LCM(a,b) = a + b$.

8 Suppose the prime numbers p and q satisfies $q^2 + 3p = 197p^2 + q$.Write $\frac{q}{p}$ in the form $l + \frac{m}{n}$, where l,m,n are positive integers $m < n\ and\ GCD(m,n) = 1$.
Find the maximum value of l+m+n.

9 Two sides of an integer sided triangle have lengths 18 and *x*. If there are exactly 35 possible integer values *y* such that 18, *x*, *y* are the side of a non-degenerate triangle, find the number of possible integer values *x* can have.

10 Consider the 10-digit number *M* = 9876543210. We obtain a new 10-digit number from *M* according to the following rule: we can choose one or more disjoint pairs of adjacent digits in *M* and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from *M* = $9\underline{87}6\underline{54}3210$, by interchanging the 2 underlined pairs, and keeping the others in their places, we get $M_1 = 9\overline{78}6\overline{45}3210$. Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from *M*.

11 Let *AB* be a diameter of a circle x and let *C* be a point on x, different from *A* and *B*. The perpendicular from *C* intersects *AB* at *D* and ω at *E*($\neq$ C). The circle with centre at *C* and radius *CD* intersects ω at *P* and *Q*. If the perimeter of the triangle *PEQ* is 24, find the length of the side *PQ*.

12 Given $\triangle ABC$ with $\angle B = 60°$ and $\angle C = 30°$, let *P*, *Q*, *R* be points on sides *BA*, *AC*, *CB* respectively such that *BPQR* is an isosceles trapezium with *PQ*||*BR* and *BP* = *QR*. Find the maximum possible value of $\frac{2[ABC]}{[BPQR]}$ where [*S*] denotes the area of any polygon *S*.

13 Let *ABC* be a triangle let *D* be a point on the segment *BC* such that *AD* = *BC*. Suppose ∠*CAD* = *x*°, ∠*ABC* = *y*° and ∠*ACB* = *z*° and *x*, *y*, *z* are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of ∠*ABC* in degrees.

14 Let x, y, z be complex numbers such that

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=9$$

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}=68$$

$$\frac{x^3}{y+z}+\frac{y^3}{z+x}+\frac{z^3}{x+y}=\ 488$$

$$if\ \frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}=\frac{m}{n}$$

where m,n are positive integers with gcd(m,n)=1, find m+n.

15 Let x, y be real numbers such that $xy = 1$. Let T and t be the largest and the smallest values of the expression $\frac{(x+y)^2-(x-y)-2}{(x+y)^2+(x-y)-2}$ if T+t can be expressed as the form $\frac{m}{n}$ where m and n are non zero integers with gcd(m,n)=1 , find the value of m+n.

16 Let a,b,c be reals satisfying 3ab+2=6b, 3bc+2=5c, 3ca+2=4a Let Q denote the set of all rational numbers.Give that product abc can take values r/s ∈Q and t/u∈Q, in lowest form, find r+s+t+u.

17 For a positive integer $n > 1$, let $g(n)$ denote the largest positive proper divisor of n and $f(n) = n - g(n)$. For example, $g(10) = 5$, $f(10) = 5$ and $g(13) = 1$, $f(13) = 12$. Let N be the smallest positive integer such that $f(f(f(N))) = 97$ Find the largest integer not exceeding N.

18 Let m, n be natural numbers such that $m + 3n - 5 = 5$ LCM(m, n) – 11 GCD(m, n) Find the maximum possible value of $m + n$.

19 Consider a string of n 1's. We wish to place some + signs in between so that the sum is 1000. For instance, if $n = 190$, one may put + signs so as to get 11 ninety times, and get the sum 1000. If a is the number of positive integers n for which it is possible to place + signs so as to get the sum 1000, then find the sum of the digits of a.

20 For an integer $n \geq 3$ and a permutation $\sigma = (p_1, p_2, \dots, p_n)$ of $\{1, 2, 3, \dots\dots, n\}$, we say p_1 is a landmark point if $2 \leq I \leq n-1$ and $(p_{I-1} - p_I)(p_{I+1} - p_I) > 0$. For example, for $n = 7$, the permutations $(2, 7, 6, 4, 5, 1, 3)$ has four landmark points: $p_2 = 7, p_4 = 4, p_5 = 5$ and $p_6 = 1$. For a given $n \geq 3$, let L(n) denote the number of permutations of $\{1, 2, 3, \dots\dots, n\}$ with exactly only landmark point. Find the maximum $n \leq 3$ for which L(n) is a perfect square.

21 An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If N is the number of one hour journey that end at the starting vertex, find the sum of the squares of the digits of N.

22 A binary sequence is a sequence in which each term is equal to 0 and 1. A binary sequence is called friendly if each term is adjacent to at least one term that is equal to 1. For example, the sequence 0, 1, 1, 0, 0, 1, 1, 1 is friendly. Let F_n denotes the number of friendly binary sequence with n terms. Find the smallest positive inter $n \geq 2$ such that $F_n > 100$.

23 In a triangle ABC, the median AD divides $\angle BAC$ in the ratio 1 : 2. Extend AD to E such that EB is perpendicular AB. Given that BE = 3, BA = 4, find the integer nearest to BC^2.

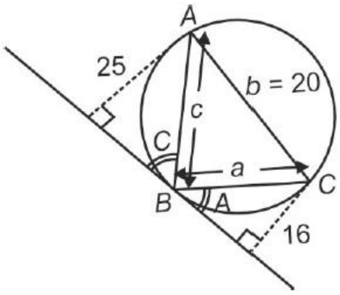
24 Let N be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If $N = 100a + b$, where a, b are positive integers less than 100, find a + b.

SOLUTIONS

1 A triangle ABC with AC=20 is inscribed in a circle ω. A tangent t to ω is drawn through B. The distance of t from A is 25 and that from C is 16. If S denotes the area of the triangle ABC, find the largest integer not exceeding S/20.

Answer: 10

Sol.



$\because \sin A = \frac{16}{a} = \frac{a}{2R} \Rightarrow a^2 = 32R$

Similarly, $c^2 = 50R$

So, $ac = 40R$

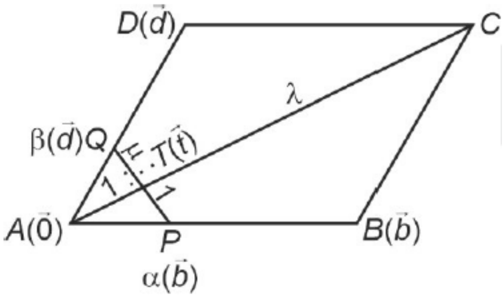
Now, $abc = 800R = 800\left(\frac{abc}{4S}\right)$

$\therefore S = 200$

2 In a parallelogram ABCD, the point P on a segment AB is taken such that $\frac{AP}{AB} = \frac{61}{2022}$ and a point Q on the segment AD is taken such that $\frac{AQ}{AD} = \frac{61}{2065}$, if PQ intersects AC at T, find $\frac{AC}{AT}$ to the nearest integer.

Answer: 67

Sol.



Here, $\alpha = \frac{61}{2022}$ and $\beta = \frac{61}{2065}$

$\vec{t} = \frac{\vec{b}+\vec{d}}{1+\lambda} = \frac{\alpha\mu\vec{b}+\beta\,\vec{d}}{1+\mu}$

So, $\frac{1}{1+\lambda} = \frac{\alpha\mu}{1+\mu}$ and $\frac{1}{1+\lambda} = \frac{\beta}{1+\mu}$

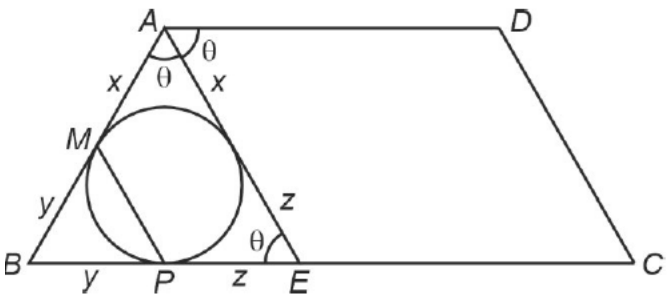
$\Rightarrow \alpha\mu = \beta$ hence, $\frac{1}{1+\lambda} = \frac{\beta}{1+\frac{\beta}{\alpha}}$

$\Rightarrow 1+\lambda = \frac{\alpha+\beta}{\alpha\beta}$

Clearly, $\frac{AC}{AT} = \frac{1+\lambda}{1} = \frac{\alpha+\beta}{\alpha\beta} = \frac{1}{\alpha}+\frac{1}{\beta} = \frac{2022}{61}+\frac{2065}{61} = \frac{4087}{61} \approx 67$

3 In a trapezoid ABCD, the internal bisector of $\angle A$ intersects the base BC(or its extension) at the point E. Inscribed in the triangle ABE is a circle touching the side AB at M and BE at P. Find the $\angle DAE$ in degrees, if AB:MP=2.

Answer: 60



Sol.

$AD||BC$

Given, $PM = \frac{x+y}{2}$

$\angle BAE = \angle AEB = \theta$

$\Rightarrow x = z$

By symmetry, M is mid point of BA

$\therefore \frac{x+y}{2} = y \Rightarrow x = y$

$\therefore \angle B = 60° \Rightarrow \angle A = 60°$

4 Starting with a positive integer M, written on the board, Alice plays the following game: In each move, if x is the number on the board, she replaces it with $3x+2$. Similarly, starting with a positive integer N, written on the board, Bob plays the following game: In each move if x is the number on the board, he replaces it with $2x+27$. Given that Alice and Bob reach the same number after playing four moves each, find the smallest value of $M+N$.

Answer: 10

Sol. In 4 steps, Alice will write

$(3(3(3(3M+2)+2)+2)+2)=\alpha$

And Bob will write

$(2(2(2(2N+27)+27)+27)+27)=\beta$

As $\alpha=\beta$, we get

$81M=16N+325$

$\therefore M_{min}=5$ and $N_{\min}=5$

$\therefore (M+N)_{\min}=10$

5 Let m be the smallest positive integer such that $m^2+(m+1)^2+\cdots+(m+10)^2$ is the square of a positive integer n. Find $m+n$.

Answer: 95

Sol.

$$\sum_{r=1}^{11}(m+r-1)^2=11m^2+(1^2+2^2+3^2+\cdots+10^2)+2m(1+2+3+..+10)$$

$$=11(m^2+10m+35)$$

$$=11((m+5)^2+10)$$

The least value of $m=18$

The required sum $=11(23^2+10)=11\times 11\times 49=77^2$

Then $m=18, n=77$

$\therefore m+n=95$

6 Let a, b be positive integers satisfying $a^3-b^3-ab=25$. Find the largest possible value of a^2+b^3.

Answer: 43

Sol. $a^3-b^3-ab=25$ for $a=4, b=3$.

Because for, any greater number $a^3-b^3-ab>25$

To prove this if $a>b$, then a^3-b^3-ab

$=(b+t)^3-b^3-b(b+t), t>0$

$=(3t-1)b^2+(3t^2-t)b+t^3$ is always >4,

Then $b\geq 3$

So, $a^2+b^3=4^2+3^3=43$

7 Find the number of ordered pair (a,b) such that $a, b\in \{110,11,\ldots\ldots 29,30\} and\ GCD(a,b)+LCM(a,b)=a+b$.

Sol. $g + l = a + b$

$g+\frac{ab}{g}=a+b$

$g^2 + (a + b)g + ab = 0$

$(g - a)(g - b) = 0$

$g = a, b$

For $a = b$, there will be 21 cases.

If a = 10, b may be 20 or 30 as well and vice-versa.
a=11 , b=22
a=12, b=24
a=13, b= 26
a=14,   b=28
a=15, b=30
Total two times = 10 ways
Hence total 35 ways

8 Suppose the prime numbers p and q satisfies $q^2 + 3p = 197p^2 + q$.Write $\frac{q}{p}$ in the form $l + \frac{m}{n}$, where l,m,n are positive integers $m < n$ and $GCD(m,n) = 1$.
Find the maximum value of l+m+n.

Sol. $q^2 + 3p = 197p^2 + q$

$\Longrightarrow$ $197p^2 = q(q-1) + 3p$

So, $q - 1 = \lambda p$

$\therefore$ $197p^2 = (\lambda p + 1)\lambda p + 3p$

$\Rightarrow$ $197p = \lambda^2 p + \lambda + 3$

$\therefore p = \frac{\lambda+3}{197-\lambda^2}$ $\therefore$ $\lambda = 14, p = 17$

So, $q = 17 \times 14 + 1 = 239$

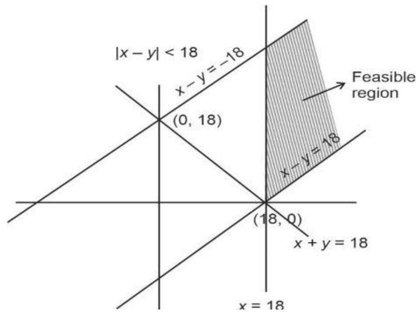
$\frac{q}{p} = 14 + \frac{1}{17}$ $\therefore l + m + n = 32$

9 Two sides of an integer sided triangle have lengths 18 and x. If there are exactly 35 possible integer values y such that 18, x, y are the side of a non-degenerate triangle, find the number of possible integer values x can have.

Sol.

$x + y > 18$

So, x can take any integer value greater than or equal to 18.
So, answer should be infinite.



10 Consider the 10-digit number M = 9876543210. We obtain a new 10-digit number from M according to the following rule: we can choose one or more disjoint pairs of adjacent digits in M and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from M = 9$\underline{87}$6$\underline{54}$3210, by interchanging the 2 underlined pairs, and keeping the others in their places, we get M_1 = $9\overline{78}6\overline{45}3210$. Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from M.

Sol. Number of ways if single pair is changed = 9 = ${}^9C1$

Number of ways if 2 pairs changed = 7 + 6 + 5 + ... + 1

= 28 = ${}^8C2$

Number of ways if 3 pairs changed = ${}^7C3$

So, total numbers that can be formed = ${}^9C1 + {}^8C2 + {}^7C3 + {}^6C4 + {}^5C5$

= 9 + 28 + 35 + 15 + 1

= 88

11 Let AB be a diameter of a circle x and let C be a point on x, different from A and B. The perpendicular from C intersects AB at D and ω at $E(\neq C)$. The circle with centre at C and radius CD intersects ω at P and Q. If the perimeter of the triangle PEQ is 24, find the length of the side PQ.

Sol.

$CPEQ$ is cyclic quadrilateral

$CP \times EQ + CQ \times PE = CE \times PQ$

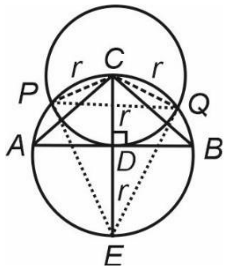
$r(EQ + PE) = 2r\ PQ$

$2PQ = EQ + PE$

$PQ + EQ + PE = 24$

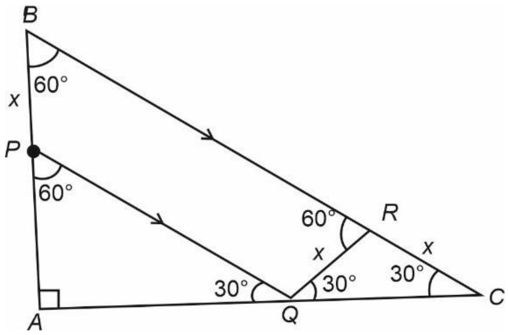
$PQ + 2PQ = 24$

∴ $PQ = 8$



12 Given $\triangle ABC$ with $\angle B = 60°$ and $\angle C = 30°$, let P, Q, R be points on sides BA, AC, CB respectively such that $BPQR$ is an isosceles trapezium with $PQ||BR$ and $BP = QR$. Find the maximum possible value of $\frac{2[ABC]}{[BPQR]}$ where $[S]$ denotes the area of any polygon S.

Sol. $BP = QR = CR = x$ & let $BC = l$



$BR = l - x$

$PQ = (l - x) - 2x\cos 60^0$

$= l - 2x$

$$2\frac{[ABC]}{[BPQR]} = \frac{2.\frac{l}{2}.\frac{l\sqrt{3}}{2}}{[l-x+l-2x]x\frac{\sqrt{3}}{2}} = \frac{l^2}{[x(2l-3x)]}$$

$$= \frac{\left(\frac{l}{x}\right)^2}{\frac{2l}{x}-3} \quad x \in \left(0, \frac{l}{2}\right)\ and\ \frac{l}{x} \in (2, \infty)$$

$let\ f(y) = \frac{y^2}{2y-3}, f'(y) = 0\ gives\ y = 3\ for\ minimum\ value.$

$minimum\ value\ of\ the\ expression$

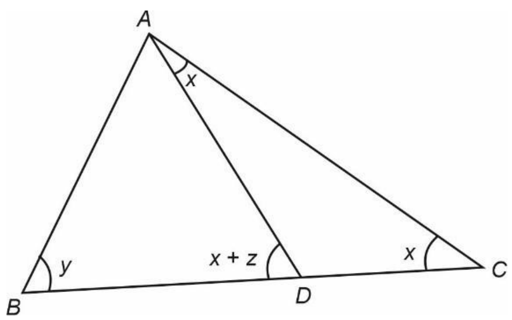
$$2\frac{[ABC]}{[BPQR]} = \frac{3^2}{2.3 - 3} = 3$$

Maximum value tends to infinity.

13 Let ABC be a triangle let D be a point on the segment BC such that $AD = BC$. Suppose $\angle CAD = x°$, $\angle ABC = y°$ and $\angle ACB = z°$ and x, y, z are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of $\angle ABC$ in degrees.

Sol.

$$\frac{AD}{siny} = \frac{BD}{\sin(x+y+z)}$$



$$\frac{AD}{sinz} = \frac{CD}{sinx}$$

$$\frac{\sin(x+y+z)}{siny} + \frac{sinx}{sinz} = 1$$

$$\frac{\sin(3y)}{siny} + \frac{\sin(y-d)}{\sin(y+d)} = 1$$

$$\frac{\sin(y-d)}{\sin(y+d)} = -2cos2y$$

$\sin(3y + d) = 0 \Rightarrow 3y + d = 180^0, 360^0$

$y\ and\ d\ are\ integers \Rightarrow y_{max} = 59^0$

$3y + d = 360^0 \Rightarrow y_{max} = 119^0 (if\ y\ is\ obtuse\ then\ z\ must\ be\ acute)$

$only\ first\ case\ is\ possible\ y_{max} = 59^0$

14 Let x, y, z be complex numbers such that

$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 9$

$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = 68$

$\frac{x^3}{y+z} + \frac{y^3}{z+x} + \frac{z^3}{x+y} = 488$

$if\ \frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} = \frac{m}{n}$

where m,n are positive integers with gcd(m,n)=1, find m+n.

Sol.

$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 9$

$(x+y+z)\left(\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y}\right) = 12$

$let\ x+y+z = S_1$

$\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y} = \frac{12}{S_1}$

$now\ (x+y+z)\left(\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y}\right) = 9S_1$

$64 + S_1 = 9S_1$

$\Rightarrow S_1 = 8$

$(x+y+z)\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) = 64S_1$

$488 + (x^2+y^2+z^2) = 64 \times 8$

$x^2+y^2+z^2 = 24$

$xy+yz+zx = \frac{64-24}{2} = 20$

$now\ \frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y} = \frac{12}{8} = \frac{3}{2}$

$\frac{1}{8-x} + \frac{1}{8-y} + \frac{1}{8-z} = \frac{3}{2}$

$(8-x)(8-y)(8-z) = 56$

$\Rightarrow xyz = 104, so\ \frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} = \frac{24}{104} = \frac{3}{13} = \frac{m}{n}, which\ \Rightarrow m+n = 16.$

15 Let x, y be real numbers such that $xy = 1$. Let T and t be the largest and the smallest values of the expression $\frac{(x+y)^2-(x-y)-2}{(x+y)^2+(x-y)-2}$ if T+t can be expressed as the form $\frac{m}{n}$ where m and n are non zero integers with gcd(m,n)=1 , find the value of m+n.

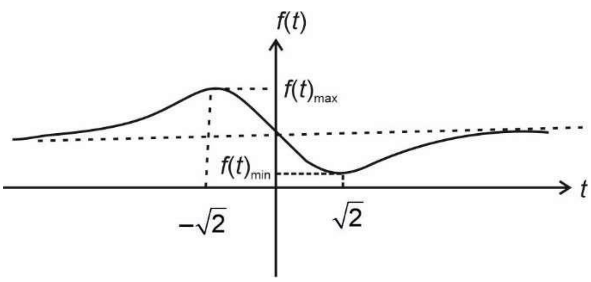
Sol. $xy = 1 \Rightarrow (x+y)^2 = (x-y)^2 + 4$

$\frac{(x+y)^2-(x-y)-2}{(x+y)^2+(x-y)-2} = \frac{(x-y)^2-(x-y)-2}{(x-y)^2+(x-y)+2}$

$= \frac{\left(x-\frac{1}{x}\right)^2 - (x-\frac{1}{x})+2}{\left(x-\frac{1}{x}\right)^2 + \left(x-\frac{1}{x}\right)+2}$

$x - \frac{1}{x} \epsilon R$

$let\ f(t) = \frac{t^2-t+2}{t^2+t+2}, t \in R$



$f(t)_{max} + f(t)_{min} = \frac{4++\sqrt{2}}{4-\sqrt{2}} + \frac{4-\sqrt{2}}{4+\sqrt{2}} = \frac{18}{7} = \frac{m}{n}$

$m+n = 25.$

16 Let a,b,c be reals satisfying 3ab+2=6b, 3bc+2=5c, 3ca+2=4a Let Q denote the set of all rational numbers.Give that product abc can take values r/s ∈Q and t/u∈Q, in lowest form, find r+s+t+u.

Sol. $3ab + 2 = 6b$ ...(i),

$3bc + 2 = 5c$ ...(ii),

$3ac + 2 = 4a$ ...(iii)

$3abc + 2c = 6bc$ ...(1)

$3abc + 2a = 5ac$ ...(2)

$3abc + 2b = 4ab$ ...(3)

Substitute bc from (ii) in (1),

$3abc + 2c = 10c - 4$

$33abc = 8c - 4$...............(4)

Substituting ac from (iii)in (2)

$3abc + 2a = 5\left(\frac{4a-2}{3}\right)$

$9abc + 6a = 20a - 10$

$9abc = 14a - 10 \quad \ldots\ldots(5)$

substituting ab from (i)in(3)

$3abc + 2b = 4\left(\frac{6b-2}{3}\right)$

$9abc + 6b = 24b - 8$

$9abc = 18b - 8 \quad \ldots\ldots(6)$

from (4), (5), (6)

$24c - 12 = 14a - 10 = 18b - 8 = \lambda$

substituting in (i)

$3\left(\frac{10+\lambda}{14}\right)\left(\frac{8+\lambda}{186}\right) + 2 = 6\left(\frac{8+\lambda}{183}\right)$

$(\lambda + 8)(\lambda + 10) + 84 \times 2 = 28(8 + \lambda)$

$\lambda^2 + 18\lambda + 80 + 168 = 224 + 28\lambda$

$\lambda^2 - 10\lambda + 24 = 0$

$(\lambda - 6)(\lambda + 4) = 0$

$\lambda = 6{,}4$

$\therefore for \;\; \lambda = 6$

$a = \frac{16}{14}, \;\; b = \frac{14}{18}, \;\;\; c = \frac{18}{24}$

$\therefore abc = \frac{16}{14} \times \frac{14}{18} \times \frac{18}{24} = \frac{16}{24} = \frac{2}{3}$

$for \; \lambda = 4, a = 1, \qquad b = \frac{12}{18} \quad c = \frac{16}{24} \;\; and \; abc = \frac{4}{9}$

hence $r + s + t + u = 18$.

17 For a positive integer $n > 1$, let $g(n)$ denote the largest positive proper divisor of n and $f(n) = n - g(n)$. For example, $g(10) = 5$, $f(10) = 5$ and $g(13) = 1$, $f(13) = 12$. Let N be the smallest positive integer such that $f(f(f(N))) = 97$ Find the largest integer not exceeding $\sqrt{N}$.

Sol. If $f(n) = x$ and x is a prime then least value of $n = 2$...(1) and if $f(n) = x$ and x is composite but $x + 1$ is a prime then least value of $n = x + 1$...(2) $\because f(f(f(n))) = 97$

Then, $f(f(n)) = 194$ [from (1)]

Now, $f(n)$ for n to be least can be 3×97 or 4×97

Case I : $f(n) = 3 \times 97$, then least value of $n = 6 \times 97$

Case II : $f(n) = 4 \times 97$, then least value of $n = 4 \times 97 + 1$ from equation (2)

The smallest positive value of $n = 4 \times 97 + 1$

$\therefore N = 389$

$\therefore \;\sqrt{N} = \sqrt{389} > 19$

18 Let m, n be natural numbers such that $m + 3n – 5 = 5$ LCM(m, n) – 11 GCD(m, n) Find the maximum possible value of $m + n$.

Sol. Let G. C. D. of $(m, n) = d$

Then for some positive coprime integers x and y $m = dx$ and $n = dy$

$\because$ $m + 3n – 5 = 2$ LCM (m, n) – 11 GCD (m, n)

$\therefore$ $dx + 3dy – 5 = 2dxy – 11d$ or, $d(x + 3y – 2xy + 11) = 5$

Now, to maximize the sum $m + n$, d must be 5

$\therefore$ $x + 3y – 2xy + 11 = 1$

or $x + 3y – 2xy + 10 = 0$

or $(2x – 3)(2y – 1) = 23$

Case I : $2x – 3 = 1$ and $2y – 1 = 23$

$\therefore$ $x = 2$ and $y = 12$

This is not possible as x, y are coprime

Case II : $2x – 3 = 23$ and $2y – 1 = 1$

$\therefore$ $x = 13$ and $y = 1$

$\therefore m + n = (13 + 1) \times 5 = 70$

19 Consider a string of n 1's. We wish to place some + signs in between so that the sum is 1000. For instance, if $n = 190$, one may put + signs so as to get 11 ninety times, and get the sum 1000. If a is the number of positive integers n for which it is possible to place + signs so as to get the sum 1000, then find the sum of the digits of a.

Answer: (10)

Sol. Since $1000 = 1.a_1 + 11.a_2 + 11.a_3 + \cdots$

Where a_1, a_2, a_3........ are non-negative integers for all a_i, when $i > 3, a_i = 0$

Therefore, $1000 = 111p + 11q + r$

If $p = 0$, then there are 91 possibilities for q, r.

For $p = 1$, there are 81 possibilities for q, r.

For $p = 2$, there are 71 possibilities for q, r.

For $p = 3$, there are 61 possibilities for q, r.

For $p = 4$, there are 51 possibilities for q, r.

…………………………………………………………………

…………………………………………………………………

…………………………………………………………………

For $p = 9$, there are 1 possibility for q, r.

Hence, total number of possibilities = 460

Sum of digit of 460 is 10.

20 For an integer $n \geq 3$ and a permutation $\sigma = (p_1, p_2, \ldots, p_n)$ of $\{1, 2, 3, \ldots \ldots, n\}$, we say p_1 is a landmark point if $2 \leq I \leq n - 1$ and $(p_{I-1} - p_I)(p_{I+1} - p_I) > 0$. For example, for $n = 7$, the permutations $(2, 7, 6, 4, 5, 1, 3)$ has four landmark points: $p_2 = 7, p_4 = 4, p_5 = 5$ and $p_6 = 1$. For a given $n \geq 3$, let L(n) denote the number of permutations of $\{1, 2, 3, \ldots \ldots, n\}$ with exactly only landmark point. Find the maximum $n \leq 3$ for which L(n) is a perfect square.

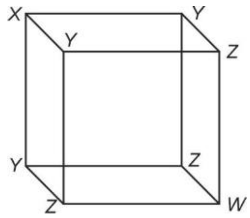
Answer (03)

Sol. For the permutations of set $\{1, 2, 3, \ldots \ldots, n\}$, the landmark point should be 1 or n to satisfy given condition.

$\overline{I^{st}}$ $\overline{II^{nd}}$ __ __ $\overline{r^{th}}$ __ __ __ $\overline{n^{th}}$

If n is at (similarly for 1) r^{th} position, there is only one permutation of the remaining number for each selection.
So, number of selections = $\sum_{r=2}^{n-1} n-1Cr-1$
$= 2^{n-1}-2$
Therefore, total number of selections = $2(2^{n-1}-2)$
L(n) = $4(2^{n-2}-1)$
Now for L(n) to be a perfect square n should be equal to 3
Therefore, $n = 3$.

21 An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If N is the number of one hour journey that end at the starting vertex, find the sum of the squares of the digits of N.
Answer (74)
Sol. We have divided vertices into four categories



X → Starting vertex
Y → Adjacent vertex
Z → Adjacent to Y but not same as X
W → Adjacent to Z but not same as Y
Let a_n = number of ways that after n steps ant is at X
b_n = number of ways that after n steps ant is at Y
c_n = number of ways that after n steps ant is at Z
d_n = number of ways that after n steps ant is at W
We need to find a_6
$a_{n+1} = 3\, b_n$ .... (i)
$b_{n+1} = a_n + 2\, c_n$ .... (ii)
$c_{n+1} = 2b_n + d_n$ .... (iii)
and $d_{n+1} = 3c_n$ .... (iv)
By eliminating b_n , c_n and d_n we get
$a_{n+3} = 10\, a_{n+1} - 9\, a_{n-1}$
Since, $a_1 = 0$, $a_2 = 3$, $a_3 = 0$ and $a_4 = 21$
Therefore $a_6 = 10\, a_4 - 9\, a_2 = 210 - 27 = 183$
So, N = 183
Sum of square of digit of N = 74.

22 A binary sequence is a sequence in which each term is equal to 0 and 1. A binary sequence is called friendly if each term is adjacent to at least one term that is equal to 1. For example, the sequence 0, 1, 1, 0, 0, 1, 1, 1 is friendly. Let F_n denotes the number of friendly binary sequence with n terms. Find the smallest positive inter $n \geq 2$ such that $F_n > 100$.
Answer (11)
Sol. Let a_n = number of friendly sequences ending with 0
b_n = number of friendly sequences ending with 1.
$F_n = a_n + b_n$ .... (i)
Now, $a_{n+1} = b_n$ .... (ii) (by adding 0 in the last)
And $b_{n+1} = a_n + b_n + a_{n-1}$ ..... (iii)
Since, $F_n = a_n + a_{n+1}$ from the equation (i) and (ii)
And $a_{n+2} = a_{n+1} + a_n + a_{n-1}$ by (ii) and (iii)
Since, $a_1 = 0$, $a_2 = 1$, $a_3 = 3$

So, $a_4 = 4$, $a_5 = 5$, $a_6 = 9$, $a_7 = 16$, $a_8 = 25$, $a_9 = 39$, $a_{10} = 64$, and $a_{11} = 105$ and so on.

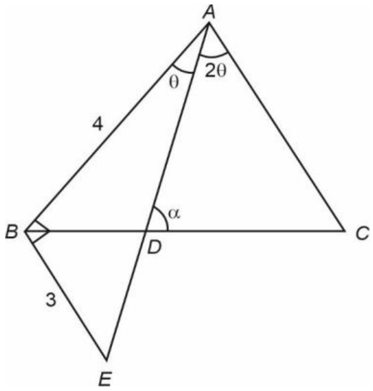
Clearly , we can see that $F_{11} = 105$.

So, $F_{11} > 100$.

23 In a triangle ABC, the median AD divides $\angle BAC$ in the ratio 1 : 2. Extend AD to E such that EB is perpendicular AB. Given that BE = 3, BA = 4, find the integer nearest to BC^2.

Answer (29)

Sol.



Here, D is the mid-point of BC, hence BD : CD = 1 : 1

$\angle ABE = 90°$

Let $\angle BAD = \theta$, then $\angle CAD = 2\theta$

Therefore, $\tan\theta = \frac{3}{4}$, and $\tan 2\theta = \frac{24}{7}$

Now, using $\cot m - n$ theorem in ΔABC

$2\cot\alpha = \cot\theta - \cot 2\theta$

$\Rightarrow \cot\alpha = \frac{25}{48}$

Now, using sine rule in ΔABC, we get

$\frac{BD}{4} = \frac{\sin\theta}{\sin(\pi-\alpha)}$

Therefore, $BD = \frac{4\times 3\sqrt{25^2+48^2}}{5\times 48}$

So, $4BD^2 = BC^2 = \frac{25^2+48^2}{100} = 29.29$

Nearest integer is 29.

24 Let N be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If $N = 100a + b$, where a, b are positive integers less than 100, find a + b.

Answer (81)

Sol. Let i^{th} box has $6\lambda_i + \mu_i$ balls where $\lambda_i, \mu_i \epsilon W$ and $\mu_i \le 5$. Also, all μ_i's are distinct.

Since, $6(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + (\mu_1 + \mu_2 + \mu_3 + \mu_4) = 52$ and $\mu_1 + \mu_2 + \mu_3 + \mu_4 \in \{6, 7, 8, 9, 10, 11, 12, 13, 14\}$

Hence, only one possibility is there

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 7$ …… (i)

And $\mu_1 + \mu_2 + \mu_3 + \mu_4 = 10$ …… (ii)

Equation (ii) has only three solution sets, which are $(5, 4, 1, 0)$, $(5, 3, 2, 0)$ and $(4, 3, 2, 1)$.

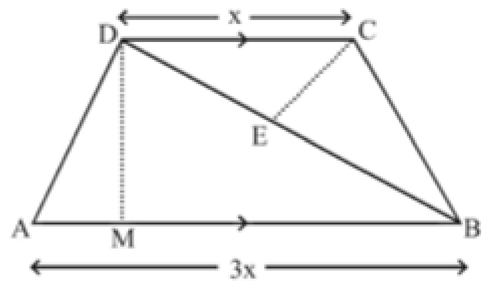
Equation (i) has total ${}^{10}C_3$ solutions and ${}^{4}C_1.\ {}^{9}C_2$ solutions when any λ_i is zero.

Therefore, a = 69 and b = 12.

IOQM 2020-21

1. Let ABCD be a trapezium in which AB || CD and AB = 3CD. Let E be the midpoint of the diagonal BD. If [ABCD] = n × [CDE], what is the value of n ? (Here [ ] denotes the area of the geometrical figure

Ans. (08)



SoL. [ABCD] = [ABD] + [BCD]

= 3xH/2 + xH/2

[ABCD] = 4xH/2

Also, [CDE] = ½*[BDC]

=1/2*1/2*xH

=1/4*xH

Therefore, $\frac{[ABCD]}{[CDE]} = 8$

Implies,[ABCD]=8*[CDE]

Implies, n = 8

2. A number N in base 10, is 503 in base b and 305 in base b + 2. What is the product of the digits of N?

Ans. (64)

Sol. $(N)_{10} = (503)_b = (305)_{b+2}$

Therefore, N = $5b^2 + 3 = 3(b+2)^2 + 5$

$5b^2 - 2 = 3(b^2 + 4 + 4b)$

Implies, $b^2 - 6b - 7 = 0$

Implies,(b-7)(b+1)=0

Implies, b=7 (since negative not possible)

Therefore, N= 5(49)+3 =248

Therefore, Product of digits of N=2.4.8=64

3. If$\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2}$ = 0.9999 ,Then determine the value of N.

Ans. (99)

Sol , $\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2}$ = 0.9999

$\sum_{k=1}^{N} \frac{k+k+1}{k^2(k+1)^2}$ = 0.9999

$\sum_{k=1}^{N} \frac{1}{k(k+1)}(\frac{1}{k} + \frac{1}{k+1})$ = 0.9999

$$\sum_{k=1}^{N} \left(\frac{1}{k} - \frac{1}{k+1}\right)\left(\frac{1}{k} + \frac{1}{k+1}\right) = 0.9999$$

$$\sum_{k=1}^{N} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) = 0.9999$$

$$1 - \frac{1}{(N+1)^2} = 0.9999$$

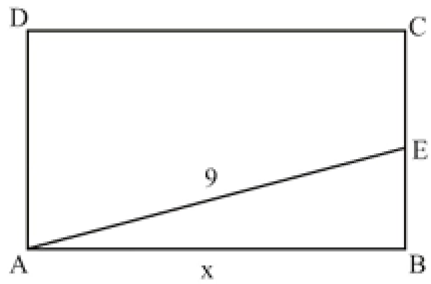
$$\frac{1}{(N+1)^2} = 0.0001$$

$(N+1)^2 = 10000$

N=99

4. Let ABCD be a rectangle in which AB + BC + CD = 20 and AE = 9 where E is the mid-point of the side BC. Find the area of the rectangle.

Ans. (19)



Sol. Let, AB=CD=x and AD=BC=y

Then, 2x+y=20,..........(i)

And $x^2 + y^2/4 = 81$ .....(ii)

Squaring Eq (i) ,we get

$(x + y/2)^2 = 100$

$X^2 + y^2/4 + xy = 100$

Xy=19 (by using ii)

5. Find the number of integer solutions to $||x| - 2020| < 5$.

Ans. (18)

Sol.

$||x| - 2020| < 5$

$-5 < |x| - 2020 < 5$

⇒ $2015 < |x| < 2025$

So, $|x| > 2015$

⇒ $-2015 > x > 2015$ ...(1)

Also, $|x| < 2025$

⇒ $-2025 < x < 2025$

So, $x \in \{-2024, -2023, \ldots\ldots\ldots 2015, 2016, 2017, 2024\}$

No of integer solution = 18 Ans.

6. What is the least positive integer by which $2^5 \cdot 3^6 \cdot 4^3 \cdot 5^3 \cdot 6^7$ should be multiplied so that, the product is a perfect square?

Ans. (15)

Sol. Let $M = 2^5 \cdot 3^6 \cdot 4^3 \cdot 5^3 \cdot 6^7$

⇒ $M = 2^{5+6+7} \cdot 3^{6+7} \cdot 5^3$

$= 2^{18} \cdot 3^{13} \cdot 5^3$

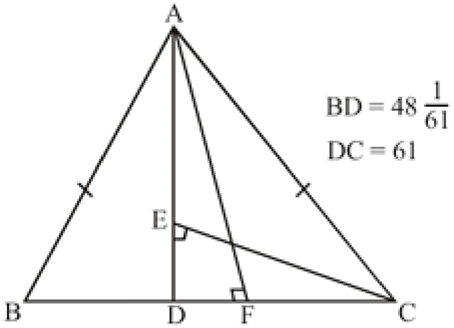
For M to be c perfect square, M should be multiplied by 3 x 5 = 15

7. Let ABC be a triangle with AB = AC. Let D be a point on the segment BC such that $\overline{BD}$ $= 48\frac{1}{6}$ and

DC = 61. Let E be a point on AD such that CE is perpendicular to AD and DE = 11. Find AE.

Ans. (25)

Sol.



In right angle triangle CED

CE= Sqrt(61^2-11^2) = 60

Hence, ABC is Isoscale Triangle

Implies,AF divide BC

Now,BC = BF + FC

Implies, 48 + $\frac{1}{61} + \mathbf{61} = \mathbf{2}\boldsymbol{BF}$ (since,BF=FC)

Implies, 109 +1/61=2BF

Implies, 108+1+1/61=2BF

Implies,54+1/2+1/122=BF

Implies,54+31/61=BF

In right ang;le Triangle AFC,

$AC^2 = (54+\frac{31}{61})^2 + AF^2$ ....(i)

In right angle Triangle AEC,

$AC^2 = 60^2 + AE^2$ ....(ii)

Now,DF=BF-BD=54+31/61 - 48 - 1/61 =6+30/61

So In Right angle Triangle AFD

$AD^2 = AF^2 + DF^2$

Implies, $(AE+ED)^2 - DF^2 = AF^2$

Implies, $(AE+11)^2 - (6+30/31)^2 = AF^2$

From Eq (i) and (ii),

$(54+31/61)^2 + (AE+11)^2 - (6+30/31)^2 = 60^2 + AE^2$

Implies,AE= 25

8. A 5-digit number (in base 10) has digits k, k + l, k + 2, 3k, k + 3 in that order, from left to right. If this number is m^2 for some natural number m, find the sum of the digits of m.

Ans. (15)

Sol. According to the question,

$m^2 = 10^4 (K) + 10^3 (K + 1) + 10^2 (K + 2) + 10^1 (3K) + K + 3$

$= 10^4 K + 10^3 K + 10^2 K + 10 (3K) + K + 10^3 + 10^2 \cdot 2 + 3$

$= K (10^4 + 10^3 + 10^2 + 31) + 1203$

= K(11131) + 1203

For K = 3

⇒ m^2 = 34,596

⇒ m = 186

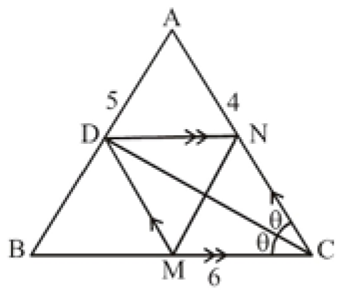
⇒ Sum of the digits of m = 15

9. Let ABC be a triangle with AB = 5, AC = 4, BC = 6. The internal angle bisector of c intersects the side AB at D. Points M and N are taken on sides BC and AC, respectively,

such that DM || AC and DN ||C. If $(MN)^2 = \frac{p}{q}$ Where p and q are relatively prime positive integer then what is the sum of the digits of |p – q|?

Ans. (02)

Sol.



Since,CD is the bisector of angle C

Implies, $\frac{AD}{BD} = \frac{4}{6} = \frac{2}{3}$

Therefore,AD=2,BD=3

Now,DH||BC

Therefore,$\frac{AD}{BD} = \frac{AN}{MC} = \frac{2}{3}$

Therefore, NC=$\frac{3}{5} * 4 = \frac{12}{5}$

Also,DM||AC

Implies,$\frac{BM}{MC} = \frac{BD}{DA} = \frac{3}{2}$

Therefore,MC=$\frac{2}{5} * 6 = \frac{12}{5}$

So,DMCM is a Rhombus

Now, cosC = $\frac{6^2+4^2-5^2}{2*4*6} = \frac{\frac{12}{5}^2 + \frac{12}{5}^2 - MN}{2(\frac{12}{5})^2}$

Implies,MN^2=126/25=p/q

|p-q|=126-25=101

Therefore, Sum of the digit of |p-q|=1+0+1=2

10.Five students take a test on which any integer score from 0 to 100 inclusive is possible .What is the largest possible difference between the median and the mean of the score?(The median of a set of score is the middle most score when the data is arranged in increasing order.It is exactly the middle score when there are an odd number of score and it is the average of the two middle score when there are an even number of scores.

Ans.(40)

Sol. 0,0,0,100,100

We know,$difference_{max}$=|median-mean|

Here,median=0 and Mean=$\frac{0+0+0+100+100}{5} = 40$

Therefore,$difference_{max}$=|40-0|=40

11. Let X = {– 5, –4, –3, –2, –l, 0, 1, 2, 3, 4, 5} and S = {(a, b) ∈ X × X : x^2 + ax + b and x^3 + bx + a have at least a common real zero}. How many elements there are in S?

Sol: x-a

$(x^2 + ax + b)\sqrt{x^3 + bx + a}$

$x^3 + ax^2 + bx$

$- ax^2 + a$

$- ax^2 - a^2x - ab$

$a^2x + a(b + 1)$

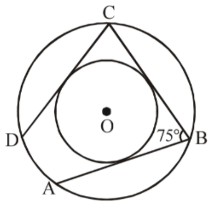
If a common root is there, then $a^2x + a(b + 1)$ must be a factor of $x^2 + ax + b$. So, for $x = -\frac{b+1}{a}$ is a root of $x^2 + ax + b$.

$\left(-\frac{b+1}{a}\right)^2 + a\left(-\frac{b+1}{a}\right) + b = 0 \quad \Rightarrow \quad a = \pm(b+1) \Rightarrow a - b = 1 \text{ or } a + b = 1$

S = { (-1,0), (0,-1), (–2, 1), (1, –2),(–3, 2), (2, –3),(–4, 3), (3, –4),(–5, 4), (4, –5),
(2, 1), (3, 2),(4, 3), (5, 4),(–1, –2), (–2, – 3),(–3, –4), (–4, –5),(1, 0)}
(0, –2), (0, –3), (0, –4), (0, –5) (0, 6) are also the solution.
therefore, total common solution = 24

12. Given a pair of concentric circles, chords AB, BC, CD, ... of the outer circle are drawn such that they all touch the inner circle. If ∠ABC = 75°, how many chords can be drawn before returning to the starting point?

Sol:



Here, central angle = 105°

So, total number of chords = $\frac{360°}{\gcd(105°, 360°)}$

$= \frac{360°}{15°} = 24$

13. Find the sum of all positive integers n for which $|2^n + 5^n - 65|$ is a perfect square.

Sol: Let $m^2 = |2^n + 5^n - 65|$

For n = 2, $m^2 = |4 + 25 - 65| = |-36| = 6^2$

For $n \geq 3$

$m^2 = |2^n + 5(5^{n-1} - 13)|$

If n = 4, $m^2 = 576 = 24^2$

If $n \geq 5$, $m^2 = 2^n + 5$ (ab 25 – 13)

$= 2^n + 5 \cdot$ ab 12

$= 2^n + x$ ab 60

So, for n = 1, 3, 5, 7 ………………… not possible as unit digit is 2 and 8.

Also, for n = 2k

$m^2 = 4^k + 5(5^{2k-1} - 13)$

$\Rightarrow \quad m^2 - (2^k)^2 = 5(5^{2k-1} - 13)$

$\Rightarrow \quad (m - 2^k)(m + 2^k) = 5(5^{2k-1} - 13)$

Here, last 2 digit always ends with 60 in RHS. So, not possible for n = 6, 8, 10, ...

⇒ n = 2 and 4

⇒ Sum of n = 6

14. The product 55 × 60 × 65 is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?

Sol: 55 × 60 × 65

N = 5 × 11 × 12 × 5 × 13 × 5

N = 5 × 11 × 13 × 15 × 20

So, least value of largest of these integers = 20

15. Three couples sit for a photograph in 2 rows of three people each such that no couple is sitting in the same row next to each other or in the same column one behind the other. How many arrangements are possible?

Sol: Couples be B_1, G_1 ; B_2, G_2 ; B_3, G_3

Case I : -

B_1, B_2, B_3 are in same row

then G_1, G_2, G_3 can be arrange in other row in Derangement (3) = 2 ways

So, (3! × 2) × 2 = 24 ways

Arrange B_1, B_2, B_3 Arrange G_1, G_2, G_3 R_1, R_2

Case II : -

Two boys one girl in a row Say if B_1, B_2 then girl cannot be G_1 or G_2 because if say B_1, B_2, G_1 in a row

B_1, B_2, G_3 in a row = 3!

In other row B_3, G_1, G_2

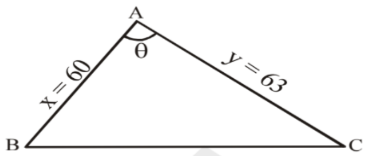
in derangement (3) = 2 ways

hence (3! × 2) × 2 = 24 ways

Similarly if B_1, B_3, G_2 = 24 ways

B_2, B_3, G_1 = 24 ways Total = 96 ways

16. The sides x and y of a scalene triangle satisfy $\boldsymbol{x + \frac{2\Delta}{x} = y + \frac{2\Delta}{y}}$ *, where* Δ *is the area of the triangle. If x = 60, y = 63, what is the length of the largest side of the triangle?*



Sol: $\boldsymbol{x + \frac{2\Delta}{x} = y + \frac{2\Delta}{y}}$

$$\Rightarrow \boldsymbol{x + 2\cdot\frac{1}{2}xy\,\frac{\sin\theta}{x} = y + 2\cdot\frac{1}{2}xy\,\frac{\sin\theta}{y}} \Rightarrow$$

⇒ x + y $\boldsymbol{\sin\theta}$ = y + x $\boldsymbol{\sin\theta}$

⇒ x − y = $\boldsymbol{\sin\theta}$ (x − y)

⇒ $\sin\theta = 1$

⇒ $\theta = 90^0$

⇒ BC = 87

⇒ Largest side of Δ ABC = 87

17. How many two digit numbers have exactly 4 positive factors? (Here 1 and the number n are also considered as factors of n.

Sol: If P_1 & P_2 are prime then

N = $P_1 \times P_2$ or N = P^3.

Primes less than so are

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

for $P_1 = 2$ $P_2 \in (5,7,\ldots\ldots 47) = 13$

$P_1 = 3$ $P_2 \in (5, 31) = 9$

$P_1 = 5$ $P_2 \in (7, 11,\ldots\ldots,19) = 5$

$P_1 = 7$ $P_2 \in (11, 13) = 2$

$P^3 = 3^3$

Total number = 13 + 9 + 5 + 2 + 1 = 30

18. If $\sum_{k=1}^{40}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = a+\frac{b}{c}$ *where a, b, c ∈ N ,b < c, gcd (b, c) = 1, then what is the value of a + b?*

Sol: $\sum_{k=1}^{40}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = a+\frac{b}{c}$

$$\Rightarrow \sum_{k=1}^{40}\sqrt{1+\frac{k^2+1+2k+k^2}{k^2(k+1)^2}} = a+\frac{b}{c}$$

$$\Rightarrow \sum_{k=1}^{40}\sqrt{1+\frac{2k^2+2k}{k^2(k+1)^2}+\frac{1}{k^2(k+1)^2}} = a+\frac{b}{c}$$

$$\Rightarrow \sum_{k=1}^{40}\sqrt{1+\frac{2}{k(k+1)}+\frac{1}{k^2(k+1)^2}} = a+\frac{b}{c}$$

$$\Rightarrow \sum_{k=1}^{40}\left(1+\frac{1}{k(k+1)}\right) = a+\frac{b}{c}$$

$$\Rightarrow \sum_{k=1}^{40}1+\sum_{k=1}^{40}\frac{1}{k(k+1)} = a+\frac{b}{c}$$

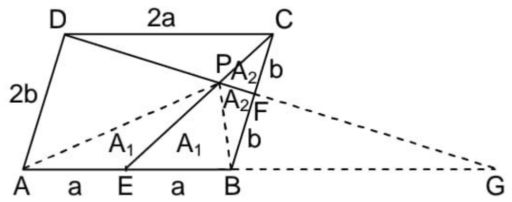
$$\Rightarrow 40+\sum_{k+1}^{40}\left(\frac{1}{k}-\frac{1}{k+1}\right) = a+\frac{b}{c}$$

$$\Rightarrow 40+\left(1-\frac{1}{41}\right) = a+\frac{b}{c}$$

$$\Rightarrow \frac{40}{41} = a + \frac{b}{c}$$
$$\Rightarrow a + b = 40 + 40 = 80$$

19. Let ABCD be a parallelogram. Let E and F be midpoints of AB and BC respectively. The lines EC and FD intersect in P and form four triangles APB, BPC, CPD and DPA. If the area of the parallelogram is 100 sq. units, what is the maximum area in sq. units of a triangle among these four triangles?

Sol:



Ar(Δ PCD)=50 –2A_1

Ar(Δ PAD)=50 –2A_2

BF|| AD and BF=$\frac{1}{2}$AD ⇒ BG= 2a

Δ BFG ≅ ΔCFD ⇒ $Ar(\Delta$ BFG) = 50 –2A_1 + A_2

Ar(Δ PAB)= *Ar(*Δ PBG) ⇒ $2A_1$=50 –2A_1 + A_2

⇒ 2A_1-A_2=25

Δ PEG ~ Δ PCD ⇒ $\sqrt{\frac{50-2A_1}{3A_1}} = \frac{2}{3}$ ⇒ A_1=15 ⇒ A_2= 5

Ar(Δ PCD)=50 –2A_1=20 ; *Ar(*Δ PAD)=50 –2A_2= 40

20. A group of women working together at the same rate can build a wall in 45 hours. When the work started, all the women did not start working together. They joined the work over a period of time, one by one, at equal intervals. Once at work, each one stayed till the work was complete. If the first woman worked 5 times as many hours as the last woman, for how many hours did the first woman work?

Sol: Let there are 'n' women

⇒Each woman's one hour work = $\frac{1}{45n}$

Also, $5[\,t-(n-1)d] = t$

⇒ 4t = 5(n – 1)d

$$\Rightarrow \frac{1}{45n}\left(\frac{n}{2}\right)[2t-(n-1)d] = 1$$

$$\Rightarrow \frac{1}{90}\left[2t - \frac{4t}{5}\right] = 1$$

⇒ t = 75 hours

21. A total fixed amount of N thousand rupees is given to three persons $\mathbf{A}, \mathbf{B}, \mathbf{C}$, every year, each being given an amount proportional to her age. In the first year, A got half the total amount. When the sixth payment was made, A got six-seventh of the amount that she had in the first year; B got Rs 1000 less than that she had in the first year; and C got twice of that she had in the first year. Find N.

Sol.

	A	B	C
Age at beginning	a	b	c
Money at first year	$\frac{\mathbf{N}}{\mathbf{2}}$	$\frac{\mathbf{b}}{\mathbf{b+c}}\left(\frac{\mathbf{N}}{\mathbf{2}}\right)$	$\frac{\mathbf{c}}{\mathbf{b+c}}\left(\frac{\mathbf{N}}{\mathbf{2}}\right)$
Age at $\mathbf{6}$ th payment	a+5	b+5	c+5
Money received	$\frac{6}{7}\left(\frac{N}{2}\right)=\frac{3N}{7}$	$\frac{\mathbf{b}}{\mathbf{b+c}}\left(\frac{\mathbf{N}}{\mathbf{2}}\right)$-1000	$\frac{c}{b+c}(\boldsymbol{N})$

Amount ∝ age ⇒ $\mathbf{a}=\mathbf{b}+\mathbf{c}$

At $\mathbf{6}^{\text{th}}$ payment, $\frac{\mathbf{a+5}}{\mathbf{a+b+c+15}}=\frac{\mathbf{3}}{\mathbf{7}}$

$$\Rightarrow \frac{b+c+5}{2b+2c+15}=\frac{3}{7} \Rightarrow b+c=10$$

For C , at 6^{th} payment : $\frac{\text{c+5}}{\text{b+c+10}} \times \frac{4\text{ N}}{7}=\frac{\text{c}}{\text{b+c}} \times \text{N}$

$$\frac{4}{7}\left(\frac{c+5}{20}\right)=\frac{c}{10}$$

$$\Rightarrow c=2 \Rightarrow b=8$$

For B , at 6^{th} payment :

$$\frac{\mathrm{b}+5}{\mathrm{b}+\mathrm{c}+10} \times \frac{4\ \mathrm{N}}{7}=\frac{\mathrm{b}}{\mathrm{b}+\mathrm{c}}\left(\frac{\mathrm{N}}{2}\right)-1000$$

$$\Rightarrow \frac{13}{20} \times \frac{4\ \mathrm{N}}{7}=\frac{8}{10}\left(\frac{\mathrm{N}}{2}\right)-1000$$

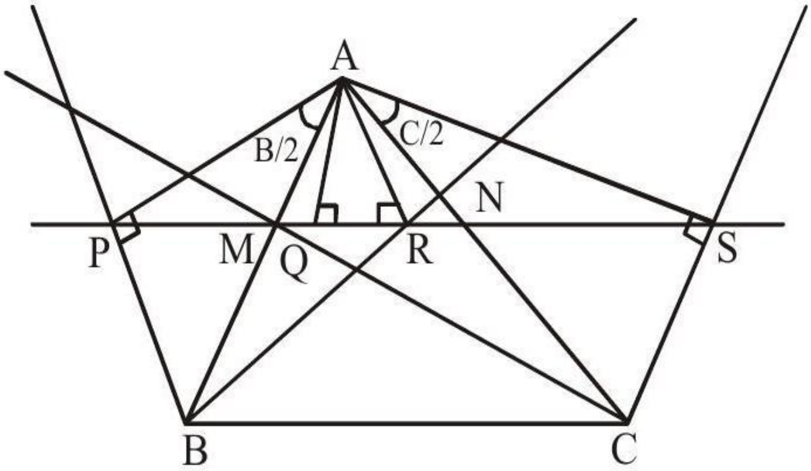
$$\Rightarrow \mathrm{N}\left(\frac{2}{5}-\frac{13}{35}\right)=1000$$

$$\Rightarrow \mathrm{N}=35{,}000$$

$$\mathrm{N}=35$$

22. In triangle ABC, let P and R be the feet of the perpendiculars from A onto the external and internal bisectors of $\angle \mathrm{ABC}$, respectively; and let Q and S be the feet of the perpendiculars from A onto the internal and external bisectors of $\angle \mathrm{ACB}$, respectively. If $\mathrm{PQ}=7, \mathrm{QR}=6$ and $\mathrm{RS}=8$, what is the area of triangle ABC ?

Sol.



Let M, N be the midpoints of AB, AC. Join PM, NS

M is circumcenter of right angled

$\Delta APB \Rightarrow \angle MAP = \angle MPA = B/2$

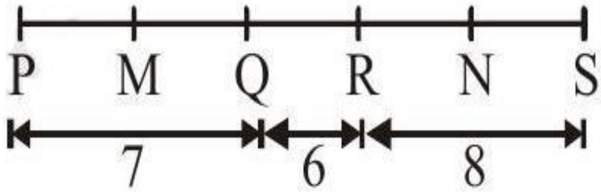
$\Rightarrow \angle BMP = B = \angle MBC$

$\Rightarrow PM || BC$

Similarly NS || BC

MN||BC (midpoint theorem)

$\Rightarrow$ P, M, N, S are collinear



Now, Join MR, NQ

M is circumcentre of right angled $\triangle ARB$

$MR = MB \Rightarrow \angle MRB = \angle MBR = B/2 = \angle RBC$

$\Rightarrow MR||BC$

Similarly NQ || BC

M, Q, R, N are collinear

$\Rightarrow$ PMQRNS is a straight line.

Also PR, QS are diameterrs of cyclic quadrilateral APBR, ASCQ $= PM = MR$ and $QN = NS$

$PM = MR = \frac{13}{2}, QN = NS = \frac{14}{2} = 7$

$\Rightarrow PA = PB = \frac{13}{2}, NA = NC = 7$

$\Rightarrow AB = 13 \Rightarrow AC = 14$

$MN = MR + QN - QR = \frac{13}{2} + 7 - 6 = \frac{15}{2}$

$\Rightarrow BC = 2MN = 15$ (Mid point Theorem)

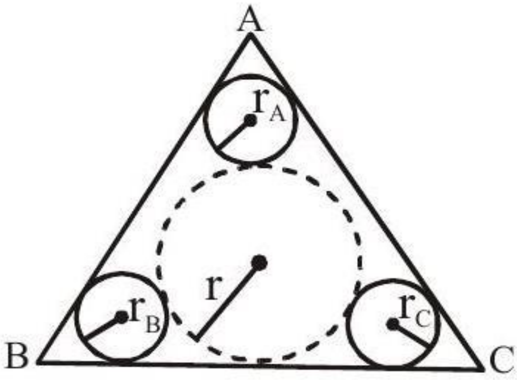
Sides are 13, 14, 15

$\Rightarrow \therefore$ area $= 84$ (Heron's formula)

23. The incircle of a scalene triangle ABC touches BC at D, CA at E and AB at F. Let r_A be the radius of the circle inside ABC which is tangent to Γ and the sides AB and AC. Define $\boldsymbol{r_B}$ and $\boldsymbol{r_C}$ similarly. If $\boldsymbol{r_A = 16, r_B = 25}$ and $\boldsymbol{r_C = 36}$, determine the radius of $\boldsymbol{\Gamma}$.

Sol. Using the formula

$$r = \sqrt{r_a \cdot r_b} + \sqrt{r_b \cdot r_c} + \sqrt{r_c \cdot r_a}$$
$$= \sqrt{16 \cdot 25} + \sqrt{25 \cdot 36} + \sqrt{36 \cdot 16}$$
$$= 20 + 30 + 24 = 74$$

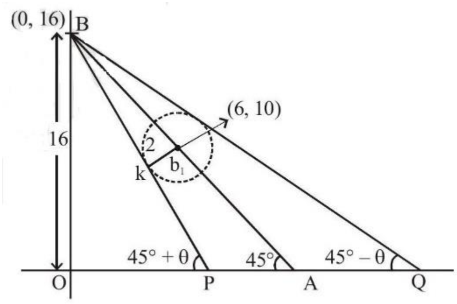


24. A light source at the point $(0,16)$ in the coordinate plane casts light in all directions. A disc (a circle along with its interior) of radius 2 with center at $(6,10)$ casts a shadow on the X axis. The length of the shadow can be written in the form $\mathrm{m}\sqrt{\mathrm{n}}$ where m, n are positive integers and n is square-free. Find $\mathrm{m}+\mathrm{n}$.

Sol. Here, PQ is the required length of shadow.
Now, slope of $\mathrm{BO}_1 = -1$
Let $\angle\mathrm{PBA} = \angle\mathrm{ABQ} = \theta$
Then,
$\angle\mathrm{OPB} = 45^\circ + \theta, \angle\mathrm{BQO} = 45^\circ - \theta$
Also, $\mathrm{OB} = 16, \mathrm{O}_1\,\mathrm{B} = 6\sqrt{2}$
$\therefore\ \mathrm{OP} = \frac{16}{\tan(45+\theta)}$ and $\mathrm{OQ} = \frac{16}{\tan(45-\theta)}$

So, $\mathrm{PQ} = \mathrm{OQ} - \mathrm{OP}$

$$= 16\left(\frac{1}{\tan(45-\theta)} - \frac{1}{\tan(45+\theta)}\right)$$
$$= 16\left(\frac{\tan\theta + 1}{-\tan\theta + 1} - \frac{-\tan\theta + 1}{\tan\theta + 1}\right)$$
$$= 16\left(\frac{4\tan\theta}{-\tan^2\theta + 1}\right) \qquad \text{........... (1)}$$



But, In $\triangle \mathrm{BO_1K}, \tan\theta = \frac{2}{2\sqrt{17}} = \frac{1}{\sqrt{17}}$

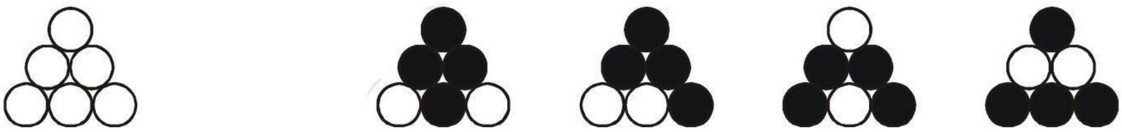
$\therefore\ \mathrm{PQ} = \frac{16.4\left(\frac{1}{\sqrt{17}}\right)}{1-\left(\frac{1}{\sqrt{17}}\right)^2}$ (from (1))

$$\mathrm{PQ} = 4\sqrt{17} = \mathrm{m}\sqrt{\mathrm{n}}$$
$$\text{So, } \mathrm{m} + \mathrm{n} = 21$$

25. For a positive integer n , let $\langle n\rangle$ denote the perfect square integer closest to n . For example, $\langle 74\rangle = 81, \langle 18\rangle = 16$. If N is the smallest positive integer such that
$\langle 91\rangle \cdot \langle 120\rangle \cdot \langle 143\rangle \cdot \langle 180\rangle \cdot \langle \mathrm{N}\rangle = 91 \cdot 120 \cdot 143 \cdot 180 \cdot \mathrm{N}$
find the sum of the squares of the digits of N .
Ans. (56)

Sol. $\langle 91\rangle = 100$
$\langle 120\rangle = 121$
$\langle 143\rangle = 144$
$\langle 180\rangle = 169$
$\therefore\ 81 \cdot 121 \cdot 144 \cdot 169 \cdot \langle \mathrm{N}\rangle = 91 \cdot 120 \cdot 143 \cdot 180 \cdot \mathrm{N}$
$\Rightarrow \langle \mathrm{N}\rangle = \frac{91 \cdot 120 \cdot 143 \cdot 180 \cdot \mathrm{N}}{100 \cdot 121 \cdot 144 \cdot 169}$
$\Rightarrow \langle \mathrm{N}\rangle = \frac{21}{22}\mathrm{N}$
Now to make $\langle \mathrm{N}\rangle$ to be a perfect square, we can take smallest N to be $2 \cdot 11 \cdot 3 \cdot 7 = 162$
$\therefore\ \langle \mathrm{N}\rangle = \frac{21}{22}\mathrm{N} = \frac{3 \cdot 7 \cdot 2 \cdot 11 \cdot 3 \cdot 7}{2 \cdot 11} = (21)^2 = 441$
Which is the nearest perfect square to 462 .
∴ Sum of square of digits of 462 is $4^2 + 6^2 + 2^2$
$$= 16 + 36 + 4 = 56$$

26. In the figure below, 4 of the 6 disks are to be colored black and 2 are to be colored white. Two colorings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same.

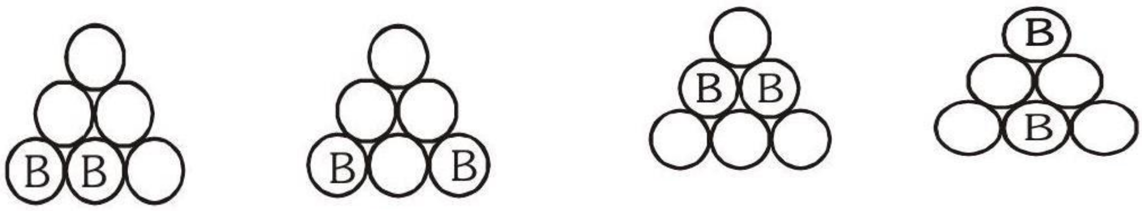


(i) (ii) (iii) (iv)

Fig. 1

There are only four such colorings for the given two colors, as shown in Figure 1. In how many ways can we color the 6 disks such that 2 are colored black, 2 are colored white, 2 are colored blue with the given identification condition?

Sol. We can colour 2 black in following four mutually exclusive ways

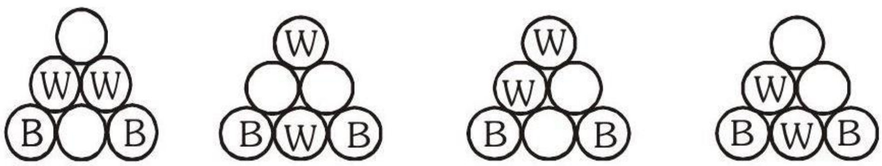


(i) (ii) (iii) (iv)

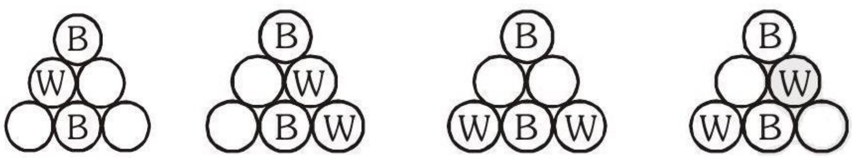
In (i) 2 W and 2 Blue can be placed in $\frac{\mathbf{4!}}{\mathbf{2!2!}} = \mathbf{6}$ ways

In (ii), (iii), (iv) we have 4 ways to colour as shown:

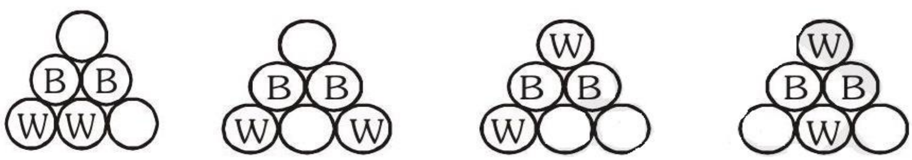
(ii)



(iii)



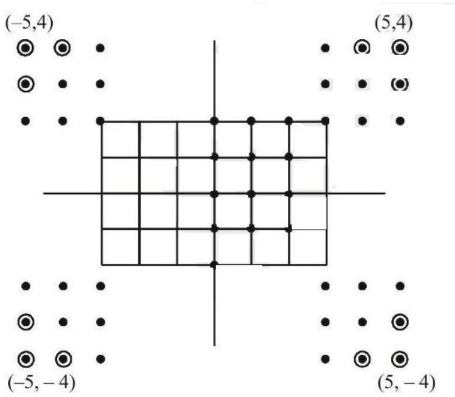
(iv)



Hence total ways $= \mathbf{6 + 3 \times 4 = 18}$ ways.

27. A bug travels in the coordinate plane moving only along the lines that are parallel to the x axis or $\boldsymbol{y}$ axis. Let $\boldsymbol{A} = (-\mathbf{3}, \mathbf{2})$ and $\boldsymbol{B}(\mathbf{3}, -\mathbf{2})$. Consider all possible paths of the bug from $\boldsymbol{A}$ to $\boldsymbol{B}$ of length at most 14. How many points with integer coordinatess lie on at least one of these paths?

Sol.



So, there are total of 11 vertical line and 9 horizontal lines creating a total of 99 integral coordinates. Now, among these 99 points, the points which are marked in red, these must be excluded, as bug has to coverup at most 14 steps (i.e., length $= \mathbf{14}$)
So, there are $\mathbf{99 - 12 = 87}$ points with integer coordinates.

28. A natural number $\boldsymbol{n}$ is said to be good if $\boldsymbol{n}$ is the sum of $\boldsymbol{r}$ consecutive positive integers, for some $\boldsymbol{r} \geq \mathbf{2}$. Find the number of good numbers in the set $\{\mathbf{1}, \mathbf{2}, \dots\dots, \mathbf{100}\}$.
Ans. (93)

Sol. Let us check several values of r starting from 2
i.e.
$\mathbf{r = 2}: \{(\mathbf{1}, \mathbf{2}); (\mathbf{2}, \mathbf{3}); (\mathbf{3}, \mathbf{4}); \dots\dots\dots\dots (\mathbf{49}, \mathbf{50})\}$
These are 49 pairs with $\mathbf{n = 3, 5, 7}$............. 99
$\mathbf{r = 3}: \{(\mathbf{1}, \mathbf{2}, \mathbf{3})$; (2, 3, 4); ....(32, 33, 34)}
These are 32 pairs with $\mathbf{n = 6, 9, 12}$, ... 99
$\mathbf{r = 4}: \{(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}); (\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}); (\mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}) \dots\dots\dots\dots (\mathbf{23}, \mathbf{24}, \mathbf{25}, \mathbf{26})\}$
These are 23 pairs with $\mathbf{n = 10, 14, \dots\dots\dots\dots 98}$
$\mathbf{r = 5}: \{(\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}): \dots\dots\dots\dots\dots\dots\dots\dots\dots (\mathbf{18}, \mathbf{19}, \mathbf{20}, \mathbf{21}, \mathbf{22})\}$
These are 18 pairs with $\mathbf{n = 15, 20}$, ... 100.
$\mathbf{r = 11}: \{(\mathbf{1}, \mathbf{2}, \dots . \mathbf{11}); (\mathbf{2}, \mathbf{3}, \dots . \mathbf{12}); \dots . (\mathbf{4}, \mathbf{5} \dots\dots\dots . \mathbf{14})\}$
$\mathbf{r = 12}: \{(\mathbf{1}, \mathbf{2}, \dots . \mathbf{12}); (\mathbf{2}, \mathbf{3}, \dots . \mathbf{13})\}$
$\mathbf{r = 13}: \{(\mathbf{1}, \mathbf{2}, \dots . \mathbf{13})\}$

Observing this, we observe that $\mathbf{n = 1, 2, 4, 8, 16, 32, 64}$ is not coming.

$\therefore$ Good numbers $= 100 - 7 = 93$

29. Positive integers a, b, c satisfy $\frac{ab}{a-b} = c$. What is the largest possible value of $a + b + c$ not exceeding 99?

Sol. $a, b, c \in \mathbb{Z}^+$

$$\frac{ab}{a-b} = c$$

$$\frac{\mathrm{a-b}}{\mathrm{ab}} = \frac{1}{\mathrm{c}}$$

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c}$$

$$\frac{1}{b} = \frac{1}{a} + \frac{1}{c}$$

Let $\gcd(a, c) = k$, then then $a = kx, c = ky$, which in turns gives b also to be a multiple of k. So, a, b, c all must be a multiple of k.

One possible solution is $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$

i.e., $a = 3, b = 2, c = 6$

So, other solutions can be taken $\mathrm{a} = 3\mathrm{k}, \mathrm{b} = 2\mathrm{k}, \mathrm{c} = 6\mathrm{k}$
Now, $\mathrm{a} + \mathrm{b} + \mathrm{c} \leq 99$

$$kx + ky + kz \leq 99$$
$$k(x + y + z) \leq 99 = 9 \times 11$$

Let us check for the largest possible value i.e., 99

$$k(x + y + z) = 99$$

Suppose $\mathrm{k} = 9$ and considering

$$a = 3 \times 9 = 27$$
$$b = 2 \times 9 = 189$$

and $c = 6 \times 9 = 54$, and checking $\frac{1}{18} = \frac{1}{27} + \frac{1}{54}$ which is satisfying and giving the largest possible value of $\mathrm{a} + \mathrm{b} + \mathrm{c}$ to be $18 + 27 + 54 = 99$

30. Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number, $a + 1$ divides $\mathrm{b} - 1$ and b divides $\mathrm{a}^2 + \mathrm{a} + 2$.
Ans. (16)

Sol. $a, b \in \mathbb{N}$
$(a+1) \mid (b-1)$
$b \mid (a^2+a+2)$
let $b-1 = k(a+1)$, where k is any positive integer.
$b = ka+k+1$
Now $(ka+k+1) \mid (a^2+a+2)$
$\Rightarrow (ka+k+1) \mid (ka^2+ka+2k)$
$(ka+k+1) \mid (ka^2+ka+2k-(ka^2+ka+a))$
$(ka+k+1) \mid (2k-a)$
Put

$$a = 2k, \text{ such that } (ka+k+1) \mid 0$$

then $b = k(2k)+k+1$
$= 2k^2+k+1$
Now, $100 \leq b \leq 999$
So, $100 \leq 2k^2+1 | \leq 999$
Checking certain values of k , we get $k \in [7,22]$
these are 16 possible values by b.
$\therefore$ 16 pairs of (a, b).

PRMO 2019

1.Consider the sequence of number $[n+\sqrt{2n}+\frac{1}{2}]$ for $n\geq 1, when\ [x]$ denotes the greatest integer not exceeding x. If the missing integers in the sequence are $n_1 < n_2 < n_3 < \cdots . then\ find$ n_{12}.

SOLUTION: $[n+\sqrt{2n}+\frac{1}{2}] = [(\sqrt{n}+\frac{1}{\sqrt{2}})^2]$

Let P = $[(\sqrt{n}+0.7)^2] \rightarrow GIF$

Given (n≥1), put n =1→ $P = 2$

n = 2 → $P = 4$

n =3 → $P = 5$

n=4 → $P = 7$

n = 5 → $P = 8$

Here we can see that missing number are: 1,3,6, 10… which is following a certain pattern

Missing number:(1→ 3→ 6 → 10 … … … 66,78)

+2 +3 +4 +5

Hence n_{12} =78.

2.If x= $\sqrt{2}+\sqrt{3}+\sqrt{6}$ is a root of $x^4+ax^3+bx^2+cx+d=0$, where a, b, c, d are integers, what is the value of $|a+b+c+d|$?

Solution: x= $\sqrt{2}+\sqrt{3}+\sqrt{6}$

$x-\sqrt{6} = \sqrt{2}+\sqrt{3}$

$(x-\sqrt{6})^2 = (\sqrt{2}+\sqrt{3})^2$

$x^2 + 6 - 2\sqrt{6x} = 5 + 2\sqrt{6}$

$x^2 + 6 - 5 = \sqrt{6}x + 2\sqrt{6}$

$x^2 + 1 = 2\sqrt{6}\ (x+1)$

$(x^2+1)^2 = 24(x^2+2x+1)$

$X^4 + 1 + 2x^2 = 24x^2 + 48x + 2$

$\Rightarrow x^4 - 22x^2 - 48x - 23 = 0$

On comparing with equation, $x^4 + ax^3 + bx^2 + cx + d = 0$ we get

a = 0, b= -22, c= -48, d= -23

Hence, $|a+b+c+d| = |0-22-48-23| = 93$.

3.Find the number of positive integers less than 101 that can not be written as the difference of two squares of integer.

Solution: Note that every odd number less than 101 can be written as

$(k+1)^2 - (k)^2 = 2k + 1$

Thus, if any even number can be written as difference of two perfect square, then that number must be a multiple of 4, because $2k = a^2 - b^2 = (a+b)(a-b) \equiv 0 \pmod 4$ if a and b are of same parity. Also, every multiple of 4 can be written as

$(k+1)^2 - (k-1)^2 = 4k$. Hence, total number of numbers less than 101 of the form $a^2 - b^2$ are 50+25 =75. Hence, the answer is 25.

4. Let a_1 =24 and form the sequence a_n, n≥ 2 by $a_n = 100a_{n-1} + 134$. The first few terms are 24,2534,253534,25353534, … What is the least value of n for which a_n is divisible by 99?

Solution: Every term in the sequence is of the form a_n = 2(53) (53)4 Where the number of 53 is the number is n-1. Also, if 99la_n ⇔ 9la_n and 11la_n.. Thus, by divisibility rule of 9, we get
0≡ a_n ≡ 2 + 8(n-1) +4(mod9) ⇔ n≡ $7(mod9)$.
Now, by divisibility of 11, we get
0≡ a_n ≡ (2+3(n-1)) -(5(n-1) +4) (mod 11) ⇔ n ≡ 0(mod 11)
But, the minimum solution to the congruences a_n ≡ 7 (mod 9) and a_n ≡ 0 (mod 11) is 88.
Thus, n= 88.

5.let N be the smallest positive integer such that N+ 2N+ 3N+…. + 9N is a number all whose digits are equal. What is the sum of the digits of N?
Solution: N + 2N + 3N + …. + 9N
=N (1 + 2 + 3 + …. + 9)
N× $\frac{9 \times 10}{2}$ =45× N
We have to multiply with '45' to a number such that, the resulting number should have all digits same.
Such N = 12345679
As 45× $12345679 = 555555555$
∴ sum of digits of N =37.
6. Let ABC be a triangle such that AB = AC. Suppose the tangent to the circumcircle of $\triangle ABC\ at$ B is perpendicular to AC. Find $\angle ABC$ measured in degrees.
Solution: Let tangent at B intersect AC at P.
Now, note that $\angle ABC = \angle ACB = \theta \ \Rightarrow \angle BAP = 2\theta$
$\Rightarrow \angle PBA = \theta,$
In $\triangle\ PBA$,
$3\theta = 90°$
$\theta = 30°$
7. Let s(n) denotes the sum of the digits of a positive integer n in base 10. If s(m)=20 and s(33m) =120, what is the value of s(3m)?
Solution: We will take sum of digits Base 10 to (mod 9)
Also, s(ab) =s(a). s (b) (mod 9)
Now, s(m)=20
S(33m) = 120 = s (11) × $s(3m)$
120=2× $s\ (m)$ [$\because s(11) = 2(mod\ 9)$]
60= s (3m)
Hence, s (3m) =60
8. Let F_k(a, b) =$(a, b)^k - a^k - b^k\ and$ let S = {1,2,3,4,5,6,7,8,9,10}. For how many ordered pairs (a, b) with a, b∈ $S\ and\ a \leq b\ is\ \frac{F_5(a,b)}{F_3(a,b)}$ an integer?
Solution: $\frac{(a+b)^5 - a^5 - b^5}{(a+b)^3 - a^3 - b^3}$

$$\Rightarrow \frac{a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^4 - a^5 - b^5}{a^3 + 3a^2b + 3ab^2 + b^2 - a^3 - b^3}$$

$$\Rightarrow \frac{5ab[a^3 + 2a^2b + 2ab^2 + b^3]}{3ab[a + b]}$$

$\Rightarrow \frac{5}{3}\frac{[a^3+3a^2b+3ab^2+b^3-a^2b-ab^2]}{(a+b)}$

$\Rightarrow \frac{5}{3}\frac{[(a+b)^3+ab(a+b)]}{(a+b)}$

$\Rightarrow \frac{5}{3}\frac{[(a+b)\{(a+b)^2-ab\}]}{(a+b)}$

$\Rightarrow \frac{5}{3}[a^2+2ab+b^2-ab]$

$\Rightarrow \frac{5}{3}[a^2+ab+b^2]$

Now, $3|a^2+ab+b^2$

Both a and b cannot simultaneously be even.

Three cases are possible

CASE 1: a$\equiv 0(mod\ 3)$

B$\equiv 0(mod\ 3)$

$\therefore (a\ ,b\ )=(3,3);(6,6);(9,9) \quad \Rightarrow 3\ pairs$

CASE 2: a$\equiv 1(mod\ 3)$

b$\equiv 1(mod\ 3)$

$\therefore (a\ ,b)$= (1,1); (1,4); (1,7) ;(1,10) ;(4,4) ;(4,7); (4,10) ;(7,7); (7,10) ;(10,10) $\Rightarrow$ $10\ pairs$

CASE 3: a$\equiv 2(mod\ 3)$

b$\equiv 2(mod\ 3)$

$\therefore$ (a, b) = (2,2); (2,5); (2,8); (5,5); (5,8) ;(8,8) $\Rightarrow 6\ pairs$

and (3,6); (3,9); (6,9) $\Rightarrow 3\ pairs$

$\therefore Total = 3+10+6+3 = 22\ pairs$

9.The center of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of the triangle ABC $\alpha°$ *and the smaller one* $\beta°$ then what is the value of $\alpha-\beta$?

Solution: The center of any circle with D and E on it must pass through the (potentially extended) bisector of $\angle A$. *For this centre to be on the circumcircle of* $\triangle ABC$ *, the only possibility is for the centre to be* A *itself*.

AF= AD since they are both radii of the same circle.

AD = BD since D is the midpoint of $\overline{AB}$

AF$\perp BC$, *since* $\triangle ABC$ *is isosceles*,

Therefore, since AB =2AF,$\angle B = 30°$, *that makes* $\angle A = 120°$, so the difference between them is 90°.

10. One day I went for a walk in the morning at x minute past 5'O clock, where x ix a two-digit number. When I returned, it was y minutes past 6'O clock, and I noticed that (i) I walked exactly for x minutes and (ii) y was a 2digit number obtained by reversing the digit of x. How many minutes did I walk?

Solution: Let x=ab

Where a and b are the unit digits

Time after 5'O clock =5$\times 60+10a+b$

= (300+10a +b) minutes

Time after 6'O clock =$6\times 60+10b+a$

=(360+10b+a) minutes

$\therefore 360+10b+a-300-10a-b=10a+b$

60 +9b – 9a =10a+b

60 + 8b =19a

$\therefore b=\frac{19a-60}{8}$

Now a & b are integers from (1 to 9) by putting different values of a, we get a= 4, b= 2

$\therefore x=42\ minutes$

11.find the largest value of a^b *such that the positive integers* $a, b>1$ *satisfy* $a^b b^a+a^b+b^a=5329$.

Solution: a, b > 1

$a^b b^a+a^b+b^a=5329$

$a^b b^a+a^b+b^a+1=5330$

$(a^b+1)(b^a+1)=2\times 5\times 13\times 41$

$=65\times 82$

$a^b+1=82$

$a^b=81=3^4$

A= 3, b = 4

Now, $b^a=4^3=64\ as\ b^a+1=65$

$\therefore Largest\ \ values\ of\ a^b=81$

12. Let N be the number of ways of choosing a subset of 5 distinct numbers from the set {10a+b: $1\leq a\leq 5, 1\leq b\leq 5\}$

Where a, b are the integers, such that no two of the selected numbers have the same units digit and no two have the same tens digit. What is the remainder when N is divided by 73?

Solution: 10a +b; $1\leq a\leq 5, 1\leq b\leq 5$

Let us divide numbers into different sets, such as

Set 1: {11,2,13,14,15}

Set 2: {21,22,23,24,25}

Set 3: {31,3,2,33,34,35}

Set 4: {41,42,43,44,45}

Set 5: {51,52,53,54,55}

Now to make a number having no two unit's digit and no two ten's digit same, we can select any 1 number from each of set 1, set 2, set 3, set 4 and set 5

$\therefore Number\ of\ ways: C_1^5\times C_1^4\times C_1^3\times C_1^2\times C_1^1$

=5! =120

$\therefore 120\div 73\Rightarrow Remainder=47$

13. Consider the sequence

1,7,8,49,50,56,57, 343, ……

Which consists of sums of distinct powers of 7 , that is $7^0, 7^1, 7^0+7^1, 7^2, \ldots\ldots$, *in increasing order* . *At what position will* 16856 *occur int this sequence* ?

Solution: Serial number	Binary	$(Base)_7$	$Series$
1	$(1)_2$	$\longrightarrow (1)_7$	7^0 = 1
2	$(10)_2$	$\longrightarrow (10)_7$	$7^1=1$

3	$(11)_2 \longrightarrow (11)_7$	$7^1 + 7^0 = 8$
4	$(100)_2 \longrightarrow (100)_7$	$7^2 = 49$
5	$(101)_2 \longrightarrow (101)_7$	$7^2 + 7^0 = 50$
6	$(110)_2 \longrightarrow (110)_7$	$7^2 + 7^1 = 56$

(16856) = $(100\ 100)_7 \rightarrow (100\ 100)_2 = 2^5 + 2^2$ =36^{th}
$\therefore 16856\ is\ 36^{th}\ term.$
14. Let R denotes the circular region int the x y-plane bounded by the circle $x^2 + y^2 = 36$ .the
Lines x=4 and y=3 divide R into four regions
$R_i, i = 1,2,3,4. If\ |R_i| denotes\ the\ area\ of\ the\ region\ \ R_i\ and\ if\ [R_1] > [R_2] > [R_3] > [R_4], determine\ [R_1] - [R_2] - [R_3] + [R_4]. (Here\ [\Omega] denotes\ the\ area\ of\ the\ region\ \Omega\ in\ the\ plane).$
Solution: We have $\sum[R_1] = 36\pi$
$[R_1] - [R_2] - [R_3] + [R_4]$
=2$([R_1] - [R_2]) - \sum[R_1]$
=2$([R_1] + [R_4]) - 36$
Now cos$\alpha = \frac{2}{3}$
also, $\theta = \frac{7\pi}{6}$-α
$[R_1] = \frac{1}{2} \times 36 \times \left(\frac{7\pi}{6} - \alpha\right) + \frac{1}{2}(4 + 3\sqrt{3}) \times 3 + \frac{1}{2} \times (3 + 2\sqrt{5}) \times 4$
Now $\angle BOP = \alpha\ as\ \cos\alpha = \frac{2}{3}$
$\Rightarrow \angle BOA = \alpha - \frac{\pi}{6}$
So,$[R_4] = \frac{1}{2} \times 36\left(\alpha - \frac{\pi}{6}\right) - \frac{1}{2} \times \left(2\sqrt{5} - 3\right) \times 4 - \frac{1}{2}(3\sqrt{3} - 4) \times 3$
Thus $[R_1] + [R_4] = 18\pi + 12 + 12 = 24 + 18\pi$
$\therefore Required\ answer = 48$
15.In base -2 notation ,digits are 0 and 1 only and the places go up in powers of -2 .for example ,11011 stands for$(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0\ \ and\ equals\ \ number\ 7\ in\ base\ 10. If\ the\ decimal\ number\ 2019\ is\ expressed\ in\ base\ -2\ how\ many\ non\ zero\ \ digits\ does\ it\ contain?$
Solution: 2019= 2048-32+4-2+1
$= 2^{12} + -2^{11} + -2^5 + -2^1 + -2^0$
=4096-2048-32+4-2+1
=1100000100111 (in base -2)
Numbers of non-zero digits =6
16.Let N denotes the number of all the natural numbers n such that n is divisible by a prime p>$\sqrt{n}\ and\ p < 20. What\ is\ the\ value\ of\ N?$
Solution: n=natural number
P=prime number
P<20
$\therefore p^2 < 400$
Also, $\sqrt{n} < p$

$\therefore n < p^2 < 400$
So, n$\in \{1,2, \ldots ..399\}$
If p=2, then n <2^2 $\Rightarrow n < 4$
$\therefore$ n=2 only case $\Rightarrow 1\ solution$
If p=3, then n<3^2 $\Rightarrow n < 9$
$\therefore n = 3, n = 6$ $\Rightarrow 2\ solution$
If p=5, then n<5^2 $\Rightarrow n < 25$
$\therefore n = 5,10,15,20$ $\Rightarrow 4\ solution$
if p=7, then n<7^2 $\Rightarrow n < 49$
$\therefore n = 7,14,21,28,35,42$ $\Rightarrow 6\ solution$
If p=11, then n<11^2 $\Rightarrow n < 121$
$\therefore n = 11,22,33,44,55,66,77,88,99,110$ $\Rightarrow 10\ solution$
If p=13, then n<13^2 $\Rightarrow n < 169$
$\therefore n = 13,26,39, \ldots \ldots ,156$ $\Rightarrow 12\ solution$
If p= 17, then n<17^2 $\Rightarrow n < 289$
$\therefore n = 17,34, \ldots \ldots ,272$ $\Rightarrow 16\ solution$
If p=19, then n<19^2 $\Rightarrow n < 361$
$\therefore n = 19,38, \ldots \ldots .. ,342$ $\Rightarrow 18\ solution$
Total 1+2+4+6+10+12+16+18$\Rightarrow$ 69
17.Let a, b, c be distinct positive integers such that b + c - a, c +a-b and a +b-c are all perfect squares. What is the largest possible value of a +b +c smaller than 1000?
Solution: Let b +c-a=x^2 … … … (i)
c +a -b =y^2 … … . . (ii)
a+ b -c =z^2 … … . (iii)
Now since a, b, c are distinct positive integers,
$\therefore x, y, z\ will\ also\ be\ positive\ integers,$
Add (i), (ii), and (iii)
a+ b + c =$x^2 + y^2 + z^2$
Now, we need to find the largest value of a + b+ c or$x^2 + y^2 + z^2$ less than 100
Now, to get a, b, c all integers x, y, z all must be of same parity, i.e. either all three are even or all three are odd.
Let us maximize $x^2 + y^2 + z^2$, for both cases.
If x, y, z are all even.
$\Rightarrow b + c - a = 8^2 = 64$
c+ a -b =$4^2 = 16$
a + b – c=$2^2 = 4$
Which on solving, give a= 5, b=34, c=40 and a + b + c =84
If x, y, z are all odd
$\Rightarrow b + c - a = 9^2 = 81$
c+ a -b =$3^2 = 9$
a + b – c=$1^2 = 1$
Which on solving, give a=5, b =41, c=45 and a + b + c<100 =91
$\therefore Maximum\ value\ of\ a + b + c < 100 = 91$

18. What is the smallest prime number p such that$p^3 + 4p^2 + 4p$ $has\ exactly\ 30\ positive\ divisors$?

Solution: $p^3 + 4p^2 + 4p$

$\Rightarrow p(p^2 + 4p + 4)$

$\Rightarrow p(p + 2)^2$

This number has 30 divisors so it can be in the form of

a.b^{14}

$a^2.b^9$

a.$b^2.c^4$

a^{29}

Out of these cases, we will check cases in which p can be minimum which seems to be possible with a.$b^2.c^4\ case$

So, by simply checking several cases,

We can put p=43

$\because \Rightarrow 43(45)^2$

$\Rightarrow 43 \times 15^2 \times 3^2$

$\Rightarrow 43 \times 5^2 \times 3^2 \times 3^2$

$\Rightarrow 43 \times 5^2 \times 3^4$

Whose number of divisors are

(1+1). (2+1). (4+1)

$2\times 3 \times 5 = 30$

$\therefore p = 43$

19. If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible are of the triangle to the nearest integer?

Solution: Area of$\Delta BGC = \frac{1}{2} \times 6 \times 10 \sin\theta$

To maximize the area of $\Delta BGC, \sin\theta$

$\therefore maximize\ area\ of\ \Delta BGC = 30$

Maximize area of $\Delta ABC = 3\Delta BGC$

=$3\times 30 = 90 sq.units$

20.How many 4-digits numbers $\overline{abcd}\ are\ there\ such\ that\ a < b < c < d\ and\ b - a < c - b < d - c$?

Solution: $\overline{abcd}$

$\therefore a < b < c < d$

$\therefore a \geq 1$

b≥ 2

c≥ 3

d≥ 4

Aso, b – a < c – b

i.e. 2b < a + c

and c - b < d – c

2c < b + d

We can make a table applying all these conditions,

a	b	c	d
1	2	4	7
1	2	4	8
1	2	4	9
1	2	5	9
2	3	5	8
2	3	5	9
3	4	6	9

So total 7 numbers are possible.

21. In parallelogram ABCD, AC =10 and BD =28. The points K and L in the plane of ABCD move in such a way that AK =BD and BL= AC. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $cot^{\theta}\left(\angle\frac{BMD}{2}\right) + tan^2(\angle\frac{ANC}{2})$.

Solution: Produce CD to K' such that CD=DK'

Then BDK'A is a parallelogram

$\because AB = CD = DK'$

AB||DK'

$\therefore AK' = BD$

Draw a circle with center A and radius BD which cuts CD produced at K' and CB produced at k' then K'' Ak' are collinear as $\angle CDA + \angle BAD = 180°$

$\angle CDA = \angle DAK' + \angle DK'A = \angle DAK' + \angle BAK'$ $\quad [\because BA||DK']$

$\therefore \angle DAK' + \angle BAK'' + \angle BAD = 180°$

Thus K'AK" is a diameter.

Let K is any point on this circle

Since M is a midpoint of CK

D is a midpoint of CK'

Then MD||KK'

In $\Delta CK'K''$,

D is a midpoint of CK'

DB|| K'A i.e., DB||KK'

$\therefore B\ is\ a\ midpoint\ of\ CK''$

In $\Delta CK''K$

B, M are the mid points of CK'' and CK respectively,

$\therefore In\ \Delta BMD\ and\ \Delta K''KK'$

BM||K''K

MD||KK'

BD||K''K'

$\therefore \angle BMD = \angle K''KK' = 90°$

$\because K''K' is\ a\ diameter\ similarly\ for\ other\ \Delta$

$\angle ANC = 90°$

$so, \frac{\angle BMD}{2} = 45°, \frac{\angle ANC}{2} = 45°$

$cot^2\frac{(\angle BMD)}{2} + tan^2\frac{(\angle ANC)}{2} = 1 + 1 = 2$

22.Let t be the area of a regular pentagon with each side equal to 1. Let P(x)=0 be the polynomial equation will least degree, having integer coefficients, satisfied by x=t and the GCD of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of P(x). What is the integer closest to $\sqrt{M}$?

(sin18° $= (\sqrt{5}-1)/2)$)

Ans: sin 18° $value\ in\ the\ question\ given\ wrong. Originally$ $\sin 18° = (\sqrt{5}-1)/4)$.

Solution: Area of regular polygon $=\frac{a^2 n}{4\tan\frac{(180)}{n}}$

n= number of sides

a=length of side

$\therefore for\ regular\ pentagon\ of\ side\ length\ 1,$

Area=t=$\frac{5}{4tan36°}$= $\frac{5}{4(0.73)} \cong 1.71$

Now, P (1.71) =0 to be found with least degree and integer coefficient such that GCD of all coefficients is 1.

Let x=1.71

100x=171

$\therefore P(x) = 100x - 171 = 0\ is\ the\ required\ polynomial$

Which satisfied all the conditions.

$\therefore m = |100| + |-171| = 271$

$\therefore \sqrt{m} = 16.46$

$\therefore nearest\ integer = 16$

But this question can have multiple solutions as student can take tan36°

$as\ 0.72, 0.726\ or\ even\ 0.7\ , every\ time\ we\ will\ get\ different\ answers.$

$So\ this\ question\ should\ be\ bonus.$

23.For n $\geq 1, let\ a_n\ be\ the\ number\ beginning\ with\ n\ 9'sfollowed\ by\ 744; e.g., a_4 = 9999744'. Define\ f(n) = \max\{m \in N\ |2^m\ divides\ a_n\}, for\ n \geq 1. Find\ f(1) + f(2) + f(3) + \ldots. + f(10)$.

Solution: $a_1 = 9744$

$a_2 = 99744$

$a_3 = 999744$

And so on….

$\because 9744\ is\ divisible\ by\ 16$

$\therefore Each\ a_n\ is\ divisible\ \ by\ atleast\ 2^4.$

Now, $a_1 = 10^4 - 256 \equiv 0(mod\ 2^4)$

$a_2 = 10^5 - 256 \equiv 0(mod\ 2^5)$

$a_3 = 10^6 - 256 \equiv 0(mod\ 2^6)$

$a_4 = 10^7 - 256 \equiv 0(mod\ 2^7)$

$a_5 = 10^8 - 256 = 256(390625 - 1)$

=256$\times$ 390264

=256$\times 32 \times 12207$

=$2^{13} \times 12207$

=0$\equiv$ $(mod\ 2^{13})$

$a_6 = 10^9 - 256 \equiv 0(mod\ 2^8)$

$a_7 = 10^{10} - 256 \equiv 0(mod\ 2^8)$
$a_8 = 10^{11} - 256 \equiv 0(mod\ 2^8)$
$a_9 = 10^{12} - 256 \equiv 0(mod\ 2^8)$
$a_{10} = 10^{13} - 256 \equiv 0(mod\ 2^8)$

$\therefore f(1) + f(2) + \ldots\ldots + f(10)$

=4+5+6+7+13+8+8+8+8+8

=75

24. Let ABC be an isosceles triangle with AB =BC.A trisector of $\angle B$ *meets* AC *at* D, AC *and* BD *are integers and* AB – BD =3, find AC.

Solution: Let B=6θ

Let BD =x$\in Z$

$\Rightarrow AB = x + 3$

Given AC $\in Z$

A=(3+x) cos 3θ $= x cos\theta$

$\Rightarrow (3 + x)\ cos3\theta = x\ cos\theta$

$$\Rightarrow sin^2\theta = \frac{3}{4(x+3)}$$

Now AL =(3+x) sinθ

$\Rightarrow AC = 2(3 + x)(3sin\theta - 4sin^3\theta)$

=2(3+ x) sin$\theta(3 - 4\frac{3}{4(x+3)})$

=6(2+x)$\frac{\sqrt{3}}{2\sqrt{x+3}} = 3(x + 2)\frac{\sqrt{3}}{\sqrt{x+3}}$

$$\Rightarrow x = 3y \qquad \Rightarrow AC = \frac{3(3y + 2)}{\sqrt{y + 1}}$$

$$\Rightarrow y + 1 = z^2 \qquad \Rightarrow AC = \frac{3(3z^2 - 1)}{z} = 9z - \frac{3}{z}$$

$\Rightarrow z = 1\ or\ 3$

But z$\neq$ $1\ as\ x = 0$ *not possible*

$\Rightarrow z = 3 \qquad \Rightarrow AC = 26$

25. A friction- less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equals the angle of reflection. The distance travelled by the ball in meters is of the form$\sqrt{N}$, *where* N *is an integer. What is the value of* N?

Solution: $x^2 = 5^2 + 1^2 - 2 \times 5 \times 1 \cos 120°$

=25+1+5

x=$\sqrt{31}$

N=31

Folding the triangle continuously each time of reflection creates the above diagram.9 points of reflection can be verified in the diagram above. Thus root(N) is the length of red line which is root (31). Thus N =31 is the answer.

26.A conical glass is in the form of a right circular cone. The slant height is 21m and the radius of the top rim of the glass is 14.An ant at the mid-point of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass (See the figure).If d the shortest distance it should crawl to reach the honey drop ,what is the integer part of d?(ignore the thickness of the glass).

Solution: Rotate ΔOAP $by\ 120°\ in\ anticlockwise\ then\ A\ will\ be\ at\ B, P\ will\ be\ at\ P'$

$\Rightarrow \Delta OAP \equiv \Delta OBP'$

$\Rightarrow PB + PA = P'B + PB \geq P'P$

Minimum PB+PA=P'P equality when P on the angle bisector of$\angle AOB$

$\Rightarrow P'P = 2(21)sin60° = 21\sqrt{3}$

[min (PB +PA)] = $[21\sqrt{3} = 36$

27.In a triangle ABC, it is known that $\angle A = 100°$ $and\ AB = AC$ $.The\ internal\ angle\ bisector\ BD\ has\ length\ 20\ units$.

$Find\ the\ length\ of BC\ to\ the\ nearest\ integer,$

Given that sin 10° ≈ 0.174

Solution: Given, BD =20 units

$\angle A = 100°$

AB=AC

InΔABD

$$\frac{BD}{sinA} = \frac{AD}{sin2\theta}$$

$$\frac{BD}{sin100°} = \frac{AD}{sin20°}$$

$$\frac{BD}{cos10°} = \frac{AD}{2sin10°cos20°}$$

$$\Rightarrow 20 = \frac{AD}{2sin10°} \qquad \Rightarrow AD = 40.sin10° = 6.96$$

InΔBDC

Also, $\frac{BD}{sin40°} = \frac{BC}{sin120°} = \frac{CD}{\sin 20°}$

$$\frac{20}{2sin20°.cos20°} = \frac{CD}{sin20°} \Rightarrow CD = \frac{20}{2cos20°} = \frac{20}{2 \times 0.9394} \cong 10.65$$

$\therefore AD + CD = AC = AB \cong 17.6$

Now, since BD is angle bisector

So,$\frac{BC}{AB} = \frac{CD}{AD} \Rightarrow BC = \frac{AB \times CD}{AD} = \frac{17.6 \times CD}{6.96} \cong 26.98 \cong 27$

28.Let ABC be an acute angled triangle with Ab=15 and BC=8.Let d be a point on AB such that BD =BC .Consider points E on AC such that $\angle DEB = \angle BEC. If\ \alpha\ denotes\ the\ product\ of\ all\ possible\ values$ of AE ,find $[\alpha]\ the\ integer\ part\ of \alpha.$

Solution: The pairs $E_1, E_2\ satisfies\ \ condition\ or\ E_1 = intersection\ of CBO\ with\ AC\ and\ E_2 = intersection\ of \angle bisector\ of\ B\ and\ AC$

$\because that\ \angle DE_2B = \angle CE_2$B and for$E_1 \angle BE_1C = \angle BDC = BCD = \angle BE_1D$

$\Rightarrow \overline{AE_1}, \overline{AC} = \overline{AD}.\overline{AB} = 7 \times 15$

$$\frac{\overline{AE_2}}{\overline{AC}}=\frac{XY}{XC}$$
(for y is midpoint of OC and X is foot of altitude from A to CD)

Also, $\frac{XD}{DY}=\frac{7}{8}$ $and\ DY=YC$

$$\Rightarrow\frac{XD+DY}{XC}=\frac{15}{7+8+8}=\frac{15}{23}$$

$$\Rightarrow\frac{XY}{XC}=\frac{15}{23}\Rightarrow\frac{AE_2}{AC}=\frac{15}{23}$$

$\Rightarrow AE_1.A\overline{E_2}=\frac{15}{23}.7.15=\frac{225\times 7}{23}$=68.

30.For any real number x, let [x] denotes the integer part of x; {x} be the fractional part of x ({x}=x-[x]). Let A denote the set of all real numbers x satisfying {x}=$\frac{x+[x]+[x+\left(\frac{1}{2}\right)]}{20}$ If S is the sum of all numbers in A, find [S].

Ans: 21

Solution: {x} =$\frac{x+[\,x]+\left[x+\frac{1}{2}\right]}{20}\Rightarrow 20f\ =2I+f+\left[x+\frac{1}{2}\right]$

$$\Rightarrow 19f\ =2I+\left[x+\frac{1}{2}\right]$$

Let x=I +F =[x] +{x}

CASE I: 0≤ $f<\frac{1}{2}$ $i.e.,\left[x+\frac{1}{2}\right]=I$

So, 19f= 3I$\epsilon\left[0,\frac{19}{2}\right)$

$$\Rightarrow I\epsilon\left[0,\frac{19}{6}\right)$$

Hence, x=I+ f= I+$\frac{3I}{19}=\frac{22I}{19}$;

I=0,1,2,3

CASE II: f$\epsilon\left[\frac{1}{2},1\right)$ $\quad i.e.,\left[x+\frac{1}{2}\right]\ =I+1$

So, 19f=3I +1$\epsilon\left[\frac{19}{2},19\right)$

$$\Rightarrow I\epsilon\left[\frac{17}{6},6\right)$$

$\Rightarrow I=3,4,5$

Hence, x= I+ f=I +$\frac{3I+1}{19}$ $\qquad \frac{22I}{19}+\frac{1}{19};I=3,4,5$

Thus, S=$\frac{22}{19}\times 18+\frac{3}{19}=21$

[S]=21.

संदर्भ / REFERENCES

1. https://nbhm.dae.gov.in/olympiad.html
2. https://olympiads.hbcse.tifr.res.in/mathematical-olympiad/
3. IMO short listed problems with solutions, 61st International Mathematics Olympiad, Saint – Petersburg, Russia
4. 250 Problems in Elementary Number Theory - Sierpinski (1970)
5. Algebra, Cengage Learning India Pvt Limited, DPP
6. Coordinate Geometry, Cengage Learning India Pvt Limited, DPP
7. Inequalities A Mathematical Olympiad Approach - Manfrino, Ortega, and Delgado (Birkhauser, 2009)
8. Integer & Polynomial Algebra, Kenneth R Davidson & Matthew Satriano
9. IOQM Previous Year's Questions & Answers – collected from different web sources
10. Basics of Olympiad Inequalities, By Samin Riast