STUDENT SUPPORT MATERIAL

PRMO

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REGIONAL OFFICE AGRA

STUDENT SUPPORT MATERIAL PRMO

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Number Theory

- An integer b is said to be divisible by a non- zero integer 'a' if there is an integer 'x' such that b=ax and we then write a/b. In case 'b' is not divisible by a we write a $\nmid$b. The property a/b may also be expressed by saying that 'a divides b' or 'a is divisor of b' or 'b is a multiple of a'.
- Let a, b, c, m, x, y be integers
  - (i) If a/b then a/bc for any integer c
  - (ii) If a/b and b/c then a/c
  - (iii) If a/b and a/c then $a/(bx+cy)$ for any integer x and y
  - (iv) If a/b and $b \neq 0$ then $|a| \leq |b|$
  - (v) If a/b and b/a then $a = \pm b$
  - (vi) If m=0 then a/b if and only if ma/mb

Congruences:-

- Let m be a non- zero integer. The integers a and b are said to be congruent modulo m if and only if $m/a-b$ and written $a \equiv b \pmod{m}$
- Let a, b, c, d, x, y denote integer then
  - (i) $a \equiv b \pmod{m}$ $b \equiv a \pmod{m}$ and $a - b \equiv 0 \pmod{m}$ are equivalent statements
  - (ii) If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $a \equiv c \pmod{m}$
  - (iii) If $a \equiv b \pmod{m}$ $c \equiv d \pmod{m}$ then $ax + cy \equiv bx + dy \pmod{m}$
  - (iv) If $a \equiv b \pmod{m}$ $c \equiv d \pmod{m}$ then $ac \equiv bd \pmod{m}$ further, if $a \equiv b \pmod{m}$ then $a^k = b^k \pmod{m}$ for every positive integer k
  - (v) If $a \equiv b \pmod{m}$ and d/m, then $a \equiv b \pmod{d}$
- The number $\phi(m)$ is the number of positive integers less than or equal to m and respectively prime to m

  for Example $\phi(6) = 2$ (1 and 6,5 and 6)

  $\phi(8) = 4$ (1 and 8,3 and 8,5 and 8)

  $\phi(11) = 10$ (1 and 8,3 and 8,7 and 8)

Note - for any prime number ‘p’

$$\phi\ (p) = p = 1$$

Formula for finding $\phi\ (m) \rightarrow$ If a number m written in the form

$$m = p_1^{n_1}.p_2^{n_2}.p_3^{n_3}............$$

Where $p_1.p_2.p_3............$ are different prime and $n_1.n_2.n_3............$ are positive integers then

$$\phi(m) = m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\left(1-\frac{1}{p_3}\right)..............$$

- Total number of divisors of m

  $$(n_1+1)(n_2+1)(n_3+1)........$$

<u>Euler’s Theorem :</u> Let a,m be integers such that (a,m)=1 (i.e. a and m are relative prime) Then $a^{\phi(m)} \equiv 1(\bmod m)$

<u>Fermat’s Theorem :</u> Let p be a prime number and a be an integer then

$$a^p \equiv a(\bmod p)$$

<u>Wilson’s Theorem:</u> Let p be a prime Then

$$(p-1)! \equiv -1(\bmod p)$$

Ex-1 Find total no. of integers n such that n^2+1 is divisible by n+1

Sol:- Let n be an integers such that $(n+1)/n^2+1$

also $n^2-1=(n+1)\ (n-1)$

so $(n+1)\ (n^2-1)$

$\Rightarrow (n+1)/(n^2+1)-(n^2-1)$

$\Rightarrow (n+1)/2$

$\Rightarrow n+1=\pm 1,\ \pm 2$

$\Rightarrow n=-3,-2,0,1$

Ex-2 If n=1!+2!+3!+............+100!

Find sum of digits of remainders when n is divided by 240

Sol:- We have 6!= 6×5×4×3×2×1

Which is divisible by 240

so all 6!, 7!, 8! ..... 100! are divisible by 240

Now 1!+2!+3!+4!+5!= 1+2+6+24+120

= 153

50 remainder = 153

50 sum of digits of remainder =9

Ex-3 Find the unit place of $(213)^{51645}$

Sol:- We have unit place of $(213)^{51645}$ is same as unit place of $(3)^{51645}$

Now divide the power of 3 by 4 and use the table

Remainders	1	2	3	0
Power	1	2	3	4

As
$2^2=(4)$
$(12)^2=14(4)$

$4\overline{)45}(11$
44
(1)

3^{51645}

$\Rightarrow 3$

$\Rightarrow 3$

Unit place =3

Ex-4 Find the remainders for $(3^{560}/8)$

Sol:- $\frac{3^{560}}{8} = \frac{(3^2)^{280}}{8} = \frac{9^{280}}{8}$

$= \frac{[9.9.9.............(280 times)]}{8}$

remainder for above expression

= remainder for (1.1.1............ (280 times)]/8

$\Rightarrow$ Remainder = 1

Ex-5 Find the tens place of the expression 3^{85}

Sol:- In order to find tens place of a number we need to divide that number by 100 and find remainder

we have $3^{85}=(3^{40})^2\times3^5$

By Euler's theorem

$a^{\phi(m)} \equiv 1(\mod m)$ if $(a,m)=1$

Here (3,100)=1

We have $100=2^2\times5^2$

$\phi(100) = 100(1-\frac{1}{2})(1-\frac{1}{5})$

= 40

Therefore

$(3^{\phi(100)} \equiv 1)/(\mod 100)$

$\Rightarrow 3^{40} \equiv 1 \ (\mod 100)$

$\Rightarrow (3^{40})^2 \equiv 1^2 \ (\mod 100)$

$\Rightarrow 3^{80} \equiv 1^2 \ (\mod 100)$

Also $3^5 = 243 \equiv 43(\mod 100)$

i.e $3^5 = 43 \ (\mod 100)$

Now $3^{80} \times 3^{5} \equiv 1 \times 43 (\text{mod}\, 100)$

$\Rightarrow 3^{85} \equiv 43 \ (\text{mod}\, 100)$

so tens place of

$\Rightarrow 3^{85} \, is \, "4"$

****

Practice Exercise

1. Find the last two digits of the number $7^{100}-3^{100}$

Ans 00

2. find the least integer n such that n does not divide 2^n-2 but n divides 3^n-3

Ans 06

3. n is a number, such that the 2n has 28 factors and 3n has 30 factors, then find no of factors of 6n

Ans. 35

4. When 4444^{4444} is written in decimal notation, the sum of its digit is A let B be the sum of the digits of A. Find the sum of the digits of B.

Ans. 07

5. Determine the number of all 3 digits number N , such that N is divisible by $11 \& \frac{N}{11}$ is equal to the sum of the squares of the digits of N.

Ans. 02(550 & 803)

6. Find the GCD of the numbers 2n+13 and n+7, Where $n \in N$

Ans 1

7. Find the remainder of 2^{100} when divided by3.

Ans 1

8. If (a,4)=(b,4)=2 and a number n divides a+b Find n(n is greatest perfect square.

Ans 4

9. Find Total number of Positive integers $x \neq 3$ such that $x-3 \mid x^3-3$

Ans 8

10. Find total number of divisors of 14400

And 63

Solution of Practice Exercise

Sol:-1 To find the last 2 digits of a number means to find the remainder when that numbers is divided by 100

Note that $100=25\times 5$ & $(25,4)=1$

We have $7 \equiv 3(mod 4) \Rightarrow 7^{100} \equiv 3^{100}(\text{mod } 4)$

now $(7,25)$ & $(3,25)=1$

By Euler's theorem $7^{20} \equiv 1(\text{mod } 25)$ & $3^{20} \equiv 1(\text{mod } 25)$

$\Rightarrow 7^{20} \equiv 3^{20}(\text{mod } 25) \Rightarrow 7^{100} \equiv 3^{100}(\text{mod } 25)$

$\because (25,4)=1, \quad so \quad 7^{100} \equiv 3^{100}(\text{mod } 100)$

Hence, the last 2 digits of the numbers $7^{100}-3^{100}$ are 00

Sol:-2 If n does not divide $2^n–2$

$\Rightarrow$ n is a composite number

The least composite number is 4

but $4 \nmid 3^4-3=78$

Next composite numbers is 6 and $6 \nmid 2^6–2$

but $6 | 3^6–3$

so least numbers is "6"

Sol:-3 Since 2 n has 28 factors & 3n has 30

so $2n = p_1^1 \times p_2^1 \times p_3^6 \ or \ p_4^3 \times p_5^6$ ($\because$ 28=2x2x7=4x7

& $3n = p_6^1 \times p_7^2 \times p_8^4 \ or \ p_9^4 \times p_{10}^5$ ($\because$ 30=2x3x5=5x6)

$\Rightarrow p_4 = p_4 = 3$ & $p_5 = P_{10} = 2$

now $6n = 3^4 \times 2^6$

so 6n has (4+1)(6+1) =35 factors

Sol:-4 We have the remainder when 4444 is divided by 9 is 7

we can also find it by adding 4+4+4+4=16, 1+6=7

Let $N=4444^{4444} < 10000^{4444}$, so N has at most 4×4444=17776 digits , each of which is at most 9. So A is at most 17776×9=159984. With the largest digits sum is 99999 so B$\leq$12 as sum of digits in B is at most 3+9=12

Now $1 \equiv A \equiv B(\text{mod } 9)$ & $4444 \equiv 7(\text{mod } 9)$

$\because\ 7^3 \equiv 1(\text{mod } 9),$ we get

$$4444^{4444} \equiv 7^{4444} \equiv 7(\text{mod } 9)$$

as $4444 \equiv 1(\text{mod } 2)$

Hence the sum of digits of $B = 7$

Sol:-5 Let N=100a+10b+c

$\therefore$ N is divisible by 11

so either a+c=b or $|\text{a+c-b}| = 11$

Also $\frac{N}{11}$ is equal to $a^2 + b^2 + c^2$

When $a + c = b$ $\quad N = 100a + 10(a + c) + c$

$$N = 11(100 + c)$$

$$= \frac{N}{11} = 100 + c = a^2 + b^2 + c^2$$

so by putting $c = 0, 1, 2.....$

we get N=550 or 803

Sol:-6 we know that for any two integers a,b not both zero

(a,b)=1 if and only if there are inegers x,y such that ax+by=1

we have $2(n+7) + (-1)(2n+13) = 1$

$\Rightarrow (n+7, 2n+13) = 1$

GCD of $n+7,\ \ 2n+13 = 1$

Sol:-7 We have $2^{100} = (2^4)^{25} = (16)^{25}$ & remainder when 16 divided by 3 is 1

so $\frac{(16)^{25}}{3} \Rightarrow \frac{(1)^{25}}{3} \Rightarrow \frac{1}{3}$

Hence remainder =1

Sol:-8 $\because (a,4)=(b,4)=2$

$\Rightarrow a=4n+2,\ b=4n+2$

$\Rightarrow a+b=4(m+n)+4$

Now $(a+b,4)=4$

so $n=4$

Sol:9-Let x be a integers such that $(x-3)/x^3-3$ observe that

$x^3-27 = (x-3)(x^2+3x+9)$

$\Rightarrow (x-3)\mid x^3-27$

$\Rightarrow (x-3)\mid (x^3-3)-(x^3-27)$

i.e $x-3\mid 24$

$\Rightarrow x-3=1,2,3,4,6,8,12,24$

Hence there are 8 such positive integers

Sol:-10 14400 =144×100

=16×9×4×25

=2^6×3^2×5^2

Total numbers of divisors of

14400 is (6+1) (2+1) (2+1)

= 63

ALGEBRA

1.1 <u>Polynomial Functions</u>:-A function $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$ is a polynomial function if $a_i (i = 0,1,2,3,........n)$ is a constant which belongs to the set of real numbers and the indices , n,n-1, -----1 are natural numbers. If $a_n \neq 0$ then we say that $f(x)$ is a polynomial of degree n.

Example : $f(x) = x^4 - x^3 + x^2 - 2x + 1$ is a polynomial , of degree 4. and 1 is a zero of this polynomial as $f(1) = 1^4 - 1^3 + 1^2 - 2(1) + 1 = 0$

Example : $f(x) = x^2 - (\sqrt{3} - \sqrt{2})\, x - \sqrt{6} = 0,$ if $x = \sqrt{3}$

as $f(\sqrt{3}) = 3 - 3 + \sqrt{6} - \sqrt{6} = 0$

i.e $f(x)$ is a polynomial with degree 2 and $\sqrt{3}$ is a zero of this polynomial

Degree	Polynomial
1	Linear
2	Quadratic
3	Cubic
4	bi-quadratic

Note : Above definition and examples refer to polynomial function in one variable,

Similarly, Polynomials in 2,3,....... n variables can be defined.

Example : $f(x, y, z) = x^2 - xy + z + 5$ is a polynomial in x, y, z i.e. three variables , of degree 2 as both x^2 and xy have degree 2 each.

1.2 <u>**Remainder Theorem**</u>:- If a polynomial $f(x)$ is divided by $(x - a)$, then the remainder is equal to $f(a)$

Proof : $f(x) = (x - a) \quad Q(x) + R$

$f\ (a) = R\ (remainder)$

If $R = o \Rightarrow\ (x-a)$ is a factor of $f(x)$

<u>Factor Theorem</u> :- $(x-a)$ is a factor of polynomial $f(x)$ if and only if $f(a)=0$

Fundamental Theorem of algebra:- Every polynomial function of degree ≥ 1 has at least one zero in the complex numbers.

From this it is easy to deduce that a polynomial function of degree 'n' has exactly n zeroes.

Note:- (i) If a polynomial equation with real coefficients has a complex root p+iq (p,q real numbers, $q \neq 0$) then it also has a complex root $p - iq$

(ii) If a polynomial equation with rational coefficients has an irrational root $p+\sqrt{q}$ (p,q rational, $q>0$) then it also has an irrational root $p-\sqrt{q}$

(iii) If rational number $\frac{p}{q}$ is a root of equation $a_0x^n + a_1x^{n-1} + ... + a_n = 0$ where $a_0, a_1 ....a_n$ are integers and $a_n \neq 0$, then p is a divisor of a_n and q is a divisor of a_0

(iv) $(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Pascal's Triangle

```
         1              row 0
       1   1            row 1
     1   2   1          row 2
   1   3   3   1        row 3
  1  4   6   4  1       row 4
 1  5  10  10  5  1     row 5
1  6 15  20  15  6  1   row 6
```

The n[th] row of triangle represents the coefficients of $(a+b)^n$ in standard order.

Each term in polynomial has the powers of its two variables adding up to n.

(v) Factoring :- It is the opposite of expansion

- or a quadratic eq[4] in the form $ax^2+bx+c=0$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ is called Quadratic formula.

- Method of completing the square
- Relation b/w roots and coefficients

(a) If $\alpha,\ \beta$ are roots of equation $ax^2+bx+c=0$ then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

(b) If α,β,γ are roots of equation $ax^3+bx^2+cx+d=0$ then $\alpha+\beta+\gamma=\frac{-b}{a}$,

$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$, $\alpha\beta\gamma=\frac{-d}{a}$

(c) If $\alpha,\beta,\gamma\ \ \delta$ be roots of equation $ax^4+bx^3+cx^2+dx+e=0$ then

$$\left\{\alpha+\beta+\gamma+\delta=-\frac{b}{a}\right\},\left\{\begin{array}{c}\gamma\delta(\alpha+\beta)+\alpha\beta(\gamma+\delta)\\ or\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-\frac{d}{a}\end{array}\right\}$$

$$\left\{\begin{array}{l}\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}\\ or\ (\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=\frac{c}{a}\end{array}\right\},\left\{\alpha\beta\gamma\delta=\frac{e}{a}\right\}$$

1.3 Identities:-

(a) If $a+b+c=0, a^2+b^2+c^2=-2(ab+bc+ca)$

(b) If $a+b+c=0,\ a^3+b^3+c^3=3abc$

(c) If $a+b+c=0,\ a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

$$=\frac{1}{2}(a^2+b^2+c^2)^2$$

- $a^3+b^3+c^3-3abc=(a+b+c)\ (a^2+b^2+c^2-ab-bc-ca)$
- $(a+b+c)^3=a^3+b^3+c^3+3(a+b)\ (b+c)\ (c+a)$
- $a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\}$

1.4 Pigeon Hole Principle (PHP): If more than 'n' objects are distributed in 'n' boxes , then at least one box has more than one object in it.

1.5 Basic Properties and Facts

1. Arithmetic Operations

$$\frac{\left(\frac{a}{b}\right)}{C}=\frac{a}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)}=\frac{ad}{bc}$$

$$\frac{a}{\left(\frac{b}{c}\right)}=\frac{ac}{b} \qquad \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

2. Exponent Properties

$$a^m a^n=a^{m+n} \qquad (ab)^n=a^n b^n$$

$$\frac{a^m}{a^n}=a^{m-n} \qquad \left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$$

$(a^m)^n = a^{mn}$ $\qquad$ $a^{-n} = \dfrac{1}{a^n}$

$a^0 = 1, \quad a \neq 0$

3. Properties of Radicals

$\sqrt[n]{a} = a^{\frac{1}{n}}$ $\qquad$ $\sqrt[n]{ab} = \sqrt[n]{a} \; \sqrt[n]{b}$

4. Properties of inequalities

If a<b then a+c< b+c and a-c<b–c

If a<b and c>0, then ac<bc and $\dfrac{a}{c} < \dfrac{b}{c}$

If a<b and c<0, then ac>bc and $\dfrac{a}{c} > \dfrac{b}{c}$

5. Properties of absolute value

$$|a| = \begin{cases} a & if\, a \geq 0 \\ -a & if a < 0 \end{cases}$$

$|a| \geq 0$ $\qquad$ $|-a| = |a|$

Triangle Inequality $|a+b| \leq |a| + |b|$

6. Complex Numbers

$i = \sqrt{-1}$, $\quad i^2 = -1$, $\quad \sqrt{-a} = i\sqrt{a}, \quad a \geq 0$

$(a+ib)+(c+id) = a+c+i\ (b+d)$

$(a+ib)-(c+id) = a-c+i\ (b-d)$

$(a+ib)(c+id) = ac-bd+(ad+bc)i$

$(a+ib)(a-ib) = a^2+b^2$

Modulus of a complex= $|a+ib| = \sqrt{a^2 + b^2}$

Conjugate of a complex $\overline{(a+ib)} = a - ib$

7. Logarithms and Log Properties

Definition : $\log_b x = y$ is equivalent to $x = b^y$

Example $\log_5 125 = 3$ because $5^3 = 125$

(natural log) $\log_e x$, (common log) $\log_{10} x$

Where e=2.718281828........

$\log_a a = 1$

$\log(a^x) = x.\log a$

$\log_b 4 = \dfrac{\log_a 4}{\log_a b}$

$\log_a b = \dfrac{1}{\log_b a}$

$\log_a 1 = 0$

$\log_b (xy) = \log_b x + \log_b y$

$\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

The domain of log $\log_b x \;\; is \;\; x > 0$

8. Factoring formulae

$x^2 - a^2 = (x+a)\ (x-a)$

$x^3 + a^3 = (x+a)\ (x^2 - ax + a^2)$

$x^3 - a^3 = (x-a)\ (x^2 + ax + a^2)$

$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + ...a^{n-1})$

$x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - ...... + a^{n-1})$

9. Absolute value Equations/ Inequalities

if b is a positive number

$|P| = b \quad \Rightarrow P = -b \quad or \quad P = b$

$|P| < b \quad \Rightarrow -b \quad < P < b$

$|P| > b \quad \Rightarrow P < -b \quad or \ P > b$

1.6 Arithmetic Progression (A.P)

a , a+d, a+2d, ..., $a + \overline{n-1}\ d$

1. $a_n = a + (n-1)d$

2. $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + a_n)$

3. Arithmetic mean between a and b = $\frac{a+b}{2}$

or in general, $A.M = \frac{a_1 + a_2 + ...a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} ai$

(B) Geometric Progression (G.P)

$a, \quad ar, \quad ar^2, \quad ar^3, ..., ar^{n-1}$

$a_n = ar^{n-1}$

$S_n = \frac{a(r^n - 1)}{r-1}$ When r>1, $S_n = \frac{a(1-r^n)}{1-r}$ When r<1

sum of infinite G.P when r<1

$S_\infty = \frac{a}{1-r}$

Geometric Mean G.M = $\sqrt{ab}$

(c) Harmonic Progression

Whose reciprocals are in A.P

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\ldots\ldots\ldots \frac{1}{a+\overline{n-1}d}$$

$$a_n = \frac{1}{a+\overline{n-1}d}$$

If Harmonic Mean of a and b = H

then $\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$

Note (i) A.M> G.M> H.M , (AM) (HM) = $(GM)^2$

1.7 MEAN , MEDIAN, MODE

Mean $\frac{\sum x_i}{n} = \frac{\sum f_i x_i}{n} = a + h \frac{\sum fiui}{\sum fi}$

Median = $\left\{\begin{matrix}\text{Middle Most} \\ \text{term when} \\ \text{n= odd}\end{matrix}\right\}$ = $\left\{\begin{matrix}\text{Mean of two} \\ \text{middle terms} \\ \text{when n= even}\end{matrix}\right\}$ = $l + \left(\frac{\frac{N}{2} - c.f}{f}\right)h$

for median, first arrange data in ascending order

Mode = term (observation)with highest frequency

$$\ell + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times h$$

Empirical Formula : Mean– Mode= 3(mean–median)

1.8 Inequalities

(i) Triangle Inequality : If x_1, x_2, y_1, y_2 be any real numbers, then

$$\sqrt{\left\{(x_1-x_2)^2+(y_1-y_2)^2\right\}} \le \sqrt{x_1^{\,2}+y_1^{\,2}}+\sqrt{x_2^{\,2}+y_2^{\,2}}$$

where '$\sqrt{\ }$' sign denotes the positive square root.

(ii) Cauchy-Schwarz Inequality : for any $a_1, a_2, ...a_n$ and $b_1, b_2, ...b_n$

$$a_1b_1+a_2b_2+......+a_nb_n \le \sqrt{a_1^{\,2}+a_2^{\,2}+.......a_n^{\,2}}\sqrt{b_1^{\,2}+b_2^{\,2}+.......+b_{n^2}}$$

or $\left(\sum a_ib_i\right)^2 \le \sum a_i^2 \ \sum b_i^2$

Proof consider f(x)=$(a_1x-b_1)^2+(a_2x-b_2)^2+.......+(a_nx-b_n)^2 \ge 0$

or f(x)= $\left(\sum a_i^{\,2}\right)x^2-2\left(\sum a_ib_i\right)x+\left(\sum bi^2\right) \longrightarrow (1)$

which is a quadratic in x. It's graph is an up parabola. Since $f \ge 0$, the graph either touches x-axis or stays in upper half of it. $\Rightarrow$ either (1) is a perfect square or doesn't have any real root. $\Rightarrow$ Discriminant $\le$ 0

$\Rightarrow$ 4 $\left(\sum a_ib_i\right)^2-4\sum a_i^2\sum b_i^2 \le 0$

$\Rightarrow$ $\left(\sum a_ib_i\right)^2 \le \sum a_i^2\sum b_i^2$ *Hence Proved*

(iii) Tcheby Chef's Inequality : If $a_1, a_2, a_3, b_1, b_2, b_3$ are any real numbers s.t. $a_1 \le a_2 \le a_3, b_1 \le b_2 \le b_3, then \ \ 3(a_1b_1+a_2b_2+a_3b_3) \ge (a_1+a_2+a_3)(b_1+b_2+b_3)$

Example 1- Let p(x)=0 be a fifth degree polynomial equation with integer co-efficients that has at least one integral root. If p(2)=13 and p(10)=5. Compute a value of x that must satisfy p(x)=0.

Sol. P(2)=13, P(10)=5 [Remainder Th^m

[Factor Th^m

Let n be required value of x, which is integral root, so (x-n) is a factor of P(x)

$\Rightarrow P(x)=(x-n)Q(x)$ [degree of $Q(x)$ =4

P(2)=(2-n) $Q(x)$ =13 [13,5 are Prime number

P(10) = (10-n) $Qx=5$

$\Rightarrow 2-n=\pm 1 or \pm 13$ and 10-n = $\pm 1 or \pm 5$

Possible values of n are

$n=\{1,3,-11,\textcircled{15}\}\ and\ n=\{11,9,\textcircled{15},5\}$

$\Rightarrow n=15$ is required value of x (root).

Example -2 The roots of $x^3+px^2+qx-19=0$ are each one more than the roots of $x^3-Ax^2+Bx-C=0$. If A,B,C,P,Q are constants, compute A+B+C.

Sol. Let $x^3+px^2+qx-19=0 \longrightarrow (1)$

and $x^3-Ax^2+Bx-C=0 \longrightarrow (2)$

Let equation (2) has roots a,b,c

then equation (1) has roots a+1, b+1, c+1

$\Rightarrow (a+1)(b+1)(c+1)=19$

To find A+B+C = $(a+b+c)+(ab+bc+ca)+abc$

= (a+1) (b+1) (c+1) - 1

= 19-1

= 18

Example 3 : The roots of the equation $x^5-40x^4+px^3+qx^2+rx+s=0$ are in G.P. The sum of their reciprocal is 10. Compute the numerical value of $|s|$.

Solution : Let the roots be $\frac{a}{r^2},\frac{a}{r},a,ar,ar^2$

$\therefore$ sum of roots = a$\left[\frac{1}{r^2}+\frac{1}{r}+1+r+r^2\right]=40 \longrightarrow (1)$

& Sum of reciprocals = $\frac{1}{a}\left[r^2 + r + 1 + \frac{1}{r} + \frac{1}{r^2}\right] = 10 \longrightarrow (2)$

Dividing (1) by (2), $a^2 = 4 \quad \therefore a = \pm 2$

since, s = - $\left(\frac{a}{r^2}.\frac{a}{r}.a.ar.ar^2\right) = -a^5$

s = , $\pm 32\, or\, |s| = 32$

Example 4 : Let p(x)=0 be the polynomial equation of least possible degree with rational coefficients, having $\sqrt[3]{7} + \sqrt[3]{49}$ as a root. Compute the product of all the roots of p(x)=0.

Solution : Let x = $\sqrt[3]{7} + \sqrt[3]{49}$

$\therefore \; x^3 = 7 + 49 + 3\sqrt[3]{7}\sqrt[3]{49}\left(\sqrt[3]{7} + \sqrt[3]{49}\right)$

i.e. $x^3 = 56 + 21x$

Thus p(x)= $x^3 - 21x - 56 = 0$

and product of the roots is 56.

Example 5: If log 2 = 0.301 and log3=0.4771, find the number of digits in 48^{12}.

Sol. we have log $48^{12} = 12x\log 48$

= 12 x log $\left(2^4 x 3\right)$

= 12 x [4 log 2 + log 3)
= 12 x [1.204+0.4771]
= 20.1732

Now, characteristic is 20, so required number of digits = 20+1=21

PRACTICE PROBLEMS

1. Evaluate :

$$y = \sqrt{\log_2 3.\log_2 12.\log_2 48.\log_2 192 + 16} - \log_2 12.\log_2 48 + 20$$

[Ans=16]

2. Suppose a,b are positive real numbers s.t.

$$a\sqrt{a} + b\sqrt{b} = 183, a\sqrt{b} + b\sqrt{a} = 182$$

find $\frac{9}{5}(a+b)$ [Ans=73]

3. The product of two of the four roots of $x^4 - 20x^3 + kx^2 + 590x - 1992 = 0$ is 24. Find k. [Ans=41]

4. If $x = 2 + \sqrt{3},$ then find the value of $2x^4 - 4\sqrt{3}x^3 - 8x + 44$ [Ans=42]

Practice Exercise

1. Find the last two digits of the number $7^{100-}3^{100}$

Ans 00

2. find the least integer n such that n does not divide 2^n–2 but n divides 3^n–3

Ans 06

3. n is a number, such that the 2n has 28 factors and 3n has 30 factors, then find no of factors of 6n

Ans. 35

4. When 4444^{4444} is written in decimal notation, the sum of its digit is A Let B be the sum of the digits of A. Find the sum of the digits of B.

Ans. 07

5. Determine the number of all 3 digits number N , such that N is divisible by $11 \& \frac{N}{11}$ is equal to the sum of the squares of the digits of N.

Ans. 02(550 & 803)

6. Find the GCD of the numbers 2n+13 and n+7, Where $n \in N$

Ans 1

7. Find the remainder of 2^{100} when divided by 3.

Ans 1

8. If (a,4)=(b,4)=2 and a number n divides a+b Find n(n is greatest perfect square.

Ans 4

9. Find Total number of Positive integers $x \neq 3$ such that $x-3 \mid x^3-3$

Ans 8

10. Find total number of divisors of 14400

And 63

Solution of Practice Exercise

Sol:-1 To find the last 2 digits of a number means to find the remainder when that numbers is divided by 100

Note that $100=25\times 5$ & $(25,4)=1$

We have $7\equiv 3(mod 4)\Rightarrow 7^{100}\equiv 3^{100}(\mathrm{mod}\,4)$

now $(7,25)$ & $(3,25)=1$

By Euler's theorem $7^{20}\equiv 1(\mathrm{mod}\,25)$ & $3^{20}\equiv 1(\mathrm{mod}\,25)$

$\Rightarrow 7^{20}\equiv 3^{20}(\mathrm{mod}\,25)\Rightarrow 7^{100}\equiv 3^{100}(\mathrm{mod}\,25)$

$\because\ (25,4)=1,\quad so\quad 7^{100}\equiv 3^{100}(\mathrm{mod}\,100)$

Hence, the last 2 digits of the numbers $7^{100}-3^{100}$ are 00

Sol:-2 If n does not divide 2^n–2

$\Rightarrow$ n is a composite number

The least composite number is 4

but $4\nmid 3^4-3=78$

Next composite numbers is 6 and $6\nmid 2^6$–2

but $6|\ 3^6$–3

so least numbers is "6"

Sol:-3 Since 2 n has 28 factors & 3n has 30

so $2n=p_1^1\times p_2^1\times p_3^6\ or\ p_4^3\times p_5^6$ ($\because$ 28=2x2x7=4x7

& $3n=p_6^1\times p_7^2\times p_8^4\ or\ p_9^4\times p_{10}^5$ ($\because$ 30=2x3x5=5x6)

$\Rightarrow p_4=p_4=3\ \&\ p_5=P_{10}=2$

now $6n=3^4\times 2^6$

so 6n has (4+1)(6+1) =35 factors

Sol:-4 We have the remainder when 4444 is divided by 9 is 7

we can also find it by adding 4+4+4+4=16, 1+6=7

Let $N=4444^{4444}<10000^{4444}$, so N has at most 4×4444=17776 digits , each of which is at most 9. So A is at most 17776×9=159984. With the largest digits sum is 99999 so B$\leq$12 as sum of digits in B is at most 3+9=12

Now $1 \equiv A \equiv B \pmod 9$ & $4444 \equiv 7 \pmod 9$

$\because$ $7^3 \equiv 1 \pmod 9$, we get

$$4444^{4444} \equiv 7^{4444} \equiv 7 \pmod 9$$

as $4444 \equiv 1 \pmod 2$

Hence the sum of digits of $B = 7$

Sol:-5 Let N=100a+10b+c

$\therefore$ N is divisible by 11

so either a+c=b or $|a+c-b| = 11$

Also $\frac{N}{11}$ is equal to $a^2 + b^2 + c^2$

When $a + c = b$ $\quad N = 100a + 10(a + c) + c$

$$N = 11(100 + c)$$

$$= \frac{N}{11} = 100 + c = a^2 + b^2 + c^2$$

so by putting $c = 0, 1, 2.....$

we get N=550 or 803

Sol:-6 we know that for any two integers a,b not both zero

(a,b)=1 if and only if there are inegers x,y such that ax+by=1

we have $2(n + 7) + (-1)(2n + 13) = 1$

$\Rightarrow (n + 7, 2n + 13) = 1$

GCD of $n + 7, \quad 2n + 13 = 1$

Sol:-7 We have $2^{100} = (2^4)^{25} = (16)^{25}$ & remainder when 16 divided by 3 is 1

so $\frac{(16)^{25}}{3} \Rightarrow \frac{(1)^{25}}{3} \Rightarrow \frac{1}{3}$

Hence remainder =1

Sol:-8 $\because (a,4)=(b,4)=2$

$\Rightarrow a=4n+2,\ \ b=4n+2$

$\Rightarrow a+b=4(m+n)+4$

Now $(a+b,4)=4$

so $n=4$

Sol:9-Let x be a integers such that $(x-3)/x^3-3$ observe that

x^3-27 = $(x-3)(x^2+3x+9)$

$\Rightarrow (x-3)\,|\,x^3-27$

$\Rightarrow (x-3)\,|\,(x^3-3)-(x^3-27)$

i.e $x-3\,|\,24$

$\Rightarrow x-3=1,2,3,4,6,8,12,24$

Hence there are 8 such positive integers

Sol:-10 14400 =144×100

=16×9×4×25

=$2^6\times3^2\times5^2$

Total numbers of divisors of

14400 is (6+1) (2+1) (2+1)

= 63

Equations and inequalities

Equations are the foundation of mathematics. All forms of maths rely on the principle of equality.

An equation states that two expressions have the same value. Expressions are what are on either side of the equation. The main rule to solve a linear equation in one variable is that " Whatever is done to one side of an equation must also be done to the other side.

A linear equation in one variable x is of the form ax+b=0, Where a and b are real numbers and $a \neq 0.$. The value of x which satisfies a given linear equation, is called its solution or root.

ax+b=0 $\Rightarrow$ ax= – b $\Rightarrow$ x=$\frac{-b}{a}$

Thus, $\frac{-b}{a}$ is a solution of given linear equation.

An equation of the form ax+by+c=0, Where a,b,c are real numbers such that $a \neq 0$ and $b \neq 0$ is called a linear equation in two variables x and y. If $x=\alpha$ and $y=\beta$ satisfy the equation ax+by+c=0 then we say that the ordered pair (α, β) is its solution. A linear equation in two variables has infinitely many solutions.

A set of multiple equations with same variables is called a system of equations. In most cases, for a system to be solvable, the number of variables must be less than or equal to the number of equations given

Quadratic Equations : The general form of a quadratic equation in x is $ax^2+bx+c=0,$ Where $a,b,c \in R$ and $a \neq 0.$ The solution of quadratic equation, $ax^2+bx+c=0$ is given by $x=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

The expression $b^2-4ac=D$ is called the discriminant of the quadratic equation.

If α and β are the roots of quadratic equation $ax^2+bx+c=0,$ then

(i) $\alpha+\beta=\frac{-b}{a}$ (ii) $\alpha\beta=\frac{c}{a}$ (iii) $\alpha-\beta=\frac{\sqrt{D}}{a}$

Nature of Roots:

(1) Consider the equation $ax^2+bx+c=0,$ Where $a,b,c \in R$ and $a \neq 0$, then ;

(i) $D>0 \Leftrightarrow$ Roots are real and distinct

(ii) D=0 $\Leftrightarrow$ Roots are real and equal

(iii) D<0 $\Leftrightarrow$ Roots are imaginary

(iv) If p+iq is one root of a quadratic equation, then the other must be the conjugate p-iq & vice versa (p,q$\in$R and i = $\sqrt{-1}$)

(2) Consider the equation $ax^2+bx+c=0, \text{ Where } a,b,c \in Q \text{ and } a \neq 0,$ then;

(i) If D>0 & is a perfect square, then roots are rational and unequal).

(ii) If $\alpha = p+\sqrt{q}$ is one root (Where p is rational and $\sqrt{q}$ is a surd) then the other root must be conjugate of it i.e $\beta = p-\sqrt{q}$ & Vice Versa.

(3) A quadratic equation whose roots are α and β is (x-α). (x-β)=0 i.e $x^2-(\alpha+\beta)x+\alpha\beta=0$

(4) Consider the quadratic expression, $y=ax^2+bx+c, a \neq 0$ and $a,b,c, \in R$ then;

The graph between x,y is always a parabola.

If a>0 then the shape of the parabola is concave upwards and if a<0 then the shape of the parabola is concave downwards.

Maximum & Minimum Value:

Maximum and minimum value of $y=ax^2+bx+c$ occurs at $x=-\left(\frac{b}{2a}\right)$ according as:

a<0 or a>0

$$y \in \left[\frac{4ac-b^2}{4a}, \infty\right) if\ a>0 \ \& \ y \in \left(-\infty, \frac{4ac-b^2}{4a}\right] if\ a<0$$

Important Results: For the quadratic equation $ax^2+bx+c=0$

(i) One root will be reciprocal of other if a=c

(ii) One root is zero if c=0

(iii) Roots are equal in magnitude but opposite in sign if b=0

(iv) Both roots are zero if b=c=0.

(v) Roots are positive if a and c are of same sign and b is of opposite sign.

(vi) Roots are of opposite sign if a & c are of opposite sign.

(viii) Roots are negative if a,b,c are of the same sign.

Theory of Equations:

If $\alpha_{1,}\alpha_{2,}\alpha_{3............}\alpha_n$ are the roots of equations

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...+ a_{n-1}x + an = 0$$

Where $a_0, a_{1,}.....a_n$ all are real and $\alpha_0 \neq 0$, then

$$\sum \alpha_1 = \frac{-a_1}{a_0}, \sum \alpha_1\alpha_2 = \frac{+a_2}{a_0}, \sum \alpha_1\alpha_2\alpha_3 = \frac{-a_3}{a_0} ....$$

$$\sum \alpha_1, \alpha_2..........\alpha_n = (-1)^n \frac{a_n}{a_0}$$

(i) If α is a root of the equation f(x)=0, then the polynomial f(x) is exactly divisible by (x-α) or (x-α) is a factor of f(x) and conversely.

(2) Every equation of n[th] degree $(n \geq 1)$ has exactly n roots & If the equation has more than n roots, it is an identity

(3) If the coefficients of the equation f(x)=0 are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root.

(4) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root. Where $\alpha, \beta \in Q$ and β is not a perfect square.

(5) If there be any two real numbers a and b such that f(a) & f(b) are of opposite signs, then f(x)=0 must have atleast one real root between a and b.

(6) Every equation f(x) = 0 of degree odd has at least one real root of a sign opposite to that of its last term.

Inequalities

We denote the set of all positive real numbers by R^+ ,the set of all negative real numbers by R^-. R is the union of R^+, R^- and {0}. Given a real number a, either a is in R^+ or a is in R^- or a=0. One and only one of these possibilities is true. We note that, given any two real numbers a and b, a is said to be less than b (or b is greater than a) is b-a is a positive real number. We write this $a<b$ (or $b>a$). We record the following important properties of this ordering in R.

(i) Given any two real numbers a and b, one and only one of the following three conditions is true

$a<b$ or $a=b$ or $a>b$

(ii) if $a<b$ and c is any real number, then $a+c < b+c$

(iii) if $a<b$ and $c>0$, then $ac<bc$

(iv) if $a>0$, $b>0$ and $a< b$, then $\frac{1}{a} > \frac{1}{b}$

(v) For any real number a, $a^2 \geq 0$

(vi) Inequalities exists only between two real numbers(not complex numbers).

(vii) If $a,b>0$ then $a+b>0$ and $ab>0$

(viii) $a>b$ if $a-b>0$

(ix) $a<b$ if $b-a>0$

(x) $a \geq b$ if either $a>b$ or $a=b$

(xi) $a \leq b$ if either $a<b$ or $a=b$

(xii) if $a>b$ then $ac>bc$ if $a,b,c>0$

(xiii) if $a>b$ and $b>c$ then $a>b>c$ and $a>c$

(xiv) if $a>b>0$ and $c>d>0$ then $ac>bd$

(xv) In a given inequality terms/coefficients from one side to other side can be transferred as in case of an equality

(xvi) We can add/ subtract the same real number on both sides of an inequality, the direction of inequality does not change.

(xvii) Two inequalities with same direction can be added(always) and multiplied (if both sides of inequality are positive). But they can never be subtracted or divided.

(xviii) Both sides of an equality can be multiplied by same positive quantity without changing the direction of inequality.

(xix) The direction of inequality changes if it is multiplied by a negative number on both sides of inequality

Some Basic Inequalities 1. AM-GM inequalities

We know that $a^2 \geq 0$ for any real number a. This is an important inequality in itself. As a consequence of this property, We can derive many inequalities.

Let c and d be any two real numbers. Then we have $(c-d)^2 \geq 0$. Expanding this, we get $c^2-2cd+d^2 \geq 0$

so $\frac{c^2+d^2}{2} \geq cd$

if a and b are non negative reals, by taking $c=\sqrt{a}, \quad d=\sqrt{b}$ in the relation (1) we get

$$\frac{a+b}{2} \geq \sqrt{ab} \qquad (2)$$

Here the number $\frac{a+b}{2}$ is called the Arithmetic Mean (A.M) of a and b. $\sqrt{ab}$ is called the Geometric Mean (G.M) of a and b. Thus the relation (2) asserts that the geometric mean of two non negative real numbers is always smaller than (and utmost) equal to) their arithmetic mean

The relation (2) holds equality iff a=b for any two non negative real numbers a and b.

i.e $\sqrt{ab}=\frac{a+b}{2}$ iff a=b

If $a_1, a_2, a_3 .... , a_n$ are n real numbers, the real numbers

$$\frac{a_1 + a_2 + a_3 + ...a_n}{n},$$

is called the arithmetic mean of $a_1, a_2, a_3 .... , a_n$. If $a_i \geq 0$ for i=1 2,3, ...n We define their geometric mean as the real number

$$(a_1.a_2.a_3.......a_n)^{\frac{1}{n}}$$

a generalization of (2) to a set of n positive numbers $a_1, a_2. a_3, ...a_n$ is the inequality

$$\frac{a_1 + a_2 + a_3 + ...a_n}{n} \geq (a_1 a_2 .....a_n)^{\frac{1}{n}} \quad (3)$$

This inequality is known as " Arithmetic Mean - Geometric mean inequality)" we write (3) also in the form

$$a_1 \; a_2 \; a_3 ....a_n \leq \left(\frac{a_1 + a_2 + a_3 + ... + a_n}{n}^{n} \right) \quad (4)$$

Equality holds in (4) iff $a_1=a_2=a_3= ....a_n$

2 For two positive numbers a and b, if A, G and H be respectively the arithmetic mean, the geometric mean and Harmonic mean then $A \geq G \geq H$

3 If A, G and H be respectively the arithmetic mean, the geometric mean and Harmonic mean of n positive integer $a_1, a_2, ...a_n$ then $A \geq G \geq H$

The equality sign holds if and only if $a_1=a_2 = ... = a_n$.

4 if $a_1 \geq a_2, \; a_2 \geq a_3 ....a_{n-1} \geq a_n$ then $a_1 \geq a_n$

5. If $a_1 \geq b_1, \; a_2 \geq b_2 ....a_n \geq b_n$ then

$a_1 + a_2 + ...a_n \geq b_1 + b_2 + ...b_n$ and equality

Sign holds iff $a_1 = b_1, \quad a_2 = b_2 ....a_n = b_n$

6. Let a>b>0, p,q>0 and let $a^{1/q}$ and $b^{1/q}$ denote positive q^{th} roots of a and b respectively then

$a^{p/q} > b^{p/q}$ and $a^{-p/q} < b^{-p/q}$

7. Let a $\geq$ b>0, p $\geq$ 0 a non negative integer and q is a postive integer and $a^{1/q}, b^{1/q}$ denotes q^{th} roots of a and b respectively then

   $a^{p/q} \geq b^{p/q}$ and $a^{-p/q} \leq b^{-p/q}$

   The equality holds iff a=b or p=0

8. Triangle inequality:- $|a| - |b| \leq |a+b| \leq |a| + |b|$

9. The lengths a,b,c can represent the sides of a triangle if and only if a+b>c, b+c>a, a+c>b

10. $\left|\sum_i a_i\right| \leq \sum_i |a_i|$

11. Weierstras's Inequality :- For positive integers $a_{1,} a_{2,} a_3 .....a_n$

    $(1+a_1)(1+a_2)(1+a_3)....... (1+a_n) > 1+a_1+a_2+a_3+...........a_n$

    If a_i are fractions (less than one) then

    $(1-a_1)(1-a_2)(1-a_3)....... (1-a_n) > 1-(a_1+a_2+a_3+...........a_n)$

12. Cauchy Schwartz Inequality:-

    $$(a_1b_1 + a_2b_2 + ...a_nb_n)^2 \leq (a_1^{\ 2} + a_2^{\ 2} + ...a_n^{\ 2})(b_1^{\ 2} + b_2^{\ 2} + ...b_n^{\ 2})$$

13. Tchebychev Inequality ::- if $x_1 \geq x_2 \geq x_3 \geq ....x_n$ and

    $y_1 \geq y_2 \geq y_3 \geq ....y_n$ or $x_1 \leq x_2 \leq x_3 \leq ....x_n$ and $y_1 \leq y_2 \leq y_3 \leq ....y_n$ then

    $$\left(\frac{x_1y_1 + x_2y_2 + ...x_ny_n}{n}\right) \geq \left(\frac{x_1 + x_2 + ...x_n}{n}\right).\left(\frac{y_1 + y_2 + ...y_n}{n}\right)$$

    If one of the sequence is increasing and other is decreasing then the direction of the inequality changes.

14. Holder's Inequality:-

    (a) $(a_1b_1 + a_2\mathbf{b}_2 + ... + a_nb_n)^{pq} \leq (a_1^{\ p} + a_2^{\ p} + ... + a_m^{\ P})^q \rightarrow (b_1^{\ q} + b_2^{\ q} + ... + b_n^{\ q})^p$

    Where $\frac{1}{p} + \frac{1}{q} = 1,$ a_i and b_i are non negative real numbers.

Equation and inequalities

Questions :-

1. Let $x_1, x_2, ...x_n$ be a sequence of integers such that

   (i) $-1 \le xi \le 2$ for i=1,2,..........n

   (ii) $x_1 + x_2 + ... + x_n = 39\, and$

   (iii) $x_1^2 + x_2^2 + ... + x_n^2 = 99$

   Determine the minimum possible value of $x_1^3 + x_2^3 + ... + x_n^3$ Ans.39

2. Let a,b,c be distinct non-zero real number such that $a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a},$ then find the value of $|abc|$ Ans = 1

3. Find natural x for which

   $$\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$$ Ans.= 1

4. Let a, b, c, d be positive integers such that abcd=a+b+c+d

   Find the maximum possible value of max (a,b,c,d) Ans.= 2

5. Find all real number x satisfying the equation. $4^x + 3^x - 16^x + 12^x - 9^x = 1$

   Ans. 0

6. Determine the number of ordered pairs of integers (m,n) for which $mn \ge 0$ and $m^3+n^3+33mn=11^3$ Ans.=13

7. Given that the real numbers a,b,c,d satisfy the condition a+b+c+d=6, and $a^2+b^2+c^2+d^2=12$, determine the maximum value of a. Ans.=3

8. If $\alpha, \beta\ \gamma$ are roots of the equations $x^3 + 2x^2 + 3x + 35 = 0$,

   Find the value of $\sum \alpha^2 \beta$ Ans. = 99

9. If α, β r are roots of $3x^3 - 4x + 2 = 0$, then find value of $\frac{1+\alpha}{1-\alpha} + \frac{1+\beta}{1-\beta} + \frac{1+\gamma}{1-\gamma}$

   Ans. 11

10. Solve for x, $x^4 - 2x^3 + 4x^2 + 200x - 10400 = 0$,

    given that sum of two of its roots is zero. for $x \in N$ Ans. 10

11. If $\alpha+\beta+\gamma=6, \alpha^2+\beta^2+\gamma^2=14$ and $\alpha^3+\beta^3+r^3=36$, Then value of $\alpha^4+\beta^4+\gamma^4$ is .... Ans. 98

12. If x,y,z be positive real numbers such that $x^2+y^2+z^2=27$ and $x^3+y^3+z^3 \geq n$ Then max value of n is Ans.81

13. If a,b,c>0 and abc=1 such that $\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c} \geq n$, then max value of n is .... Ans=3

14. If a,b,c>0 and a+b+c=1, such that

$\left(\frac{1}{a}+1\right)\left(\frac{1}{b}+1\right)\left(\frac{1}{c}+1\right) \geq n$, Then max value of n is .... Ans. 64

15. If a, b, c > 0 and a+b+c=1, such that $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) \geq n$, Then max value of n is .... Ans. 8

16. If a, b, c > 0 and $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1$, such that abc $\geq$ n, Then max value of n is Ans.8

17. If a, b, c, d are real number satisfied the conditions a+b+c+d=6, $a^2+b^2+c^2+d^2=12$, such that $4(a^3+b^3+c^3+d^3)-(a^4+b^4+c^4+d^4) \leq n$
   (a) Then least value of n is .... Ans. 48
   (b) Then most value of n is Ans.36

18. If a, b, c > 0 such that

$\frac{(a+b)(b+c)(c+a)}{abc} \geq n$, Then max n is .... Ans 8

19. If a,b,c, d are (+) ve real such that

$(ab+cd)(\frac{1}{ac}+\frac{1}{bd}) \geq n$ Then max n is .... Ans. 4

20. If a, b, c, d > 0 satisfying the condition $(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq n$, Then max n is Ans16

21. The smallest value of the sum $a^3+b^3+c^3$ for a+ b, c $\in$ R satisfying the condition $a^2+b^2+c^2=27$ is Ans.81

Solution

1. Let a,b,c denotes the number -1s, 1s and 2s in the sequence respectively (Not consider zero)

   Then a,b,c are non-negative integers satisfying

   -a+b+2c=39 and a+b+4c=99

   $\rightarrow$ a=30-c and b=69-3c

   where $0 < c \le 39[\because b \ge 0]$

   so $x^3_1 + x_2^3 + ... + x_n^3 = (-1)^3 a + 1^3 b + 2^3 c = -a + b + 8c$

   = -30+c+69-3c+8c=39+6c

   when c=0 $\Rightarrow$ a=30, b=69 $\therefore$ min=39

2. It's given $a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}$

   $\Rightarrow$ a-b = $\frac{b-c}{bc}, b - c = \frac{c-a}{ca}, c - a = \frac{a-b}{ab}$

   on multiplying $|abc| = 1 Ans$

3. Let $2^x = a \ and \ 3^x = b$, then given equation became

   $$\frac{a^3 + b^3}{a^2 b + ba^2} = \frac{\cancel{(a+b)}(a^2 + b^2 - ab)}{ab\cancel{(a+b)}} = \frac{7}{6}$$

   $$\Rightarrow 6a^2 + 3ab + 6b^2 = 0$$

   $$\Rightarrow (2a - 3b)(3a - 2b) = 0$$

   $$\Rightarrow 2a = 3b \quad OR \quad 3a = 2b$$

   $$\Rightarrow 2x2^x = 3 \times 3^x \ or \ \frac{1}{2}x2^x = \frac{1}{3}x3^x$$

   $$\Rightarrow x = 1 \ OR -1 \ \ Ans\, x = 1$$

4. Let $a \le b \le c \le d$, so only required the max. value of d

   $$\text{since} \quad d \le a + b + c + d \le 4d$$

   $$then \quad d < abcd \le 4d$$

   $$\Rightarrow 1 < abcd \le 4$$

Hence (a,b,c)=(1,1,1) (1,1,2) (12,1) (2,1,1)

Total=4 max d = 2

5. Let $4^x = a \,\&\, 3^x = b$ then equation become

1+$a^2 + b^2 - ab - a - b = 0$

$\Rightarrow 2 + 2a^2 + 2b^2 - 2a - 2b - 2ab = 0$

$\Rightarrow \left(a^2 + 1 - 2a\right) + \left(b^2 + 1 - 2b\right) + \left(a^2 + b^2 - 2ab\right) = 0$

$\Rightarrow (1-a)^2 + (b-1)^2 + (a-b)^2 = 0$

Therefore $1 = 4^x = 3^x \Rightarrow x = 0$ only equation

6. We know that $(m+n)^3 = m^3 + n^3 + 3mn(m+n)$

If m+n=11 then

11^3 $(m+n)^3 = m^3 + n^3 + 3mn(m+n) = m^3 + n^3 + 33mn$

Here m+n–11 is a factor of $m^3 + n^3 +$ +33mn–11

We have $m^3+n^3+33mn-11^3=(m+n-11)\,(m^2+n^2-mn+11m+11n+11^2)$ (Identity)

$= \frac{1}{2}(m+n-11)\,[(m-4)^2+(m+11)^2+(n+11)^2]$

so solution are (0,11) (1,10) (2,9) (3,8) (4,7) (5,6) (6,5) (7,4) (8,3) (9,2) (10,1) (11, 0) (–11,–11) Total =13 Ans.

7. Since the b,c,d is 6–a, their average is $x = \frac{6-a}{3}$

Let $b = x + b_1, c = x + c_1 \,\&\, d = x + d_1$

Then $b_1 + c_1 + d_1 = 0\ [b - x + c - x + d - x = [b + c + d - 3x =$

$[6-a] - (\cancel{3} \times -\frac{6-a}{\cancel{3}} = 0)$

a=0

and $12 = a^2 + 3x^2 + b_1^2 + c_1^2 + d_1^2 \ge a^2 + 3x^2$

$$=a^2+3\left(\frac{6-a}{3}\right)^2=a^2+\left(\frac{6-a}{3}\right)^2$$

$$\Rightarrow 36\geq 3a^2+36+a^2-12a$$

$$\Rightarrow 0\geq 4a^2-12a=4a(a-3)$$

$$\Rightarrow 0\leq a\leq 3, where\, a=0\, iff\; b=c=d=2$$

and a=3 iff b=c=d=1 Max=3

8. In general equation x^3+px^2+qx+r=0

$\alpha+\beta+\gamma=-p$ $\alpha\beta+\beta\gamma+r\alpha=q,\ \alpha\beta\gamma=r$

$\therefore \sum\alpha^2\beta=\alpha^2\beta+\beta^2\alpha+\gamma^2\alpha+\alpha^2\gamma+r^2\beta+\beta^2\gamma$

$=(\alpha\beta+\beta\gamma+\alpha\gamma)(\alpha+\beta+\gamma)-3\alpha\beta\gamma=-(pq)-3\gamma$

=3r-pq

Here p=2, q=3,r=35

$\therefore \sum\alpha^2\beta=3\times 35-2\times 3=99$

9. Let y=$\frac{1+x}{1-x}\Rightarrow x=\frac{y-1}{y+1}$

therefore $3\left(\frac{y-1}{y+1}\right)^3-4\left(\frac{y-1}{y+1}\right)+2=0\Rightarrow y^3-11y^2+19y+3=0$

$\therefore$ Sum of roots = 11

10. If $\alpha,\beta,\gamma,\delta$ are roots of equation then

$\alpha+\beta+\gamma+\delta=2\Rightarrow \gamma+\delta=2\ \therefore\ \alpha+\beta=0$

so equations whose roots are α,β is x^2+px+p=0$\Rightarrow$ x^2+p=0

so equations whose roots are γ,δ is x^2–2x+q=0

$\therefore x^4-2x^3+4x^2+200x-10400=(x^2+p)(x^2-2x+q)=0$

$=x^4-2x^3+(p+q)x^2-2px+pq$

$\therefore\ p+q=4,\ pq=25,\ -2p=200\Rightarrow P=-100$

q=104

Hence $x^4-2x^3+4x^2+200x-10400=(x^2-100)(x^2-2x+104)=0$

$\Rightarrow x = 10$

11. Let α,β,γ be the roots of the equation $x^3+px^2+qx+r=0-(1)$

Here $a_0=1, a_1=p, a_2=q, a_3=0$

$\therefore a_0s_1+a_1=s_1+p=0\Rightarrow s_1=-p$

But $s_1=\sum\alpha=6\Rightarrow -p=6\Rightarrow p=-6$

Also $a_0s_2+a_1s_1+2a_2=s_2+ps_1+2q=0\Rightarrow s_2=-ps_1-2q$

=36-2q

But $s_2=\sum d^2=14\Rightarrow 36-2q=14\Rightarrow q=11$

Now putting $x=\alpha$, β,γ in (1) & adding

$$s_3+ps_2+qs_1+3r=0\Rightarrow s_3+(-6)\times 14+11x6+3r=0$$

$$\Rightarrow s_3=18-3r$$

But $s_3=\sum\alpha^3=36\Rightarrow 18-3r=36\Rightarrow r=-6$

Equation (1) is now $x^3-6x^2+11x-6=0-(2)$

on multiplying by x $\rightarrow$ $x^4-6x^3+11x^2-6x=0-\ (2)$

Putting $x=\alpha,\beta,\gamma$ in (3) & Adding

$$s_4-6s_3+11s_2-6s_1=0$$

$$\Rightarrow s_4=6x36-11x14+6x6=98\ Ans$$

12. Applying Cauchy-Schwraz inequality - to the numbers

$$a_{1,}a_2...a_n \ \& \ b_{1,}b_2,......b_n \sum R$$

$$then\left(a_1b_1+a_2b_2+a_nb_n\right)^2\le\left(a_1^2+a_2^2+........an^2)(b_1^2+b_2^2+.....bn^2\right)$$

$$iff\ \frac{a_1}{b_1}=\frac{a_2}{b_2}.......=\frac{a_n}{b_n}$$

$x^{3/2}, y^{3/2}, z^{3/2}, x^{1/2}, y^{1/2}, z^{1/2}$ *we have*

$$\left(x^2+y^2+z^2\right) \le \left(x^3+y^3+z^3\right)\left(x+y+z\right)-(1)$$

Again applying C-s inequality to x,y,z,1,1,1

$$\left(x+y+z\right)^2 \le \left(x^2+y^2+z^2\right)(1+1+1)$$

$$\Rightarrow \left(x+y+z\right)^2 \le 3\left(x^2+y^2+z^2\right)-(2)$$

Squaring both sides $\left(x+y+z\right)^4 \le 9\left(x^2+y^2+z^2\right)-(3)$

Multiplying (1) by 9 both sides

$9\left(x^2+y^2+z^2\right) \le 9\left(x^3+y^3+z^3\right)$(x+y+z) - (4)

from (3) & (4) $\left(x+y+z\right)^4 \le \left(x^3+y^3+z^3\right)(x+y+z)$

Squaring (1) $\left(x^2+y^2+z^2\right)^4 \le \left(x^3+y^3+z^3\right)\left(x+y+z\right)^2-(5)$

from (5) & (2) $\left(x^2+y^2+z^2\right)^4 \le 3\left(x^3+y^3+z^3\right)^2\left(x^2+y^2+z^2\right)$

$$\Rightarrow \frac{\left(x^2+y^2+z^2\right)^3}{3} \le \left(x^3+y^3+z^3\right)^2$$

Putting $x^2+y^2+z^2=27 \quad x^3+y^3+z^3 \ge 81$

13. $\dfrac{1+ab}{1+a}=\dfrac{abc+ab}{1+a}, \ \sin ce\, abc=1$

= $ab\left(\dfrac{1+c}{1+a}\right)$

$$\therefore \frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+ac}{1+c}=ab\left(\frac{1+c}{1+a}\right)+bc\left(\frac{1+a}{1+b}\right)+ca\left(\frac{1+b}{1+a}\right)$$

$$\geq 3\sqrt[3]{(abc)^2} = 3 \quad [AM \geq GM]$$

14. Since abc $\leq \left[\frac{a+b+c}{3}\right]^3 = \frac{1}{27}$, since $a+b+c=1$ Ans = 3

$$\left(\frac{1}{a}+1\right)\left(\frac{1}{b}+1\right)\left(\frac{1}{c}+1\right) = 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}+\frac{1}{abc}$$

$$\geq 1+\frac{3}{\sqrt[3]{abc}}+\frac{3}{\sqrt[3]{(abc)^2}}+\frac{1}{abc} \quad (AM \geq GM)$$

$$= \left(1+\frac{1}{\sqrt[3]{abc}}\right)^3 \geq 4^3 = 64 \; Ans$$

15. Here

$$\frac{1}{a}-1 = \frac{1-a}{a} = \frac{a+b+c-a}{a} = \frac{b+c}{a}$$

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right) = \left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right)$$

$$= \sqrt{bc} x \sqrt{ca} x \sqrt{ab} \qquad AM \geq GM$$

= 2x2x2=8 Ans

16. Let $\frac{1}{1+a} = x, \frac{1}{1+b} = y, \frac{1}{1+c} = z \Rightarrow x+y+z=1$

Now $\frac{1}{1+a} = x \Rightarrow \frac{1}{x} = 1+a \Rightarrow a = \frac{1}{x}-1$ Now proceed as 15

Ans = 8

17. Here

$$4\left(a^3+b^3+c^3+d^3\right)-\left(a^4+b^4+c^4+d^4\right)$$

$= - \left[(a-1)^4+(b-1)^4+(c-1)^4+(d-1)^4\right]+6\left(a^2+b^2+c^2+d^2\right)$

-4 (a+b+c+d)+4

$=- \left[(a-1)^2+(b-1)^4+(c-1)^4+(d-1)^4\right]+52$ (By given conditions)

Now let a-1=x, b-1=y, c-1=z, d-1=t

$$x^2+y^2+z^2+t^2=\left(a^2+b^2+c^2+d^2\right)-2(a+b+c+d)+4=4$$

$\therefore\ x^4+y^4+z^4+t^4 \geq \left(\dfrac{x^2+y^2+z^2+d^2}{4}\right)^2+q$ where q is non negative

$$x^4+y^4+z^4+t^4 \leq \left(x^2+y^2+z^2+t^2\right)=16$$

again a+b+c=6-d & $a^2+b^2+c^2=12-d^2$

Hence by power mean inequality (Power inequality)

$\Rightarrow a^2+b^2+c^2 \geq \dfrac{(a+b+c)^2}{3}$ $\left(t_1x_1^x+...+t_nx_n^r\right)^{1/r} \geq \left(t_1x_1^s+-+t_nx_n^s\right)^{1/s}$

$$12-d^2 \geq \frac{(6-d)^2}{3} \Rightarrow 2d\ (d-3) \leq 0$$

$$\therefore 0 \leq a,b,c,d \leq 3$$

Now $x^2(x-2)^2 \geq 0\ for\ x \in R$

$$\Rightarrow 4x^3-x^4 \leq 4x^2$$

$$\Rightarrow \left(4a^3-a^4\right)+\left(4b^3-b^4\right)+\left(4c^3-c^4\right)+\left(4d^3-d^4\right) \leq 4\left(a^2+b^2+c^2+d^2\right)$$

= 48

Also $x \in [0,3] \Rightarrow (x+1)(x-1)^2(x-3) \leq 0$

$$\Rightarrow 4x^3-x^4 \geq 2x^2+4x+3$$

$$\Rightarrow \left(4a^3 - a^4\right) + \left(4b^3 - b^4\right) + \left(4c^3 - c^4\right) + \left(4d^3 - d^4\right) \ge 2\left(a^2 + b^2 + c^2 + d^2\right)$$

+ 4 (a+b+c+d)

= 36 Ans

18. $$\frac{(a+b)(b+c)(c+a)}{abc} = \left(\frac{a+b}{a}\right)\left(\frac{b+c}{b}\right)\left(\frac{c+a}{c}\right)$$

$$= \frac{a+b}{\sqrt{ab}} \times \frac{b+c}{\sqrt{bc}} \times \frac{c+a}{\sqrt{ca}} \ge 2 \times 2 \times 2 = 8$$ AM ≥ GM Ans=8

19. $$(ab+cd)\left(\frac{1}{ac} + \frac{1}{bd}\right) = \frac{ab}{ac} + \frac{ab}{bd} + \frac{cd}{ac} + \frac{cd}{bd}$$

$$= \frac{b}{c} + \frac{a}{d} + \frac{d}{a} + \frac{c}{b}$$ AM ≥ GM

$$= \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{a}{d} + \frac{d}{a}\right) \ge 2 \times 2 =$$ Ans4

20. By cauchy's inequality

$$(u_1v_1 + u_2v_2 + ... + u_nv_n)^2 \le (u_1^2 + u_2^2 + ... + u_n^2)\ (v_1^2 + v_2^2 + ... + v_n^2)$$

equality occurs iff

$v_n = tu_n$ when $1 \le k \le n,\ t \in R$ Here $u_n = \frac{1}{\sqrt{a_x}},\ k = 1,\ 2,\ 3, 4$

$$= \sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}$$

Since $U_k\ \&\ V_k \in R$ so $V_k = tu_k$

$$\sqrt{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} = t\sqrt{a+b+c+d}$$

It is possible if a= b= c= d= t=4

$$\therefore (a+b+c+d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \ge t^2 = 4^2 = 16$$

21. By Power means of degree 2,& 3 a,b,c satisfy the condition

$$2\sqrt{\frac{a^2+b^2+c^2}{3}} \le 3\sqrt{\frac{a^3+b^3+c^3}{3}} \quad = 2\sqrt{\frac{27}{3}} \le 3\sqrt{\frac{a^3+b^3+c^3}{3}} \Rightarrow 3 \le 3\sqrt{\frac{a^3+b^3+c^3}{3}}$$

$$\Rightarrow a^3+b^3+c^3 \ge 81$$

Triangles

1. **<u>Basic Proportionality Theorem</u>**

   If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points the other two sides are divided in the same ratio.

2. **<u>Converse of BPT :</u>**

   If a line divides any two sides of a triangle in the same ratio, then the line is parallel to third side.

3. **<u>Pythagoras Theorem :</u>**

   In a right triangle the square of hypotenuse is equal to the sum of squares of other two sides.

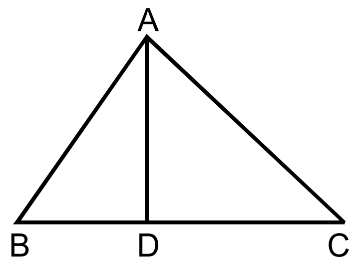
4. **<u>Converse of Pythagoras Theorem :</u>**

   In a triangle if square of one side is equal to the sum of squares of other two sides then the angle opposite to first side is right angle.

5. **<u>Apellonius Theorem :</u>**

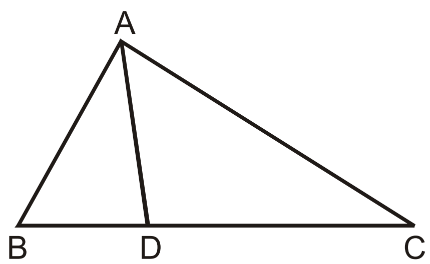
   In a triangle ABC if AD is the median bisecting BC then

   $$AB^2 + AC^2 = 2\left(BD^2 + AD^2\right)$$



6. **<u>Stewart's Theorem :</u>**

   In a triangle ABC if AD is the angle bisector then



   $$DC.AB^2 + BD.AC^2 = BC(AD^2 + BD.CD)$$

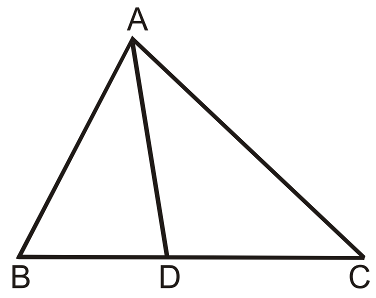
7. **<u>Theorem 01 :</u>**

   In △ ABC, if AD is the angle bisector then $\frac{AB}{AC} = \frac{BD}{DC}$

△

8. Theorem 02 :

In $\triangle$ ABC, if AD is angle bisector then $AD^2 = \frac{4bc\ s(s-a)}{(b+c)^2}$



Where a,b & c are the sides of Δ ABC and s is the semi-perimeter.

9. Steiner Lehmus Theorem :

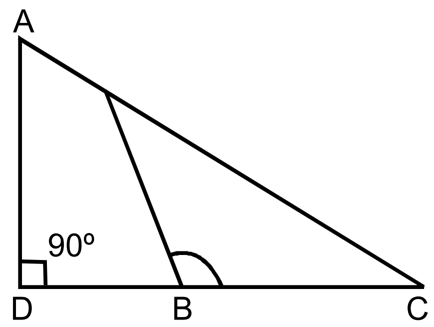
In a triangle, if two angle bisectors are equal then the triangle is isosceles triangle.

10. Theorem-03

The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

11. Theorem - 04

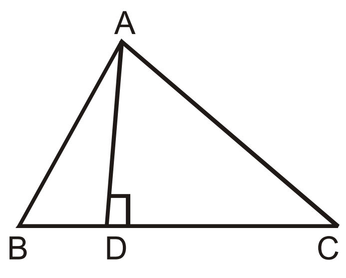
In an obtuse angled triangle ABC if $AD \perp CB$ produced, then $AC^2 = AB^2 + BC^2 + 2BC.BD$



12. **Theorem-05**

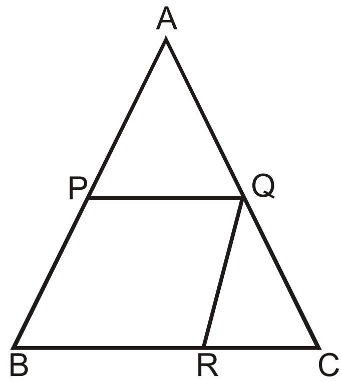
In an acute angled triangle ABC if $AD \perp BC$ then

$AC^2 = AB^2 + BC^2 - 2BC.BD$



Some Solved Problems

1. Rhombus PQRB is inscribed in a Δ ABC as shown in the figure. If AB=12 cm and BC=6 cm find PQ.



Sol. Let PQ=a then In Δ ABC, PQ||BC as PQRB is a rhombus.

Hence ΔAPQ ~ Δ ABC

then $\frac{AP}{AB} = \frac{PQ}{BC}$

$\frac{12-a}{12} = \frac{a}{6}$

72-6a = 12 a

72 = 18a

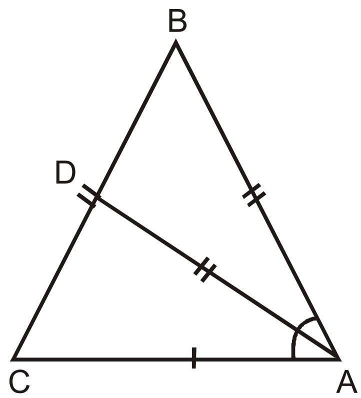
a=4 cm.

2. In an isosceles triangle ABC, AC=BC, $\angle BAC$ is bisected by AD where D lies on BC. If AD=AB then find $\angle ACB$

Sol. In $\Delta ABC, AC = BC$ and AD=BD where AD is angle bisector of $\angle A.$ Let $\angle BAC$ =2x then $\angle BAD$ =x

$\because$ AC=BC Hence $\angle BAC = \angle ABC$

$\angle ABC = 2x$



In ΔABD, AD=AB Hence $\angle ADB = \angle ABC = 2x\, then\, in\, \Delta ABD$

$2x + 2x + x = 180°$

$5x = 180°$

$x = 36°$

so $\angle BAC = 2x\ 36° = 72^\circ$

$\angle ABC = 72^\circ$

Now in ΔABC

$\angle BAC + \angle ABC + \angle ACB = 180°$

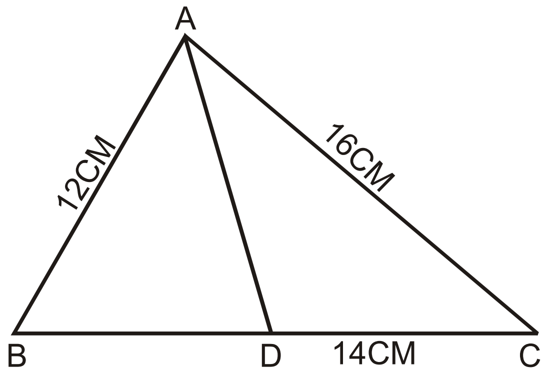
$72° + 72° + \angle ACB = 180°$

$\angle ACB = 180° - 144°$

$\angle ACB = 36°$

3. In a triangle ABC, AB=12 cm, BC=14 cm, AC=16 cm then find AD where AD is the angle bisector of $\angle BAC$

Sol. Let ΔABC having a=14 cm, b=16 cm, c=12 cm



s=$\dfrac{a+b+c}{2}$

= $\dfrac{14+16+12}{2}$

= $\dfrac{42}{2} = 21cm$

$AD^2 = \dfrac{4bcs(s-a)}{(b+c)}$

= $\dfrac{4x16x12x21(21-14)}{(12+16)^2}$

= $\dfrac{4x16x4x3x3x7x7}{(28)^2}$

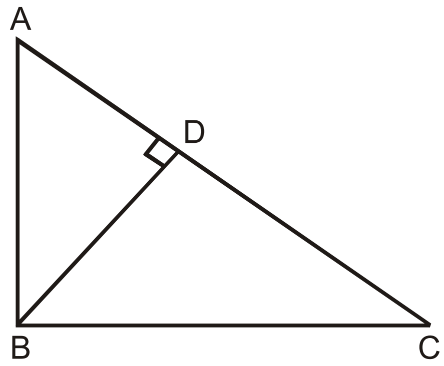
AD = $\sqrt{\frac{4x4x4x4x3x3x7x7}{(28)^2}}$

AD = $\frac{4x4x3x7}{28} = 12\,cm$

4. In adjoining figure ABC is a triangle right angled at B and $BD \perp AC$. If AD=4 cm and CD=5 cm then find AB.

Sol $\because BD \perp AC$

$\therefore \Delta ADB \sim \Delta BDC$



then $\frac{AD}{BD} = \frac{BD}{CD}$

$AD.CD = BD^2$

4.5 = BD^2

BD^2 =20

Now in ΔABD By Pythagoras theorem

$AB^2 = AD^2 + BD^2$

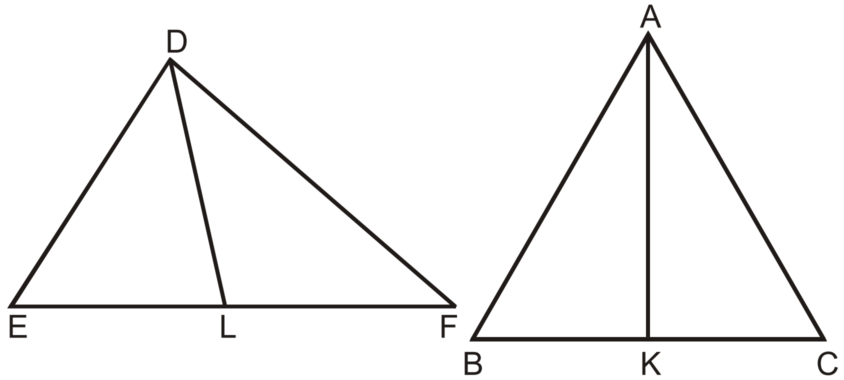
= $(4)^2 + 20$

$AB^2 = 16 + 20$

$AB = \sqrt{36} = 6\,cm$

5. Let $\Delta ABC \sim \Delta DEF$ and their areas be respectively $64cm^2$ and $121cm^2$ if a median of ΔDEF is 16.5 cm find the corresponding median of ΔABC.

Sol. Let AK be the median of ΔABC. and DL be the median of ΔDEF



$\because$ Ratio of areas of two similar triangle is equal to the ratio of squares of their corresponding sides.

$$\frac{ar\ ABC}{ar\ DEF} = \frac{AB^2}{DE^2}$$

$$\frac{64}{121} = \left(\frac{AB}{DE}\right)^2$$

$$\frac{AB}{DE} = \sqrt{\frac{64}{121}}$$

AB : DE = 8:11

Now $\because$ $\Delta ABC \sim \Delta DEF$

$$\therefore \quad \frac{AB}{DE} = \frac{BC}{EF}$$

$\frac{AB}{DE} = \frac{2BK}{2EL}$ As AK and DL are medians

$$\frac{AB}{DE} = \frac{BK}{EL}$$

& $\angle B = \angle E$ By CPST

$\Delta ABK \sim \Delta DEL$

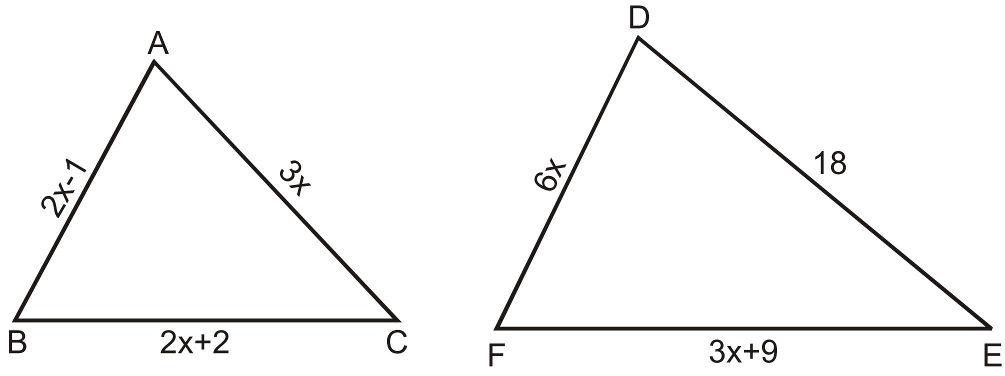
$\therefore \frac{AB}{DE} = \frac{AK}{DL}$ By CPST

$$\frac{8}{11} = \frac{AK}{16.5}$$

AK = $\frac{8}{11} x 16.5 = 12\, cm$

Problems for Practice

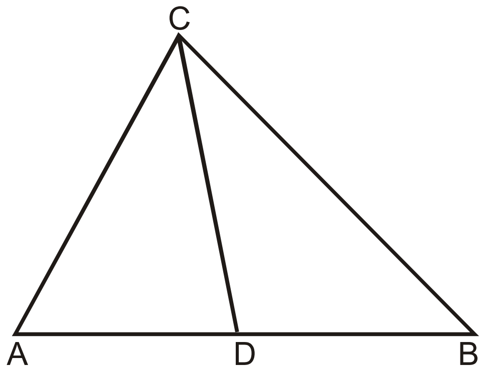
1. In adjoining figure, if $\Delta ABC \sim \Delta DEF$ and their sides are of lengths (in cm) as marked along them, then find the perimeter of ΔABC



[Ans : 36 cm]

2. For going to a city B from city A, there is a route via city C such that $AC \perp CB$, AC=2x km and CB=$2(X+7)Km.$ It is proposed to construct a 26 Km highway which directly connects two cities A and B. Find how much distance will be saved in reaching city B from city A after the construction of the highway. [Ans : 8 km]

3. In adjoining figure if $\angle ACB = \angle CDA$, AC=8 cm and AD=4 cm find BD.



[Ans : 12 cm]

4. ABCD is a trapezium in which AB||DC and P and Q are points on AD and BC respectively such that PQ||DC. If PD=18 cm, BQ=35 cm and QC=15 cm, find AD. [Ans : 60 cm]

5. In an isosceles ΔABC, AB=AC=13 cm and the median AD=12 cm. Find BC.

[Ans : 10 cm]

6. If $\Delta ABC \sim \Delta DEF$, AB=4 cm, DE=6 cm, EF=9 cm and FD=12 cm, find the perimeter of ΔABC. [Ans : 18 cm]

7. Two sides of a triangle are 10 cm and 5 cm in length, and the length of the median to the third side is 6.5 cm. The area of the triangle is $6\sqrt{x}\,cm^2$. Determine the value of x. [Ans : 14]

8. Perimeters of two similar triangles ABC and PQR are in the ratio 4:5. If the sum of their areas is $164 cm^2$. Find the area of smaller triangle.

   [Ans : $64\,cm^2$]

9. The areas of two similar triangles ABC and PQR are in the ratio 4:9. If the sum of the perimeters is 30 m, find the perimeter of larger triangles.

   [Ans : 18 cm]

10. A ladder 25 m long is placed so as to reach a road side window 24 m high and returning the ladder over to the other side of the road, it reaches a point 7m high. Find the breadth of the road. [Ans : 31 m]

Solution to Unsolved

1. $\because \Delta ABC \sim \Delta DEF$

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$$

$$\frac{2x-1}{18}=\frac{2x+2}{3x+9}=\frac{3x}{6x}$$

Taking

$$\frac{2x-1}{18}=\frac{3x}{6x}$$

$$\frac{2x-1}{18}=\frac{3}{6}$$

2x-1=9

2x=10

x=5

Taking

$$\frac{2x+2}{3x+9}=\frac{3x}{6x}$$

12x+12=9x+27

12x-9x=27-12

3x=15

x=5

Taking

$$\frac{2x-1}{18}=\frac{2x+2}{3x+9}$$

$6x^2-3x+18x-9$

= 36x+36

$6x^2+15x-9-36x$

-36=0

$6x^2 - 21x - 45 = 0$

$2x^2 - 7x - 15 = 0$

$2x^2 - 10x + 3x - 15 = 0$

(x-5) (2x+3)=0

x=5, x=-3/2 NA

So x=5 is acceptable

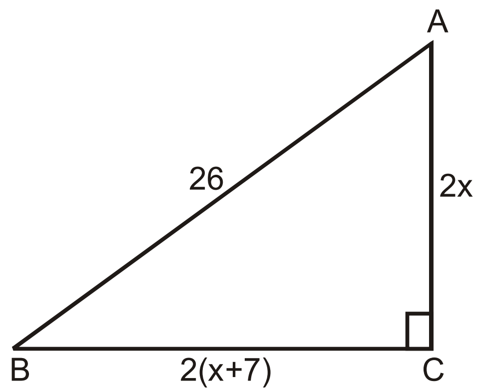
AB=2x5-1=9

BC=2x5+2=12

AC=3x5=15

Perimeter = 9+12+15=36 cm

2. In ΔABC By Pythagoras theorem



$AB^2 = AC^2 + BC^2$

$(26)^2 = (2x)^2 + [2(x+7)]^2$

676= $4x^2 + 4x^2 + 196 + 56x$

$4x^2 + 4x^2 + 56x + 196 - 676 = 0$

$8x^2 + 56x - 480 = 0$

$2x^2 + 14x - 120 = 0$

$2x^2 + 24x - 10x - 120 = 0$

2x(x+12)-10(x+12)=0

(x+12) (2x-10)=0

either x+12=0

x=-12 N.A.

As distance cannot be negative

2x-10=0

x=5

Then AC+BC=2x5+2(5+7)

= 10+24

= 34 Km

Distance saved

= 34-26=8 km

3. In $\Delta ACB \; and \; \Delta ADC$

$\angle ACB = \angle CDA \; Given$

$\angle A = \angle A$ Common

$\Delta ACB \sim \Delta ADC$ By AA Similarity

$$\frac{AC}{AD} = \frac{AB}{AC}$$

$$AC^2 = AB.AD$$

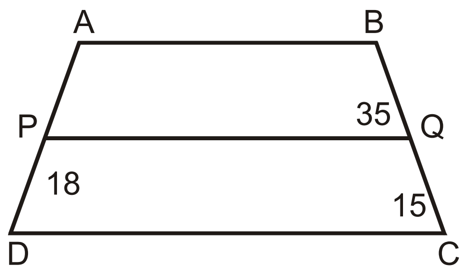
$$(8)^2 = (AD + BD)AD$$

64 = $(4 + BD)x4$

16=4+BD

BD=12 cm

4. $\because$ PQ||DC||AB

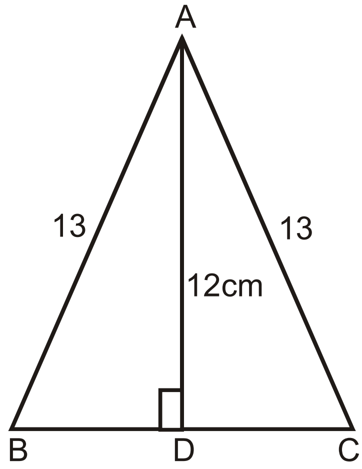


$$\therefore \frac{AD}{PD} = \frac{BC}{QC}$$

$$\frac{AD}{18} = \frac{50}{15}$$

$$AD = \frac{{}^{10}\cancel{50} \times \cancel{18}^{6}}{\cancel{15}\,\cancel{}_{3}} = 60\,cm$$

5. $\because$ In an isosceles triangle median to non-equal side is $\perp r$ to that side.
$\therefore$ ΔABD is a right triangle.
In ΔABD By Pythagoras theorem

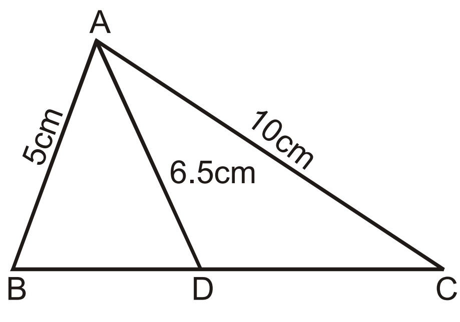


$$AB^2 = AD^2 + BD^2$$
$$(13)^2 = (12)^2 + BD^2$$
$$BD^2 = 169 - 144$$
$$BD = \sqrt{25} = 5$$
$$\therefore BC = 2BD = 10\ cm$$

6. $\because \Delta\ ABC \sim \Delta\ DEF$

$$\frac{Perimeter\ of\ \Delta\ ABC}{Perimeter\ of\ \Delta\ DEF} = \frac{AB}{DE}$$
$$\frac{Perimeter\ of\ \Delta\ ABC}{6+9+12} = \frac{4}{6}$$
$$Perimeter\ of\ \Delta\ ABC = \frac{4}{6} \times 27$$
$$= 18\ cm$$

7. In $\Delta\ ABC$ By appolonius theorem



$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$
$$(5)^2 + (10)^2 = 2[(6.5)^2 + BD^2]$$
$$25 + 100 = 2[42.25 + BD^2]$$
$$\frac{125}{2} = 42.25 + BD^2$$
$$62.5 = 42.25 + BD^2$$
$$BD^2 = 62.5 - 42.25$$
$$BD^2 = 20.25$$
$$BD = \sqrt{20.25} = 4.5$$

$BC = 9cm$

s $= \frac{5+10+9}{2} = \frac{24}{2} = 12\ cm$

area of $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$6\sqrt{x} = \sqrt{12(12-5)(12-10)(12-9)}$

$6\sqrt{x} = \sqrt{12 \times 7 \times 2 \times 3}$

$6\sqrt{x} = 6\sqrt{14}$

$x = 14$ Ans.

8. $\because \Delta\ ABC \sim \Delta\ PQR$

$Hence\ \frac{area\ \ ABC}{area\ \ PQR} = \left(\frac{Perimeter\ \Delta\ ABC}{Perimeter\ PQR}\right)^2$

$= \left(\frac{4}{5}\right)^2$

$\frac{area\ \ ABC}{area\ \ PQR} = \frac{16}{25}$

$area\ \ ABC = 16x$

$area\ \ PQR = 25x$

ATQ

$16x + 25x = 164$

$41x = 164$

$x = 4$

area of smaller $\Delta = 16 \times 4$

$= 64\ cm^2$

9. $\Delta\ ABC \sim \Delta\ PQR$

$\frac{arABC}{arPRQ} = \left(\frac{PerimeterABC}{PerimeterPQR}\right)^2$

$$\frac{4}{9} = \left(\frac{PerimeterABC}{PerimeterPQR}\right)^2$$

$$\frac{2}{3} = \frac{PerimeterABC}{PerimeterPQR}$$

$Perimeter\ \ ABC = 2x$

$Perimeter\ \ PQR = 3x$

$ATQ.\quad 2x + 3x = 30$

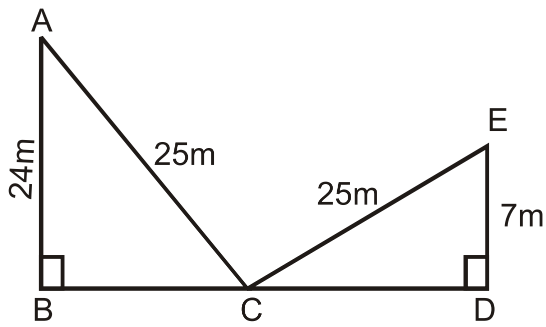
$5x = 30$

$x = 6$

$Perimeter\ \ of\ \ l\arg er\Delta = 3 \times 6$

$= 18cm$

10. In $\Delta\ ABC$ By Pythagoras Theorem



$AC^2 = AB^2 + BC^2$

$(25)^2 = (24)^2 + BC^2$

$625 - 576 = BC^2$

$BC^2 - 49$

$BC = 7m$

In $\Delta\ CDE$ By Pythagoras theorem

$CE^2 = ED^2 + CD^2$

$(25)^2 = (7^2) + CD^2$

$625 - 49 = CD^2$

$CD^2 = 576$

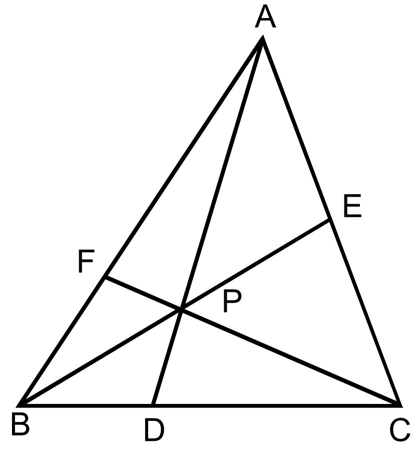
$CD = 24\,m$

breadth of road= 24+7=31 m Ans

Leva's Theorem

Let P be a point inside the ΔABC continue lines AP, BP, CP to hit BC, CA, AB at D, E, F respectively , then

$$\frac{|AF|}{|FB|}.\frac{|BD|}{|DC|}.\frac{|CE|}{|EA|}=1$$



Converse of Ceva's Theorem-

If D, E, F are on sides BC, CA, AB respectively s.t.

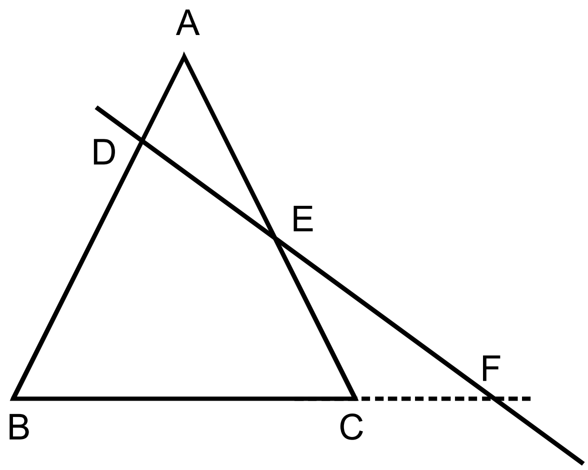
$$\frac{AF}{FB}.\frac{BD}{DC}.\frac{CE}{EA}=1$$, then

lines AD, BE & CF are concurrent at a point P.

Menelaus's Theorem–

Let ABC be a triangle and let line l cut the sides of the triangle(extended if necessary) at points D, E and $F.$, then

$$\frac{AD}{BD}.\frac{BE}{EC}.\frac{CF}{FA}=1$$



Converse of Menelaus's Theorem

Suppose three points D, E, F are on sides (or extension) AB, BC, AC respectively such that 1 or 3 of them are in the extensions of the sides, then points D, E, F are collinear if and only if

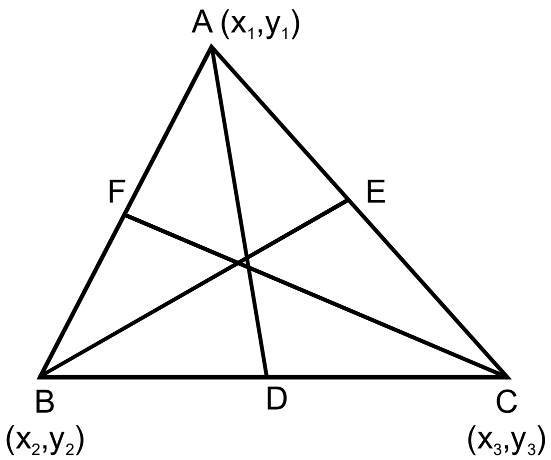
$$\frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA} = 1$$

Centroid of a Triangle

The centroid of a triangle is a point where all three medians meets/ Intersect.

If $A(x_1, y_1), B(x_2, y_2)$ and $C(x_{3,} y_3)$ are vertices of triangle, then co-ordinate of centroid

$$G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$



Some Properties of centroid of Triangle

Centroid divide the median in $2:1$

$$\therefore \quad AG : GD = 2:1$$

- $AG = \frac{2}{3} AD, \quad GD = \frac{1}{3} AD$

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$$

- $= \frac{4}{3}(AD^2 + BE^2 + CF^2)$

- For any point inside the triangle

$$PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2$$

- The three medians divides the triangle into six triangles, each of which have the same area.
- If $BC = a, AC = b$ and $AB = c$ then length of median

$$AD = \sqrt{\frac{2b^2 + 2c^2 - a^2}{4}}$$

$$BE = \sqrt{\frac{2a^2 + 2c^2 - b^2}{4}}$$

$$CF = \sqrt{\frac{2a^2 + 2b^2 - c^2}{4}}$$

- As Centroid divides medians in 2 : 1 , therefore

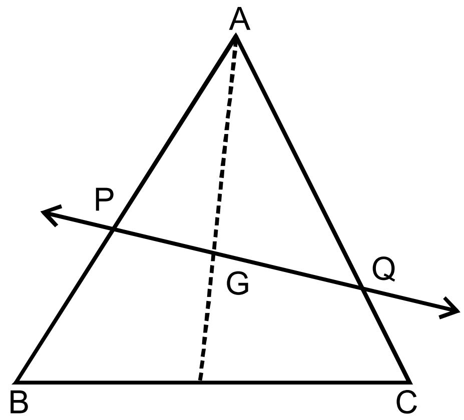
$$AG = \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$$

$$BG = \frac{1}{3}\sqrt{2a^2 + 2c^2 - b^2}$$

$$CG = \frac{1}{3}\sqrt{2a^2 + 2b^2 - c^2}$$

- If any line through the centroid hits AB at P and AC at Q, then

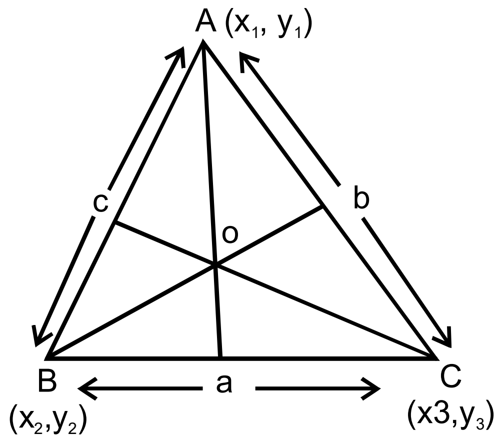
$$\frac{BP}{PA} + \frac{CQ}{QA} = 1$$



Incenter of Triangle

The incenter of a triangle is a point where all three internal angle bisector intersect.

– Incenter is equidistant from triangle's side



– If $A(x_1, y_1), B(x_2 y_2)$ and $C(x_3, y_3)$ are vertices of triangle, then co-ordinate of incenter

$$o \equiv \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$$

if s is semiperimeter of triangle, r is inradius , then

area of Triangle = $\boxed{\Delta = s.r}$

Circumcenter of Triangle

The circumcenter of a triangle is a point where all three perpendicular bisector of sides meets.

– Circumcenter is equidistant from all vertices of triangle

– Circumradius

$$R = \frac{a}{2\sin A} = \frac{abc}{4s}$$

– Euler's Theorem [Relation between inradius (r) and circumradius(R)]

$$\boxed{(R-r)^2 = d^2 + r^2}$$

Where R = circumradius

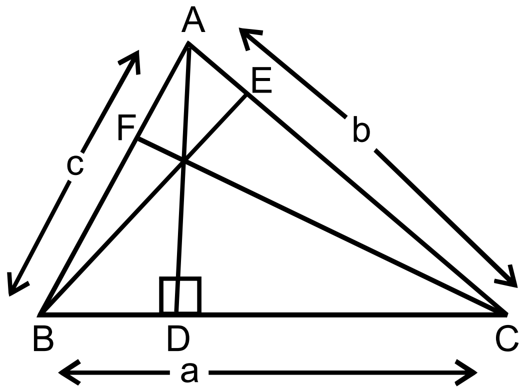
r = Inradius

d = distance between the incenter and the circumcenter

Orthocenter of Triangle

The orthocentre of a triangle is a point where all three altitudes from vertices of opposite side meets

In ΔABC , AD, BE and CF are altitudes on sides BC, AC and CA respectively.



$$\Delta = \frac{1}{2} ah1$$

$$\Delta = \frac{1}{2} bh2$$

$$\Delta = \frac{1}{2} ch3$$

$$\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta}$$

$$= \frac{a+b+c}{2\Delta} = \frac{\cancel{2}s}{\cancel{2}\Delta}$$

$$= \frac{s}{s.r} = \frac{1}{r}$$

$$\boxed{\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r}}$$

Inequalities in a Triangle-

If three line segments with lengths a,b and c are given, then triangle with sides a,b and c exist if and only if

(i) $a+b>c$

(ii) $a+c>b$

(iii) $b+c>a$

these relations call " Triangle existence inequalities"

- If two sides of a triangle are unequal, the angle opposite to the longer side is larger (or greater)

eg if $a \geq b \geq c$

$\Rightarrow \angle A \geq \angle B \geq \angle C$

- In any triangle, the side opposite to the larger (greater) angle is longer.
- In a right angled triangle, the hypotenuse is the longest side.

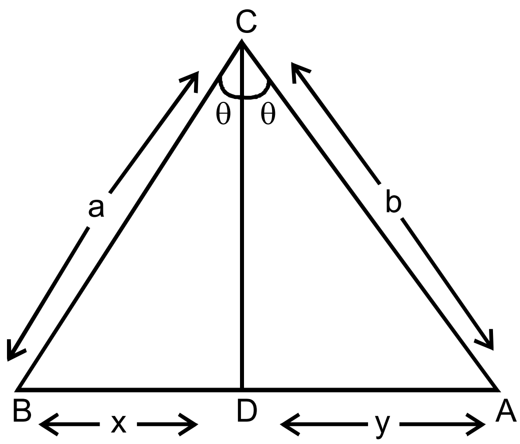
Angle bisector Theorem

An angle bisector of a triangle divides the opposite side in to two segments that are proportional to other two sides of triangle.

Let CD be the angle bisector in ΔABC, then

$$\frac{BD}{AD} = \frac{BC}{AC}$$

$$\frac{x}{y} = \frac{a}{b}$$



Length of Angle bisector CD

$= \sqrt{ab - xy}$

Some Solved Problem

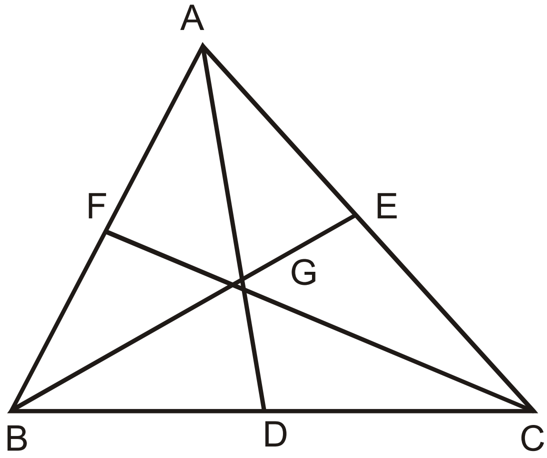
Problem (1)

In ΔABC, G is the centroid and AB= 6 cm, AC=8cm and medians

$AD = 5\ cm$

$BE = \sqrt{52}\ cm$

$CF = \sqrt{73}\ cm$



Find the length of BC.

Solution -

$$GA = \frac{2}{3}AD = \frac{10}{3}cm$$

$$GB = \frac{2}{3}BE = \frac{2\sqrt{52}}{3}cm$$

$$GC = \frac{2}{3}CF = \frac{2\sqrt{73}}{3}cm$$

$$AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$$

$$36 + BC^2 + 64 = 3\left(\frac{100}{9} + \frac{208}{9} + \frac{292}{9}\right)$$

$$BC^2 + 100 = 3 \times \frac{600}{9}$$

$$\boxed{BC^2 = 100}$$

$BC = 10cm$

Problem- 2

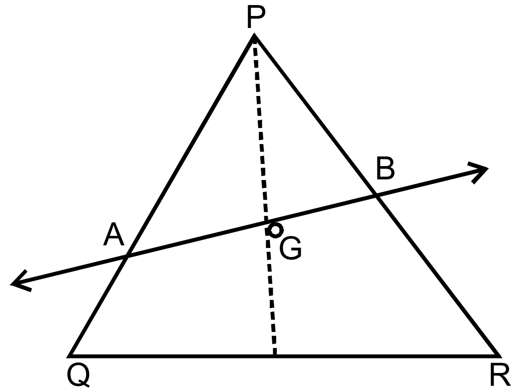
In $\Delta ABC,\ a = \sqrt{70}\ cm,\ b = 6cm, c = 7cm,$ Find the length of median AD

Sol. $AD = \sqrt{\dfrac{2b^2 + 2c^2 - a^2}{4}}$

$= \sqrt{\dfrac{72 + 98 - 70}{4}} = 5cm$

Problem- 3

If G is the centroid of ΔPQR and a line passing through G meet PQ at A and PR at B such that



$AQ = 2cm$

$AP = 5cm$

$BR = 3cm$

Find the length of PR

Sol. Using $\dfrac{AQ}{AP} + \dfrac{BR}{PB} = 1$

$\dfrac{2}{5} + \dfrac{3}{PB} = 1$

PB=5 cm

$\therefore$ PR= 5+3 = 8 cm

Problem-4

Let $o\ (a,b)$ be the circumcenter of $\Delta\ ABC$, Whose vertices are $A(1,4), B(-2,3)$ and $c\ (5,2)$, Find $(a+b)$

Sol. As circumcenter is equidistant from vertices

$\therefore \quad OA = OB = OC$

$OA^2 = OB^2$

$(a-1)^2 + (b-4)^2 = (a+2)^2 + (b-3)^2$

$3a + b = 2$ _______(1)

$OB^2 = OC^2$

$(a+2)^2 + (b-3)^2 = (a-5)^2 + (b-2)^2$

$7a - b = 8$ _______(2)

on solving eg (1) and (2)

$a = 1, \ b = -1$

circumcentre $\equiv (1, -1)$

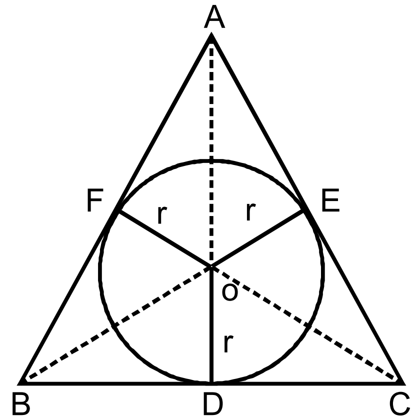
$a + b = 1 - 1 = 0$

Problem- 5

In $\Delta\ ABC$, $AB = 15cm,\ BC = 14cm\ AC = 13cm$

Find inradius r.

Sol.



$s = \dfrac{a+b+c}{2}$

$= \dfrac{15+14+13}{2}$

$= 21\ cm$

area $(ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{21(21-15)(21-14)(21-13)}$

$= 84\ cm^2$ _______________(1)

area $(ABC) = ar\ (BOC) + ar\ (AOC) + ar\ (AOB)$

$$= \frac{1}{2}.14.r + \frac{1}{2}.13.r + \frac{1}{2}.15.r$$

$$= \frac{14r + 13r + 15r}{2}$$

$$= .21r \qquad cm^2 \quad \_\_\_\_\_\_\_\_\_\_ (2)$$

from (1) and (2)

$$21\,r = 84$$

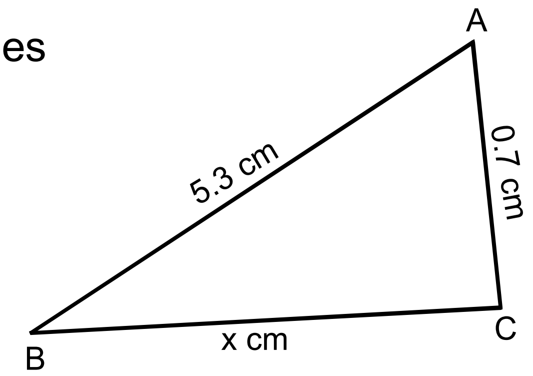
$$r = 4\ cm$$

Problem- 6

One side of some triangle is 5.3 cm, and the other is 0.7 cm, Find the third side of triangle, if its length is a natural number

Sol. Let third side be x cm.

By Triangle existence inequalities



$$x < 5.3 + 0.7$$

$$x < 6 \quad \_\_\_\_\_\_\_\_ (1)$$

$$5.3 < x + 0.7$$

$$x > 4.6 \_\_\_\_\_\_\_\_ (2)$$

$$0.7 < x + 5.3$$

$$x > 4.6 \quad \_\_\_\_\_\_\_\_ (3)$$

from eq (1), (2) and (3)

$$4.6 < x < 6$$

or $\qquad x \in (4.6, 6)$

As x is a natural number

$$\therefore \qquad x = 5\ cm$$

Some Problems for Practice

1. In ΔABC, $BC = 8cm$, $AC = \sqrt{82}\ cm$ median $BE = 6cm$

   Fine the length of AB.

   [Ans AB=7cm]

2. In ΔABC, $AB = 5\ cm$, $BC = 6\ cm$, $CA = 8\ cm$ median $BE = 7\ cm$,

   Median $CF\ \frac{\sqrt{35}}{2} cm$

   Find the length of median AD. [Ans $AD = 6cm$]

3 In ΔABC

equation of side AB is

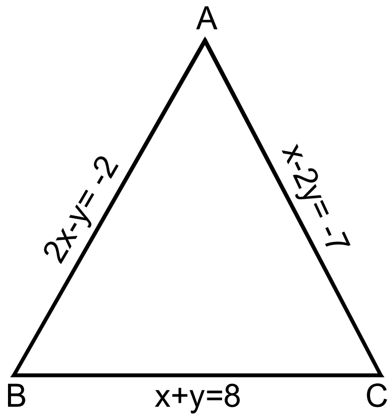
$2x - y = -2$

eq. of side BC is

$x + y = 8$

egs of side AC is

$x - 2y = -7$



If centroid of ΔABC, is $G(x, y)$ then Find the value of x.y

Ans [x.y=10]

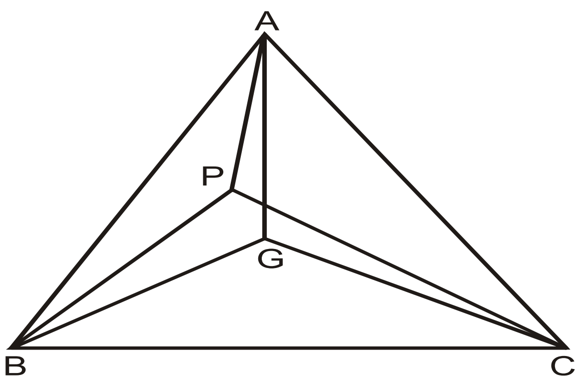
4. In ΔABC, G is centroid and p be any point in the interior of ΔABC, such that

   $GA = 5\ cm$, $GB = 5\ Cm$

   $GC = 4\sqrt{3}\ cm$, $PA = 3cm$

   $PB = 5cm$

   Find length of PC



Ans PC=8cm

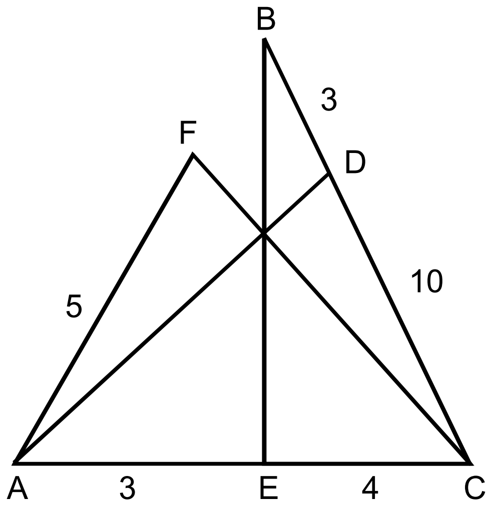
5. In ΔABC, Altitude $AD = 4\,cm$ Altitude $BE = 6\,cm$ and Altitude $CF = 12\,cm$

Find in radius

Ans r-2cm

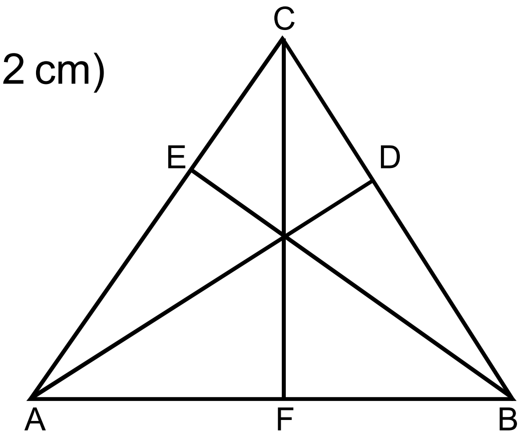
6. An artist has created a triangular stained glass window and has one strip of siding left before completing the window.

Find out the length of the last side (FB) based on the length of other sides as shown in figure



Hint :- Use ceva's Theorem [Ans FB=2 cm)

7. In given figure



AF = 6 cm

BD= 7 cm

CE= 8 cm

AE=6 cm

Find CD×FB

[Ans FB×CD=56]

8. A right triangle with legs of length 3 cm and 4 cm. Find the radius of inscribed circle. [Ans r-1 cm]

9. In the following fig. CD in angle bisector of $\angle ACB$ such that

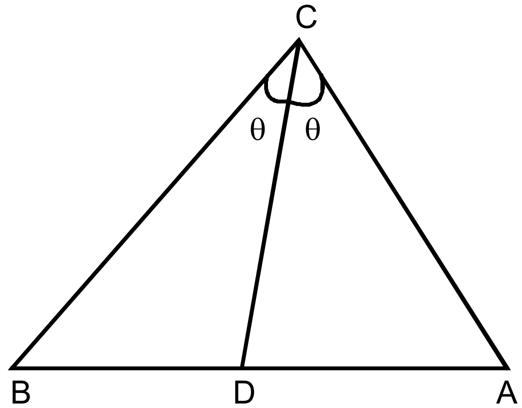
$BC = 8\ cm$

$AC = 12\ cm$

$AB = 10\, cm$

Find (i) length of AD

(ii) length of CD



$\left[\begin{array}{l} Ans AD = 6cm \\ \quad CD = 6\sqrt{2}\ cm \end{array}\right]$

10. For $\Delta\ ABC$ inradius = $r = 5/2\, cm$

circum radius $R = 9\ cm$

Find the distance between incenter and circumcenter.]

[Ans d = 6 cm]

Solution of problems for Practice

1. $a = 8\ cm,\ b = \sqrt{82}cm,\ C = ?$

median $BE = 6cm$

$$BE = \sqrt{\frac{2a^2 + 2c^2 - b^2}{4}}$$

$$36 = \frac{128 + 2c^2 - 82}{4}$$

$2c^2 = 98$ $\boxed{C = 7\ cm}$

2. $AB = 5\ cm, BC = 6\ cm,\ CA = 8\,cm$

Median $BE = 7cm$, Median $CF\ \frac{\sqrt{35}}{2}cm$

median $AD = ?$

$$AB^2 + BC^2 + CA^2 = \frac{4}{3}(AD^2 + BE^2 + CF^2)$$

$$25 + 36 + 64 = \frac{4}{3}(AD^2 + 49 + \frac{35}{4})$$

$$375 = 4AD^2 + 196 + 35$$

$$4AD^2 = 144 \Rightarrow AD^2 = 36 \Rightarrow AD = 6\ cm$$

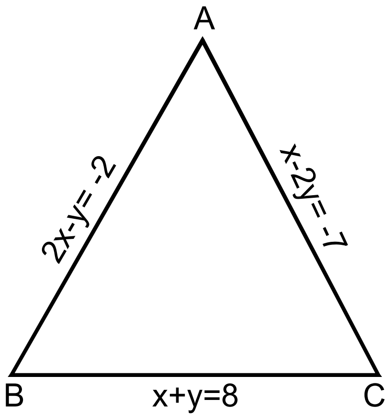
3. <u>For vertex A</u>

2x-y=-2 ______(1)

x-2y=-7 ______(2)

on solving $A \equiv (1,4)$

<u>For vertex B</u>



2n-y = –2 _______(1)

x+y=8 ______(2)

on solving B $\equiv$ (2,6)

For vertex C

x-2y=-7 ______(2)

x+y=8 ______(3)

on solving $C \equiv (3,5)$

Centroid $G(x, y) = \left(\frac{1+2+3}{3}, \frac{4+6+5}{3}\right) = (2,5)$

$\therefore \ x.y = 2 \times 5 = 10$

4. $GA = 5\ cm$, $GB = 5\ cm$, $GC = 4\sqrt{3}\ cm$

$PA = 3\ cm$ $PB = 5\ cm$ $PC = ?$

$PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2$

$9 + 25 + PC^2 = 25 + 25 + 48$

$PC^2 = 64 \ \Rightarrow \ PC = 8\ cm$

5. Altitude $AD = h_1 = 4\,cm$

Altitude $BE = h_2 = 6\,cm$

Altitude $CF = h_3 = 12\,cm$

Inradius= $r = ?$

$$\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = \frac{1}{r}$$

$$\frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{1}{r}$$

$$\frac{3+2+1}{12}=\frac{1}{r}\Rightarrow r=2\ cm$$

6. By Ceva's Th

$$\frac{FB}{FA}.\frac{EA}{EC}.\frac{DC}{DB}=1$$

$$FB.\ EA.\ DC=FA.EC.DB$$

$$FB\times 3\times 10=5\times 4\times 3$$

$$\boxed{FB=2cm}$$

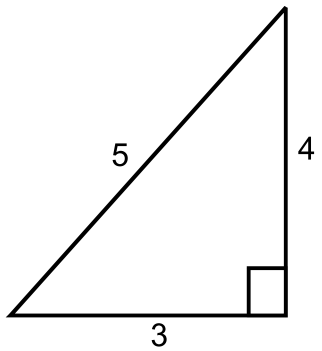
7. BY Ceva's Th

$$\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{EC}{EA}=1$$

$$\frac{6\times 7\times 8}{FB\times CD\times 6}=1$$

$$FB\times CD=56$$

8. Inradius $r=?$



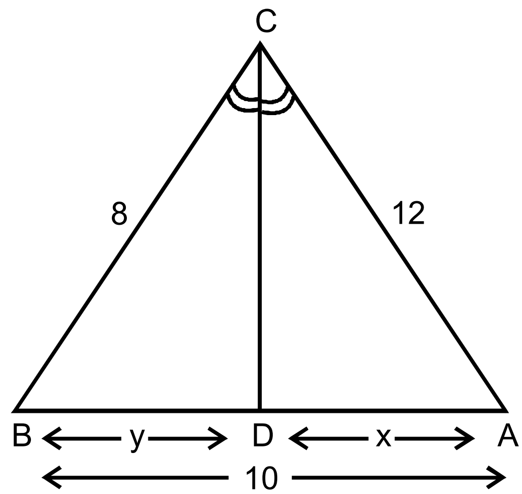
Area of $\Delta=\frac{1}{2}\times 3\times 4=6\,cm^2$

$$s=\frac{3+4+5}{2}=6cm$$

$$\Delta=sr$$

$$r=\frac{\Delta}{s}=\frac{6}{6}=1\ cm$$

9. By Angle insector Th



$$\frac{12}{8} = \frac{x}{10-x}$$

$$8x = 120 - 12x$$

$$20x = 120$$

$$x = 6$$

$$\boxed{AD = 6\ cm}$$

(ii) $CD = \sqrt{ab - xy}$

$$= \sqrt{12 \times 8 - 6 \times 4} \quad = \sqrt{72}$$

$$\boxed{CD = 6\sqrt{2}\ cm}$$

10. Inradius $r = 5/2\ cm$, circum radius R=9 cm Distance between incenter & circumcenter d=?

By Euler's Th

$$(R - r)^2 = d^2 + r^2$$

$$\left(9 - \frac{5}{2}\right)^2 = d^2 + \left(\frac{5}{2}\right)^2$$

$$d^2 = \frac{169}{4} - \frac{25}{4} = \frac{144}{4} = 36$$

$$d = 6\ cm$$

(a) A circle is symmetric about any of its diameter.

(b) Given any three non-collinear points A, B, C there exist a unique circle passing through A, B, C.

(c) Equal chord of a circle are equidistant from center.

(d) Given any two chords of a circle, the one which is nearer to the center, is greater than one more remote.

(e) For any arc of a circle the angle substended at the center is double the angle substended at any point on the remaining part of circumference.

(f) Angle in the same segment of a circle are equal.

(g) If a straight line segment joining two points substends equal angle at two other points on the same side of it, than four points are concyclic.

(h) One and only one tangent can be drawn to a circle at any point on its circumference and this tangent is perpendicular to the radius through the point of contact.

(i) If two circles touch one another than the point of contact lies on the line joining the center.

(j) If two tangents are drawn to a circle from an exterior point then :

 (1) The length of tangent are equal.

 (2) They subtend equal angles at the center.

 (3) The angle between them is bisected by line joining the points and the center.

(k) In equal circles of two chords are equal than they cut-off equal arc on the circle.

(l) If any circle of angle between a tangent and chord through the point of contact of tangent is equal to the angle in alternate segment.

(m) A common tangent of two circles divides the line segment joining center externally or internally in the ratio of their radii.

(n) The opposite angle of a cyclic quadrilateral are supplimentary.

(o) If ABCD is cyclic quadrilateral then any exterior angle of ABCD is equal to interior opposite angle.

(p) If two opposite angle of a quadrilateral are supplimentary then it is cyclic.

(q) If AB and CD are any two chords of a circle meeting at point, P then :

PA.PB = PC.PD

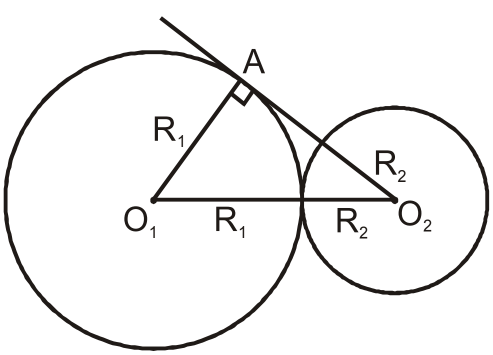
(r) If, p is any point on a chord AB of a circle with center, O and radius, r then:

$AP.PB = r^2 - OP^2$

Example : 1

Two circles are tangent to each other through the center of the second is drawn a tangent to the first. The distance from the point of tangency to the center of the second circle equals 3 times its radius. What is the ratio of radii of the first circle to the second.

Solution :



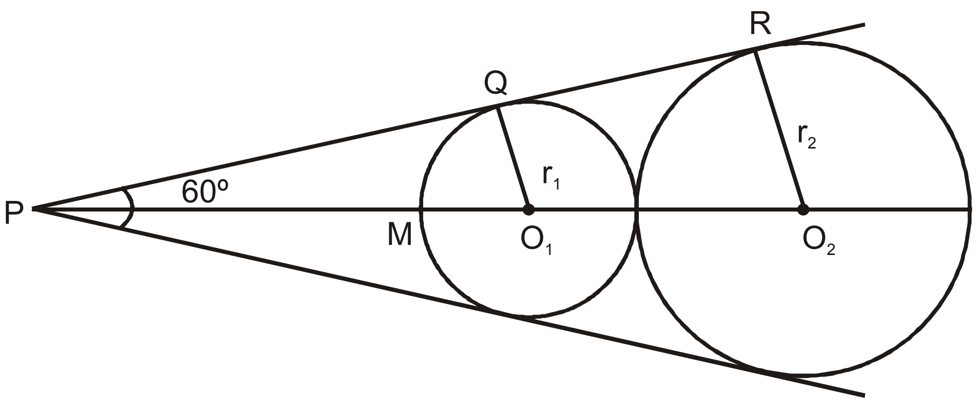
According to question :

$$9R_2^2 + R_1^2 = (R_1 + R_2)^2 \text{ on solving } \frac{R_1}{R_2} = 4$$

Example : 2

Two circles are inscribed in a 60° angle, so that second circle is tangent to first one. How much is the combined area of two circles greater than that of first.

Solution :



Let PM = x

$\therefore$ $\quad \sin 30° = \dfrac{r_1}{x + r_1} \Rightarrow r = x$

also $\quad \sin 30° = \dfrac{r_2}{x + 2r_1 + r_2} \Rightarrow r_2 = x + 2r_1 = 3r_1$

According to question,

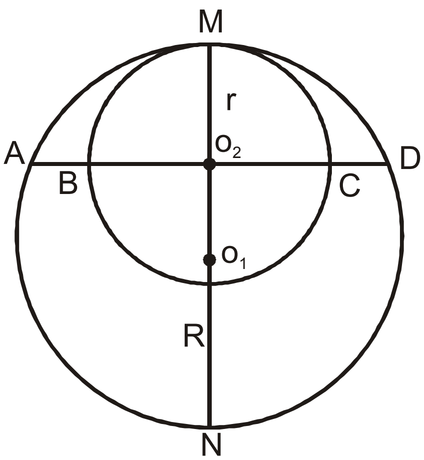
$$\frac{\pi(r_1^2 + r_2^2)}{\pi r_1^2} = \frac{r_1^2 + 9r_1^2}{r_1^2} = 10$$

Example : 3

Consider two tangent circles such that one is within the other. A line passes through point B,C and the center of the smaller circle and crosses the bigger one at point A and D. Find the ratio of radii of circles of :

| AB | : | BC | : | CD | = 2 : 4 : 3

Solution :



Let r be the radius of smaller circle and R of bigger

$$\frac{AB}{BC} = \frac{2}{4} \text{ or } AB = r\ (\because\ BC = 2r)$$

Similarly $$CD = \frac{3r}{2}$$

Now $$AO_2 . O_2D = MO_2 . O_2N$$

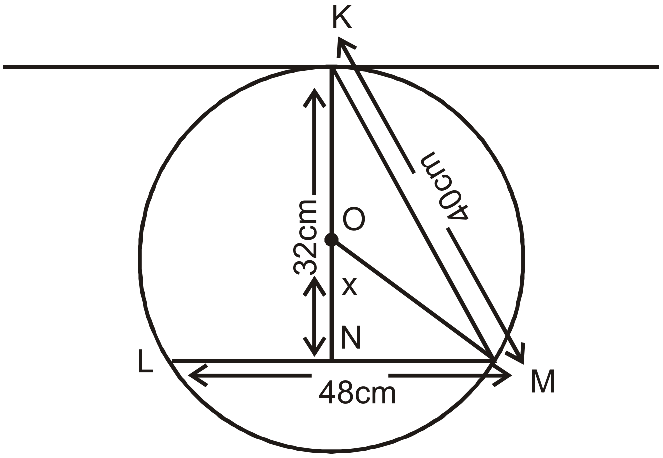
$$\Rightarrow \quad 2r . \frac{5r}{2} = r . (2R - r)$$

$$\Rightarrow \quad \frac{R}{r} = 3$$

Example : 4

The tangent to a circle at point K is parallel to chord LM of circle it is known that LM = 48 cm and KM = 40 cm. Find radius of circle.

Solution :



Using the pythagoras theorem is ΔKMN, KN = 32 cm.

Again $\quad ON = x$, so $OK = 32 - x = r = OM$ (radius)

$$x^2 + (24)^2 = (32 - x)^2$$

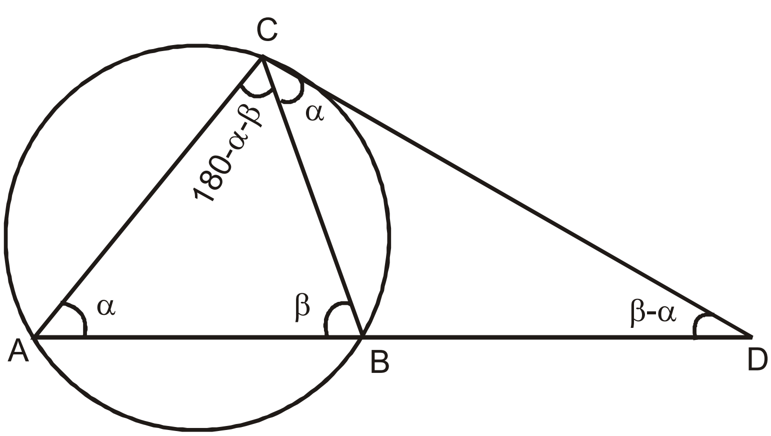
$\Rightarrow \quad x = 7$

$\therefore \quad r = 32 - 7 = 25$ cm

Example : 5

Triangle ABC is inscribed in a circle. A tangent dropped from vertex C of the triangle intersects the extension of side AB from vertex B at point D. It is known that : $\angle CDA + \angle ACB = 2\angle BAC$. Find the angle between tangent and chord CB.

Solution :



Let $\angle A = \alpha$ and $\angle B = \beta$

Using properties of a angle between tangent CD and chord CB are equal to the angle BAC.

And $\angle CDA + \angle ACB = 2\angle BAC$

$\Rightarrow \beta - \alpha + 180 - \alpha - \beta = 2\alpha$

$\Rightarrow \alpha = 45°$

Question/Answer

1. The tangent point of a right triangle and an inscribed circle divides a leg into segment of length 3 cm and 2 cm.

   Ans. 12

2. A circle is inscribed into a triangle with sides AB = 8, BC = 6 and AC = 4. Find the length of segment, DE in nearest integer of D and E are tangent to sides AB, AC res.

Ans. 2

3. An isosceles triangle, ABC, with sides of length 2 cm and 120° vertex angle is given. Find the radius of the circle in nearest integer that can be inscribed in ΔABC.

Ans. 5

4. An isosceles, trapezium of base 9, and 16 cm circumscribed about a circle, then find is height.

Ans. 12

5. ABCD is a cyclic quadrilateral such that angles between sides CB and DA be 20° and between sides AB and DC be 30°, find $\angle D$.

Ans. 65°

6. Two circles of same radius $\sqrt{7}+1$ intersect each other so that distance between their centers equal $\sqrt{7}+1$. In a figure formed by intersection of two

circles, a square is inscribed. Find the side of square.

Ans. 3

7. Four congruent circles are placed in a square of sides 40 cm, so that each is tangent to two sides of the square and to two sides of other circle. A smaller fifth circle is drawn tangent to each of the four circles. Find its radius in nearest integer.

Ans. 4

8. The tangent to a circle at point, K is parallel to chord, LM of the circle it is known that $LM = 6$ and KM = 5. Find BR where, R is radius of circle.

Ans. 25

9. Chord AP and CQ of circle circumscribed about acute triangle ABC contain its height dropped from verticle A and C. Find the radius of the circle circumscribed about triangle PLO where L is the point of intersection of chords AP and CQ, $\angle BAC = 65°$, $\angle ACB = 70°$ and PQ = $2\sqrt{2}$.

Ans. 2

Circle : A circle is the path travelled by a point which moves in such a way that its distance from a fixed point remain constant.

The fixed point is known as center and the fixed distance is called radius.

(a) Circumference or perimeter of circle = $2\pi r = \pi d$

(b) Area of circle = $\pi r^2 = \frac{\pi d^2}{4} = \frac{c^2}{4\pi} = \frac{1}{2}cr$

(c) r = $\sqrt{\frac{A}{r}} = \frac{\text{Circumference}}{2\pi}$

(d) Ratio of area of the two circles :

$$\frac{\textit{Area of circle circumscribing square}}{\textit{Area of circle inscribing same square}} = \frac{2}{1}$$

(e) Ratio of area of two squares is :

$$\frac{\text{Area of square circumscribing circle is}}{\text{Area of square inscribing in same circle}} = \frac{2}{1}$$

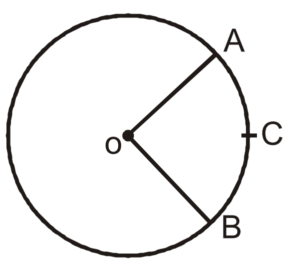
Sector : A sector is a figure enclosed by two radii and an arc lying between them :

$$\text{Arc} = \frac{2\pi n\theta}{360}$$

Area of sector : $\frac{1}{2} \times \text{Arc} \times r = \frac{\pi r^2\theta}{360}$

Semi-circle : Semi-circle is a figure enclosed by a diameter and the part of circumference cut off by it.

Segment : Segment of a circle is a figure enclosed by chord and an arc which it cut off. Any chord of a circle which is not a diameter divides the circle into two segments, one greater and one less than a semi-circle.



•Area of segment ACB = Area of sector ACBO : Area of Δ OAB

•Area of segment ADB = Area of circle – Area of segment ACB.

•A circular ground of radius, r, has a path of width, w around it on outside. The area of circular pathway is given by πw (2r + w).

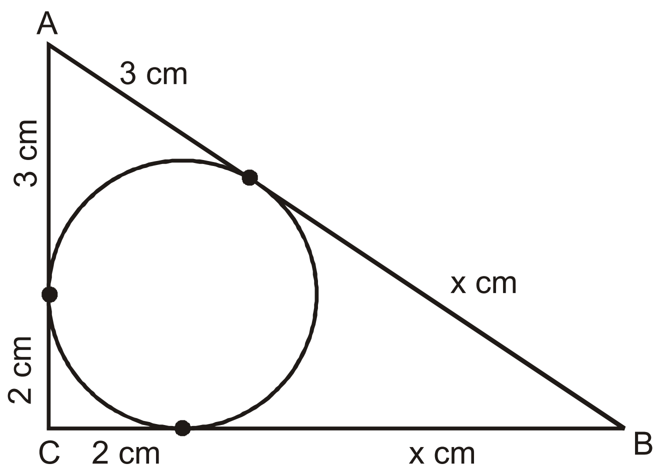
•A circular ground of radius, r has a pathway of width, w around it as inside. The area of circular pathway is given by πw (2r – w).

•The area of largest circle can be inscribed in a sqaure of side, a is $\frac{\pi a^2}{4}$.

•Area of square inscribed in a circle of radius, r is $2r^2$. Area of largest triangle inscribed in a semi-circle of radius, r is r^2.

•The number of revolutions made by a circular wheel of radius, r is travelling distance d is given by $\left(\frac{d}{2\pi r}\right)$.

1.



Using pythagoras theorem

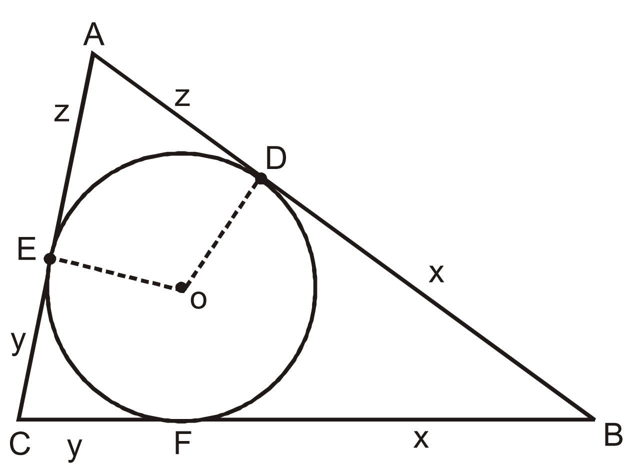
$$(5)^2 + (2 + x)^2 = (3 + x)^2$$

$$\Rightarrow 25 + 4 + 4x + x^2 = 9 + 6x + x^2$$

$$\Rightarrow x = 10$$

$\therefore$ Second leg CB = 2 + 10 = 12

2.



Let us introduce the variable x = FB = BD, Y = CF = EC and Z = DA = AE, and r = OD = OE.

We can find the radius r, using the relationship between the area of triangle ABC, the radius of inscribed circle and the half perimeter, $p=\frac{6+4+8}{2}=9$. By Heron's formula,

$$s = \sqrt{P(P-a)(P-b)(P-c)} = \sqrt{9(9-6)(9-8)(9-4)} = 3\sqrt{15} \quad ...(1)$$

and s = pr = 9r

....(2)

From (1) and (2) $r = \frac{\sqrt{15}}{3}$

Now x + y = 6, y + z = 4, z + x = 8 $\Rightarrow$ x = 5, y = 1, z = 3

Now area of AEOD in two different ways.

$$\frac{DEOA}{2} = 2.\frac{r.z}{2} \Rightarrow DE =$$

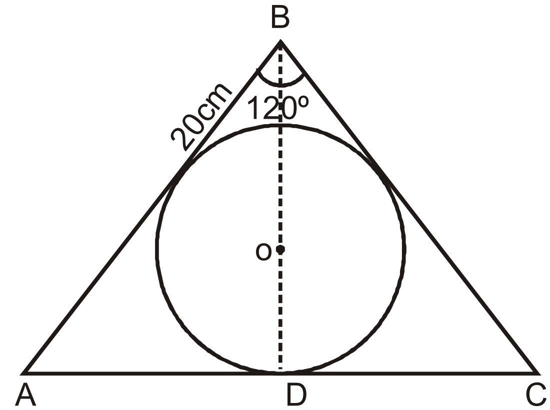
$$\frac{2\sqrt{15.3}}{3.0A} = \frac{2\sqrt{15}}{OA} ......(3)$$

Again $OD^2 = OD^2 + AD^2 \Rightarrow OA\frac{4\sqrt{6}}{3}$ ....(4)

From (3) and (4)

$$DE = \frac{3\sqrt{10}}{4} = 02$$

3.



Using the formula S = Pr.

In ΔABD sin 60° = $\frac{AD}{AB} \Rightarrow \frac{\sqrt{3}}{2} = \frac{AD}{20} \Rightarrow AD = 10\sqrt{3}$

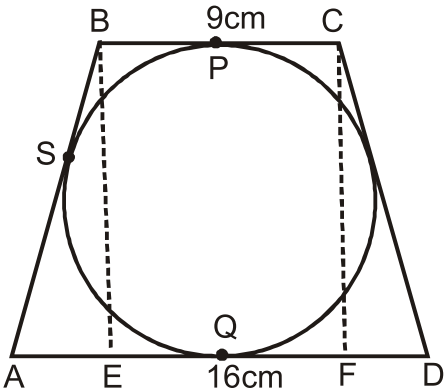
$\Rightarrow$ AC = $20\sqrt{3}$

P = $\frac{20+20+20\sqrt{3}}{2} = 20+10\sqrt{3}$

S = $\frac{1}{2} \times 20\sqrt{3} \times 10 = 100\sqrt{3}$

$\Rightarrow$ r = $\frac{100\sqrt{3}}{20+10\sqrt{3}} = \frac{10\sqrt{3}}{2+\sqrt{3}} \cong 05$

4.



$$AE \quad \frac{7}{2}, AB = AS + SB = AQ + BP = \frac{16}{2} + \frac{9}{2}$$

$$\Rightarrow AB = \frac{25}{2}$$

In Δ AEB

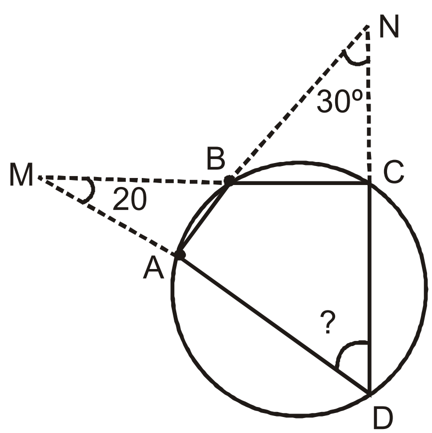
$$AB^2 = BE^2 + AE^2$$

$$\frac{625}{4} = h^2 + \frac{49}{4}$$

$(\because BE = h)$

$$h = 12$$

5.



We know that $\angle B + \angle D = 180°$ ...(i)

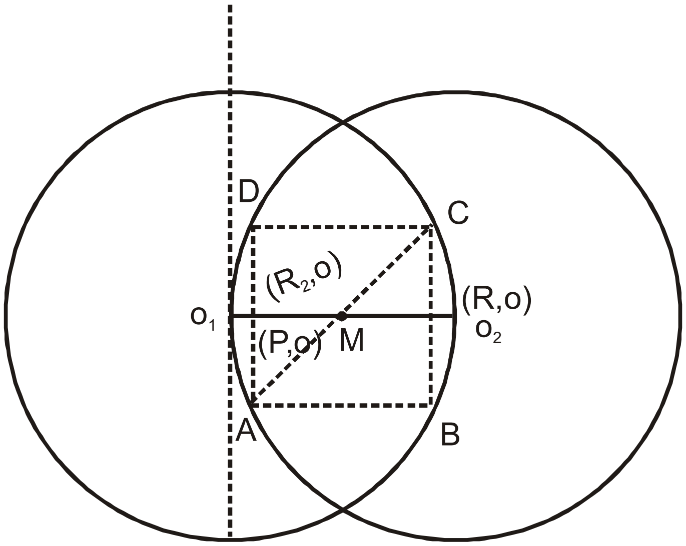
and $\angle A + \angle C = 180°$ ...(ii)

In ΔAND $\angle A + \angle D = 150°$ ...(iii)

In Δ MCD $\angle C + \angle D = 160°$ ...(iv)

On solving these four equation

$$\angle D = 65°$$



6.

Radius of both circles are same centre of both the circle lie on one another.

$$\therefore O_1O_2 = R$$

Let first circle's centre is on (0, 0), $\therefore$ (R, 0) is the centre of second circle. Equation of first circle will be $x^2 + y^2 = R$.

Let ABCD be the required square then AC diagonal inclined at 45° with O_1O_2. So equation of AC will be y – 0 = tan 45 (x – R/2)

$\Rightarrow y = x - \frac{R}{2}$. Now point of intersection of (1) and first circle $x^2 + y^2 = R^2$ is

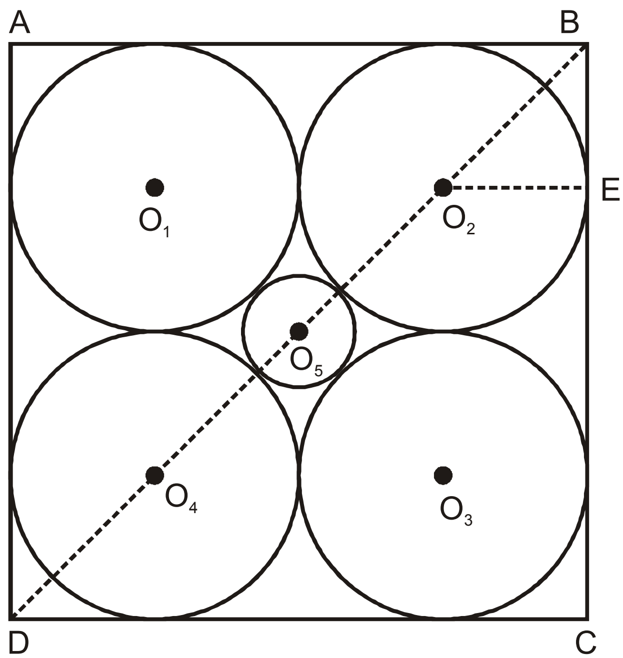
$$\left(\frac{R}{4}\left(\sqrt{7}+1\right), \frac{R}{4}\left(\sqrt{7}-1\right)\right)$$

$\therefore$ Length of one side of square will be $\frac{R}{2}\left(\sqrt{7}-1\right)$ here

$$R = \sqrt{7}+1$$

$\therefore$ Side = $\dfrac{(\sqrt{7}+1)(\sqrt{7}-1)}{2} = 3$

7.



Radius of each big circle will be 10 cm. Let r be that of 5m one.

In ΔBCD right angled at C

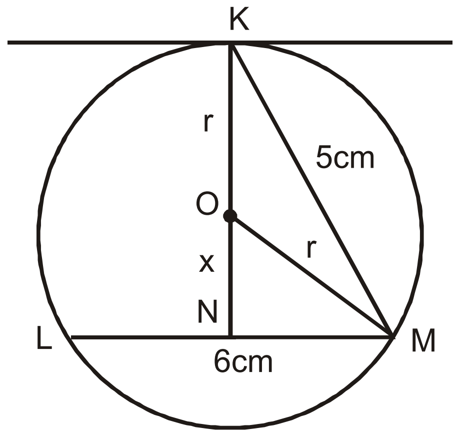
$BD^2 = BC^2 + CD^2$ and in $BO^2 E$

$O^2B^2 = BE^2 + O^2E^2$ ....(i)

$x = 10(\sqrt{x}-1)$

and BD = 2x + 2r + 40 = 20 $(\sqrt{2}-1)$ = 20 $(\sqrt{2}+1)+2r$

from (i) r=10 $\left(\sqrt{2}-1\right)r = 4$



In ΔKMN

$$25 = (r + n)^2 + 9$$

$\Rightarrow r + x = 4$ ...(i)

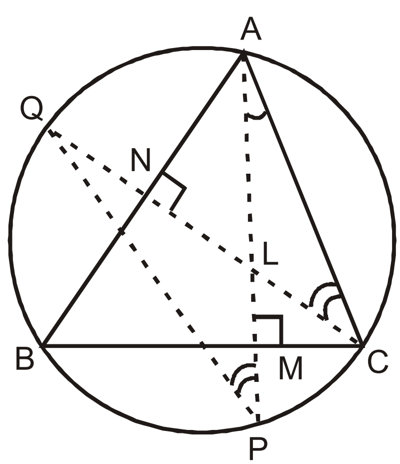
again in ΔMNO

$$r^2 - n^2 = 9$$

$$(r + x)(r - x) = 9$$ ...(ii)

from (i) and (ii)

$$r = \frac{25}{8} = 8R = 25$$



CAN and AMC are right triangles so $\angle ACQ = 25°$ and $\angle CAP = 20°$

From the inscribed angle theorem

$$\angle APQ = \angle ACQ = 25, \ \angle CQP = \angle CAP = 20°$$

$$\Rightarrow \angle QLP = 180 - 25 - 20 = 135°$$

Finally, applying law of sine to the triangle PLQ and find the radius of its circumscribed circle

$$R = \frac{PQ}{2\sin\angle PLQ} = \frac{2\sqrt{2}}{2\sin + 135°} = 2$$

Combinatorics

Rule of sum : If one event can take place in m ways and 2^{nd} event can take place in n ways then any one out of these 2 events (either first only or 2^{nd} only) can take place in m+n ways.

Rule of multiplication : If one event can take place in m ways and 2^{nd} event can take place in n ways then both the events (one after the other or simultaneously) can take place in m n ways:

Difference between permutations & Combinations:

(between arrangements & selections)

No. of permutations means no. of different arrangements. Order of objects is important in this case. No of combinations means no. of different groups/ selections. Order doesn't have any importance in this case.

Example: If there are 3 objects a, b & c, then different permutations of 3 objects taken 2 at a time are ab, ba, bc, cb, ac & ca i.e. 6 arrangements while for these 3 objects different combinations (selections) if 2 are taken at a time are ab, bc & ca i.e. 3 only. Please note ab & ba are 2 different permutations but it is only one combination.

- No. of permutations of n objects taken r at a time (repetition not allowed)

$$= {}^{n}P_{r} = {}^{n}_{r}p = \frac{\lfloor n}{\lfloor n-r}$$

Where $\lfloor n$ = n! = factorial n

= 1×2×3.......n.

- No. of permutations of n objects taken all at a time if out of these n objects, p_1 are of same kind, p_2 are of same kind,.....,......, p_n are of same kind is

$$\frac{\lfloor n}{\lfloor p_1 . \lfloor p_2 .... \lfloor p_n}.$$

- No. of ways in which m+n things can be divided into 2 groups containing m & n objects respectively is $\frac{\lfloor m+n}{\lfloor m \ \lfloor n}$.

- If m=n then ways of subdivision is $\frac{2m!}{m!\,m!\,2!}$.
- No of ways in which m+n+p objects can be divided into 3 groups containing m, n & p things is $\frac{(m+n+p)!}{m!\,n!\,p!}$
- No. of permutations of n things taken r at a time, when each object may be repeated once, twice, thrice, ... r times = n^r
- No. of combinations/selections of n different objects taken r at a time $= {}^nC_r = \frac{n!}{r!\,(n-r)!} = {}^nC_{n-r}$
- No. of combinations of n different objects taken 1 or 2 or 3, ...or n objects at a time = 2^n-1.
- No. of objects, when no. of combinations out of n objects is maximum is:

  (i) $\frac{n}{2}$ if n is even & no. of combinations = ${}^nC_{\frac{n}{2}}$

  (ii) $\frac{n+1}{2}$ or $\frac{n-1}{2}$, if n is odd.

  & no. of combinations = ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$.
- No. of selections by taking some or all out of p+q+r+..... objects, where p are all of one kind, q are alike of a II kind, r are alike of III kind is (p+1).(q+1). (r+1)........–1.

Q1. Find the number of ways in which a selection of 4 letters can be made from the letters of the word ‘PROPORTION’.

Ans. There are 10 letters of 6 different sorts.

O, O, O, P, P, R, R, T, I, N

Possible groups of 4 letters are as follows.

(a) <u>3 alike 1 different</u> is 3 O’s & 1 different letter out of P,R,T,I,N i.e in 5 ways.

(b) <u>2 alike & 2 different</u> : Selection of 1 set of 2 alike out of 3 pairs in ${}^3C_1$ =3 ways & selection of 2 out of 5 = ${}^5C_2$ =10, $\therefore$ No. of ways = 3×10=30

(c) <u>2 alike & 2 alike</u> : Selection of 2 alike groups out of 3 groups = ${}^3C_2$=3.

(d) <u>All four different</u> : No. of selections = ${}^6C_4$ =15

$\therefore$ Total no. of selections = 5+30+3+15 = 53.

Q2. If words formed by all the letters of the word MOTHER are arranged as in a dictionary, If there are total n words before mother then find $\frac{n}{4}$.

Ans. In alphabetical sequence,

Words starting with	**No. of words**
E	$\lfloor 5 = 120$
H	$\lfloor 5 = 120$
ME	$\lfloor 4 = 24$
M H	$\lfloor 4 = 24$
M O E	$\lfloor 3 = 6$
MOH	$\lfloor 3 = 6$
MOR	$\lfloor 3 = 6$
MOTE	$\lfloor 2 = 2$
MOTHER	______.

$\therefore$ n=120+120+24+24+6+6+6+2=308

$\therefore \frac{n}{4}$=77

Q3. If total n rectangles can be drawn on a chess board then find $\frac{n}{36}$.

Ans. To form a rectangle we need 2 horizontal and 2 vertical lines. 2 lines out of 9 lines can be chosen in $^9C_2$ ways

$\therefore$ n= $^9C_2 \times {^9C_2}$ = 36×36

$\therefore \frac{n}{36} = 36$

Q4. In how ways can 6 speakers A, B, C, D, E and F address a gathering if A speaks after B.? If answer is n then find n/6.

Ans. Total ways of addressing the gathering = $\lfloor 6$

= 720

Using symmetry it is clear that out of 720, $\frac{720}{2}$ i.e. 360 ways are there when A speaks after B. n/6=60

Q5. If n distributions are possible of 5 identical objects into 7 distinct boxes. If there is no restriction on how many each box many contain. Find n/7.

Ans. Let boxes are repressented by $B_1, B_2, ...B_7$. And identical objects are represented by xls. Let one of the distributions is

$B_2 B_2\ B_4\ B_7\ B_7$

Then we can write is as |x x|| x||| x x

there are 6 vertical separators to separate 7 boxes

Now the question is in how many ways can we arrange these 6 separators & 5 x's.

i.e. $\frac{\lfloor 11}{\lfloor 6 \lfloor 5} = 66 \times 7 = n$

$\therefore \frac{n}{7} = 66.$

Q6. Find the number of all rational numbers $\frac{m}{n}$ s.t. (i) $0<\frac{m}{n}<1$, (ii) m & n are relatively prime and (iii) mn= $\underline{|25}$. If answer is n find n/16.

Ans. We have to resolve $\underline{|25}$ into 2 relatively prime factors. Out of $p/q \& q/p$ any one out of these 2 will be less than 1. The distinct primes occuring in the expression of $\underline{|25}$ are the exponents of 2,3,5,7,11,13,17 19 & 23. As these are 9 distinct primes so there will be $\frac{1}{2}\underline{|9}$ =256 ways expressing $\underline{|25}$.

$\therefore$ There are 256 required numbers. $\therefore \frac{n}{16}=16$

Q7. If in any set of n distinct integers chosen from the set {1,4,7,.......100} there will always be 2 distinct integers whose sum is 104. Find minimum value of n.

Ans. No.s {1,4,7.....100} can be divided into 17 pairs (4,100), (7, 97), (10,94)......., (49,55) and (1,52). The sum of integers in first 16 pairs is 104. Sum of last pair is not 104.

Now we construct largest set, sum of whose any 2 integers is always other than 104. So we will choose exactly 1 no. from each of the first 16 pairs and 2 no.s from the last pair i.e 18 nos can be chosen. Now if we choose and 19th no. then it will be 2 no. of any one pair sum will be 104.

$\therefore$ Answer to this questions =19

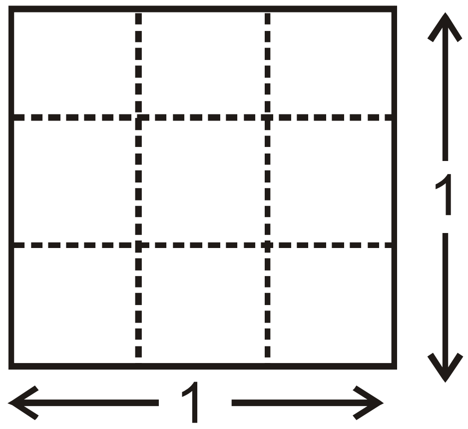
Q8. Ten points are taken at random within a square of side 1 unit. If maximum possible distance between 2 nearest paints is d. Find the value of $9d^2$.

Ans. First 9 points can be placed in 9 small squares having side $\frac{1}{3}$ unit, Now 10^{th} point will be in any 1 of these 9 squares.

$\therefore$ Maximum distance between 2 nearest points

= length of diagonal of smaller square

$\therefore\ d=\frac{\sqrt{2}}{3}$ $\therefore 9\ d^2=2.$



9. If 12 things can be divided equally among 4 persons in n ways then find $\frac{n}{4620}$.

Ans. $n=\frac{\lfloor 12}{(\lfloor 3)^4}=11\times 10\times 6\times 7\times 5\times 4\times 4$

$\therefore \frac{n}{4620}=80$.

Questions (1to15)

1. How many strings of digits from 1–9 of length 5 have an odd number in the odd positions and an even number in the even positions? suppose the answer is n then find $\frac{n}{100}$. **Ans-20**

2. Rob has 4 blue socks, 7 red socks, 5 white socks and 3 black socks. Rob likes to wear either a red sock on his left foot with a blue sock on his right foot or a white sock on his left foot with a black sock on his right foot. How many ways are there Rob to choose his socks? **Ans-43**

3. There are 5 candidates for 2 posts. In how many ways can the posts be filled? **Ans-20**

4. The students in a class are seated according to their marks in the previous examination. Once it so happens that four of the students got equal marks and therefore the same rank. To decide their seating arrangement, the teacher wants to write down all possible arrangements one in each of separate bits of paper in order to choose one of these by lots. How many bits of paper are required? **Ans-24**

5. For a set of five true or false questions, no student has written all the correct answers, and no two students have given the same sequence of answers. What is the maximum number of students in the class for this to be possible. **Ans-31**

6. How many different numbers of two digits can be formed with the digit 1,2,3,4,5,6, no digits being repeated? **Ans-30**

7. The number of ways in which none gets his umbrella of the 5 gentlemen. at the end of the party. **Ans-44**

8. Find the number of positive integers not greater than 100, which are not divisible by 2,3, or 5. **Ans-26**
9. Find the number of ways of dealing a five cards hand from a regular 52 card deck such that the hand contains at least one card in each suit If number of ways are equal to z. then find $\left(\frac{z}{57122}\right)$.**Ans-12**
10. How many integers from 1 to 10^6 (both inclusive) are neither perfect squares nor perfect cubes nor perfect fourth powers? If answer is P then find $\left(\frac{P}{10090}\right)$.
Ans-99
11. If n is a positive integers, the number of integers less than n and prime to it called the Euler function ϕ(n). Calculate the value of ϕ(100) using the IEP.
Ans-40
12. In how many ways can we arrange three objects into five slots if exactly two of the objects are identical? **Ans-30**
13. Find the number of three elements subsets of a set with 9 distinct elements.
Ans-84
14. A girl has three markers of different colours and one black pen. She is going to colour a grid of six white squares. Three of the squares are going to be coloured with the markers and the remaining three will be either coloured black or left white. The black pen can be used multiple times, but each marker can only be used to colour one square. How many different ways are there to colour the grid? If number of ways be N then find $\left(\frac{N}{20}\right)$. **Ans-48**
15. Ram has 12 bricks, each painted with a different pattern he chooses 4 of these bricks, one at a time, and stacks the bricks in the order that she choose them. How many different stacks can she obtain? If number of different stacks be z. then find $\left(\frac{Z}{990}\right)$. **Ans-12**

Solutions of question (1-15)

Ans. 1. Total strings $(n) = 5^3 \times 4^3 = 2000$

$$\therefore \frac{n}{100} = \frac{2000}{100} = 20 \ Ans$$

Ans. 2 Number of ways to choose his socks =(7×4)+(5×3)

= 28+15

=43

Ans. 3. Number of ways $= {}^5P_2 = \frac{5!}{3!} = 20$

Ans. 4 Number of bits of papers are required = $4! = 24$

Ans. 5. Total number of different sequence of answer = $2^5 = 32$ there is only one correct answer.

Here the maximum number of students = (32-1)=31 Ans

Ans. 6. Number of ways $= {}^6P_2$

$$= \frac{6!}{4!} = 6.5. = 30$$

Ans. 7. Total number of ways $= 5!\,(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}$

$$= 5!\,(\cancel{1} - \cancel{1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!})$$

$$= \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - \frac{5\cancel{!}}{5\cancel{!}}$$

=5.4.3-5.4+5-1

=60-20+4

=44

Ans. 8 Positive integers less than or equal to 100 that are divisible by at least one of 2,3 or 5

= 50+33+20-16-6-10+3

=74

Total numbers are not divisible by 2,3 or 5 = 100-74 =26

Ans. 9. Number of all 5 cards be ${}^{52}C_5$

If n(1) is calculated by removing one suit from the deck and dealing the rest n(1)= $4\times{}^{39}C_5$

Similarly $n(2)={}^{4}C_2\times{}^{26}C_5 = 6\times{}^{26}C_5$

$n(3)={}^{4}C_3\times{}^{13}C_5 = 4\times 35$

$n(4)={}^{4}C_4\times 0=0$

Hence $z=n(o) = {}^{52}C_5-4\,{}^{39}C_5+6\,{}^{26}C_5-4\,{}^{13}C_5$

$=685464$

$$\therefore \frac{Z}{57122}=\frac{685464}{57122}=12\ Ans$$

Ans. 10 Total perfect square be $\leq(10^3)^2$

Let n(1) be perfect squres , n(2) be perfect cubes and n(3) be perfect fourth power.

Then $P=10^6-(n(1)-n(2)+n(3))$

$= 10^6-(10^3+10^2+31)+(10+3+31)-3$ ________(1)

This is because the perfect squares less than or equal to $10^6=(10^3)^2$

are $1^2.2^2.3^2.4^2.-$ —$(10^3)^2$

So that no. of perfect squares is 10^3.So also the no of perfect cubes is the perfect fourth powers are

$1^4.2.^4\,3^4 ---[(10^6)^{1/4})^4=(10^{3/2})^4=31^4$

i.e $1^4.2^4.3^4$- - - 31^4

So that number of such perfect fouth power $\leq 10^6\, is\, 31.$Again to calculate n(2), we look for integers which are perfect squares as well as perfect cube e.g.$8^2=64=2^6$. The number are in fact

$1^6.2^6.3^6$ - - - $10^{6.}$

Which means there are only 10 such.

If we look for perfect cubes which are also perfect forth power there are $1^{12}, 2^{12}, 3^{12}$ only since $4^{12}>10^6$

From eq(1). $\therefore$ P = 100000-1131+44-3 = 998910

$\therefore$ P/10090=99 Ans.

Ans 11. Let the prime decomposition of n be

n= $P_1^{\alpha 1}.\ P_2^{\alpha 2} ----- P_{12}^{\alpha 12}$

Where P is distinct primes and α's are all positive integers for each i = 1,2 ---- k define property of having P_i^0 then

$$\phi(n) = n - n(1) + n(2) - n(3) + ---$$

$$= n - \sum_i \frac{n}{Pi} + \sum_{\substack{i,j \\ i<j}} \frac{n}{P_i P_j} - \sum_{\substack{i.j.\hat{k} \\ i<j<k}} \frac{n}{P_i . P_j . P_k} + ---$$

$$= n(1-\frac{1}{p_1})\ (1-\frac{1}{p_2})(1-\frac{1}{P_3}) ---$$

To illustrate we have,

100 = $2^2.5^2$

$$\phi\ (100) = 100\ (1-\frac{1}{2})(1-\frac{1}{5})$$

$$= 100 \times \frac{1}{2} \times \frac{4}{5}$$

$$= 40$$

Ans. 12. We choose two slots to put the identical objects and then one of the remaining slots to place the other objects total number of possible arrangement = (3) ${}^5C_2$

$$= \frac{3 \times 5 \times 4}{2 \times 1}$$

= 30

Ans.13 In subsets the order of the elements does not matter

Total number of elements $= {}^{9}C_3 = \frac{9\times8\times7}{3\times2\times1} = 84$

Ans. 14. (i) 3 out of 6 squares to be coloured = ${}^{6}C_3 = 20$

(ii) 3 squares we choose a different colour = 3×2×1=6

(iii) Each of the three remaining squares can be either coloured black or left white = 2×2×2

=8

Total distinct ways to colour the grid =20×6×8

N= 960

$\therefore \frac{N}{20} = \frac{960}{20} = 48.$

Ans. 15. Let number of ways that she can make a stack of bricks is number of ways that she choose from bricks. From the set of 12. Where order is not matter be ${}^{12}P_4$.

$${}^{12}P_4 = \frac{12!}{8!} = 11880$$

If Z = 11880

then $\frac{Z}{990} = 12$ Ans.

TRIGONOMETRY

<u>Basic Concepts:</u>

1. **Measurement of an angle**:

(i) **Sexagesimal System**: In this system angle is measured in term of **'degrees'**

One right angle = 90 degree; 1 degree = 60 minutes; 1 minute = 60 seconds

(ii) **Circular measure of an angle**: In this system, angle is measured in terms of "radians". A radian is a constant angle is defined as angle formed at the centre of the circle by an arc whose length is equal to the radius of the circle.

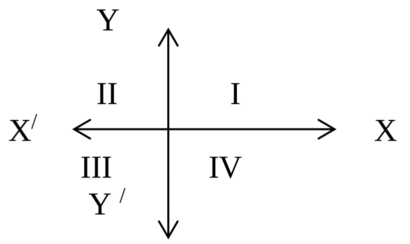
2. Relation between Sexa-gesimal system and Circular system: $180^\circ = \pi^c$

$$1^0 = \left(\frac{\pi}{180}\right)^c \quad \text{and} \quad 1^c = \left(\frac{180}{\pi}\right)^0$$

3. An angle is formed by rotating a ray about its fix point in clock and anti-clock wise direction. If rotation is anti clock-wise, angle is termed as positive and if it is in clock-wise direction it is taken as negative.

4. In a circle of radius 'r', if radian measure of an arc of length 'l' is θ , then $l = r.\theta$

5.



I : All are positive
II: Only Sin and Cosine are positive
III: Only Tan and Cot are positive
IV: Only Cos and Sec are positive

I Quadrant: θ, $360 + \theta$, $90 - \theta$; II Quadrant: $90 + \theta$, $180 - \theta$

III Quadrant: $180 + \theta$, $270 - \theta$; IV Quadrant: $270 + \theta$, $360 - \theta$, $-\theta$

6. For the values $90 \pm \theta$ and $270 \pm \theta$ Trigonometry values are changed as per table

Sin $\longleftrightarrow$ Cos ; tan $\longleftrightarrow$ cot ; Cosec $\longleftrightarrow$ Sec

Illustrations: $\sin(90-\theta) = \cos\theta$, $\cot(270+\theta) = -\tan\theta$

7. Some Basic Formulas :

$sin^2\theta + cos^2\theta = 1,$ $\quad 1+tan^2\theta = sec^2\theta,$ $\quad 1+cot^2\theta = cosec^2\theta$

8. **Sum and Difference Formulas**:

Sin (A + B) = Sin A Cos B + Cos A Sin B; Sin (A - B) = Sin A Cos B - Cos A Sin B

Cos (A + B) = Cos A Cos B - Sin A Sin B; Cos (A - B) = Cos A Cos B + Sin A Sin B

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A.\tan B}; \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A.\tan B}$$

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}; \qquad \cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

$$Sin^2 A - Sin^2 B = Sin(A+B).Sin(A-B) \qquad Cos^2 B - Sin^2 A = Cos(A+B).Cos(A-B)$$

9. **Product to Sum & Difference Formulas**:

2 Sin A. Cos B = Sin (A + B) + Sin (A - B); 2 Cos A. Sin B = Sin (A + B) - Sin (A - B)

2 Cos A. Cos B = Cos (A + B) +Cos (A - B); 2 Sin A. Sin B = Cos (A - B) – Cos (A + B)

10. **Sum/ Differecne to Product Formulas**:

$SinC + SinD = 2Sin\frac{C+D}{2}.Cos\frac{C-D}{2}$; $SinC - SinD = 2Cos\frac{C+D}{2}.Sin\frac{C-D}{2}$

$Cos\,C + Cos\,D = 2Cos\frac{C+D}{2}.Cos\frac{C-D}{2}$: $Cos\,C - CosD = -2Sin\frac{C+D}{2}.Sin\frac{C-D}{2}$

11. **Multiple angle formulas**:

Sin 2 A = 2 sin A. Cos A = $\frac{2\tan A}{1+\tan^2 A}$

Cos 2A = Cos^2 A - Sin^2 A = 2 Cos^2 A -1 = 1 – 2 Sin^2 A = $\frac{1-\tan^2 A}{1+\tan^2 A}$

tan 2 A = $\frac{2\tan A}{1-\tan^2 A}$

Sin 3 A = 3 Sin A – 4 Sin^3 A ; Cos 3 A = 4 Cos^3 A – 3 Cos A , tan 3 A = $\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

12. Domain and Range of Trigonometric Functions

Function	**Domain**	**Range**
sin A	R	$[-1,1]$
cos A	R	$[-1,1]$
tan A	R - $\left[(2n+1)\frac{\pi}{2}, n \in Z\right]$	R
cosec A	R - $[n\pi, n \in Z]$	$(-\infty,-1] \cup [1,\infty)$
sec A	R - $\left[(2n+1)\frac{\pi}{2}, n \in Z\right]$	$(-\infty,-1] \cup [1,\infty)$
cot A	R - $[n\pi, n \in Z]$	R

Hence

$|\sin A| \le 1$ i.e. $-1 \le \sin A \le 1$

$|\cos A| \le 1$ i.e. $-1 \le \cos A \le 1$

$\sec A \ge 1$ or $\sec A \le -1$ and $\operatorname{cosec} A \ge 1$ or $\operatorname{cosec} A \le -1$

Problems on Basic Concepts

Illustration 1. If $10\ sin^4x + 15cos^4x = 6$ then calculate value of $27cosec^6x + 8\ sec^6x$.

Sol.: $10\ sin^4x + 15cos^4x = 6$

$10\ sin^4x + 15cos^4x = 6\ (sin^2x + cos^2x)^2$

$10tan^4x + 15 = 6(tan^2x + 1)^2$

$(2\ tan^2x - 3)^2 = 0$ implies $tan^2x = \frac{3}{2}$

$27cosec^6x + 8\ sec^6x = 27(1 + cot^2x)^3 + 9(1 + tan^2x)^3 = 250$

Illustration 2. If $\sin x + \text{cosec}\, x = 2$, calculate value of $sin^{10}x + cosec^{10}x$.

Sol.: $\sin x + \text{cosec}\, x = 2$ Implies $sin^2x + 1 = 2\sin x$

$sin^2x - 2\sin x + 1 = 0$ implies $(\sin x - 1)^2 = 0$

$\sin x = 1$ Hence $sin^{10}x + cosec^{10}x = 1 + 1 = 2$

Problem set 1.

1. If $\sin x + sin^2x + sin^3x = 1$ calculate value of $cos^6x - 4\ cos^4x + 8\ cos^2x$ [Ans.04]
2. If $\sin 2x = \frac{3}{4}$ and value of $sin^3x + cos^3x = \frac{a}{b}$ then calculate value of b [Ans.]
3. If $\frac{sin^4 x}{2} + \frac{cos^4 x}{3} = \frac{1}{5}$ and $\frac{sin^8 x}{8} + \frac{cos^8 x}{27} = \frac{1}{abc}$ then calculate value of (a + b) . c [Ans.15]
4. If $\sin A . \sin B \sin C + \cos A . \cos B = 1$ then calculate value of $23 \sin C$ [Ans.23]
5. For $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$ with $0 < \alpha, \beta < \frac{\pi}{4}$ and $\tan 2\alpha = \frac{a}{b}$ then calculate (a + b). [Ans.89]
6. Calculate $\tan 1^o . \tan 2^o \tan 3^o \dots\dots\dots\dots . \tan 89^o$. [Ans.01]
7. Calculate $\cos 1^o . \cos 2^o \cos 3^o \dots\dots\dots\dots . \cos 179^o$ + 34 [Ans.34]
8. In a triangle ABC, $\sin A - \cos B = \cos C$ then calculate $28 \sin B - 2\cos B$ [Ans.28]
9. If $x \sin\theta = y \sin\left(\frac{2\pi}{3} + \theta\right) = z \sin\left(\frac{4\pi}{3} + \theta\right)$, then calculate value of $(xy + yz + zx)$. [Ans.00]
10. If $3\sin\theta + 4\ \cos\theta = 5$, calculate value of $3\cos\theta - 4\ \sin\theta$ [Ans.00]

Important Result

Maximum and Minimum Values

For trigonometric expression $a\sin x + b\cos x$

$$-\sqrt{a^2 + b^2} \le a\sin x + b\cos x \le \sqrt{a^2 + b^2}$$ (How? Explore)

Hence minimum value of $a\sin x + b\cos x = -\sqrt{a^2 + b^2}$

Maximum value of $a\sin x + b\cos x = \sqrt{a^2 + b^2}$

Illustration 1. If a and b are respectively minimum and maximum values of the expression $6\sin x\cos x + 4\cos 2x$, then calculate the value of (2 a + 7 b)

Sol.: $6\sin x\cos x + 4\cos 2x = 3\sin 2x + 4\cos 2x$ as $\sin 2x = 2\sin x . \cos x$

a = Minimum value of $3\sin 2x + 4\cos 2x = -\sqrt{3^2+4^2} = -5$

b = Maximum value of $3\sin 2x + 4\cos 2x = \sqrt{3^2+4^2} = 5$

Now (2 a + 7 b) = -20 + 35 = 15

Illustration 2. Calculate the maximum value of expression $5\cos\theta + 3\,cos\left(\theta + \frac{\pi}{3}\right) + 37$

Sol.: $5\cos\theta + 3\,cos\left(\theta + \frac{\pi}{3}\right) = 5\cos\theta + 3\cos\theta . \cos\frac{\pi}{3} - 3\sin\theta . \sin\frac{\pi}{3}$

$= \frac{13}{2}\cos\theta + \frac{3\sqrt{3}}{2}\sin\theta$

Maximum value of $\frac{13}{2}\cos\theta + \frac{3\sqrt{3}}{2}\sin\theta = \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = 7$

Hence Maximum value of $5\cos\theta + 3\,cos\left(\theta + \frac{\pi}{3}\right) + 37$ = 7 + 37 = 44

Illustration 3. The maximum and minimum values of $4sin^2x - \cos 2x$ are m and n respectively. Calculate value of $(2a + b^2)$.

Sol.: Let $4sin^2x - \cos 2x = 2(1 - cos2x) - \cos 2x = 2 - 3cos2x$

$-1 \le \cos 2x \le 1$ $-3 \le 3\cos 2x \le 3$ $2 - 3 \le 2 + 3\cos 2x \le 2 + 3$

$-1 \le 2 + 3\cos 2x \le 5$

Hence m = minimum value = -1 and n = maximum value = 5

Hence $(2a + b^2)$ = - 2 + 25 = 23

Problem set 2.

1. The least value of $5\cos x + 3\cos\left(x + \frac{\pi}{3}\right) + 15$ is m. Calculate value of (3 m – 2) [Ans.22]
2. Calculate the difference in maximum and minimum value of the function $4\sin x - 3\cos x + 7$ [Ans.10]
3. The ratio of greatest value of $2 - \cos x + sin^2x$ to its least value is $\frac{a}{b}$. Calculate value of $(b^2 + 3a)$ [Ans.55]
4. Find sum of mini. and max values of expression $cos^2x - 6\sin x\cos x + 3\,sin^2x + 2$ [Ans.08]
5. Calculate the maximum value of the expression $\frac{1}{sin^2\theta + 3\sin\theta\cos\theta + 5cos^2\theta}$ [Ans.02]
6. If maximum value of $cos^2\left(\frac{\pi}{3} - x\right) - cos^2\left(\frac{\pi}{3} + x\right) = \frac{a}{b}$ then calculate $(a^4 + 10\,b)$ [Ans.29]
7. Calculate minimum value of expression $9\,tan^2\theta - 4\,cot^2\theta$ [Ans.12]
8. Calculate maximum value of expression $12\sin\theta - 9\,sin^2\theta$ [Ans.04]
9. If $u = \sqrt{16\,cos^2\theta + 25\,sin^2\theta} + \sqrt{25\,cos^2\theta + 16\,sin^2\theta}$, Calculate maximum value of u^2 [Ans.82]
10. Evaluate $\sqrt{3}cosec\,20^0 - \sec 20^0$ [Ans.04]

TRIGONOMETRIC EQUATIONS AND INEQUATIONS

Solutions of Trigonometric Equations

(i) Principal solution of trigonometric equations lie between 0° and 2π

(ii) General Solution of the trigonometry equations

S. No.	Equation	General Solution
1	*Sin x = 0 or tan x = 0*	$x = n\pi$, $n \in z$
2	*Cos x = 0 or Cot x = 0*	$x = (2n + 1)\pi/2, n \in z$
3	*Sin x = Sin* α	$x = n\pi + (-1)^n \alpha, n \in z$
4	*Cos x = Cos* α	$x = 2n\pi \pm \alpha, n \in z$
5	*tan x = tan* α	$x = n\pi + \alpha, n \in z$
6	$sin^2 x = sin^2\alpha$ $cos^2 x = cos^2\alpha$ $tan^2 x = tan^2\alpha$	$x = n\pi \pm \alpha, n \in z$

Illustration 1. Find the maximum value of "k" for which equation $k\cos x - 3\sin x = k + 1$ is solvable.

Sol.: We know that $|a\sin x + b\cos x| \le \sqrt{a^2 + b^2}$ for all x

Hence $|k + 1| \le \sqrt{k^2 + 9}$ implies $(k + 1)^2 \le k^2 + 9 \implies k \le 4$

Hence Maximum value is 04.

Illustration 2. If 'm' is the number of solutions of the equation $2\,sin^3 x + 2\,cos^3 x - 3\sin 2x + 2 = 0$ in $[0,4\pi]$, then calculate value of "$2\text{ m}^2 + 3$".

Sol.: $2\,sin^3 x + 2\,cos^3 x - 3\sin 2x + 2 = 0$

$\implies$ $sin^3 x + cos^3 x - 3\sin x\cos x + 1 = 0$

$\sin x + \cos x + 1 = 0$ $\quad 2\sin\frac{x}{2}\cos\frac{x}{2} + 2cos^2\frac{x}{2} = 0$ $\quad 2\,cos\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) = 0$

$cos\frac{x}{2} = 0$ $\quad x = \pi, 3\pi$

$\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right) = 0$ $\quad \tan\frac{x}{2} = -1$ $\quad x = \frac{3\pi}{2}, \frac{7\pi}{2}$

Hence number of solutions "m" = 4 Hence $2\text{ m}^2 + 3 = 35$

Problem Set 3

1. Find the integral number of solutions of the equation $1 + \sin x\, sin^2\frac{x}{2} = 0$ in $[-\pi, \pi]$ [Ans. 00]
2. Calculate the integral value of "k" so that the equation $7\cos x + 5\sin x = 2\,k + 1$ has a solution. [Ans.08]
3. If 'm' is the number of solutions of the equation $3\,sin^3 x - 7\sin x + 2 = 0$ in $[0,5\pi]$, then calculate value of "$2\text{ m}^2 + 3\text{ m} + 7$". [Ans.: 97]
4. If $2\,tan^2 x - 5\sec x = 1$ has exactly seven solutions in $\left[0, n\,\frac{\pi}{2}\right]$, then calculate the sum of least and largest value of n. [Ans.: 28]
5. The number of solutions in $[0,2\pi]$ of equation $16^{sin^2 x} + 16^{cos^2 x} = 10$ is [Ans.: 08]

SOLUTIONS PROBLEM SET 1

1. $\sin x + sin^2x + sin^3x = 1 \Longrightarrow \sin x + sin^3x = 1 - sin^2x \Longrightarrow \sin x\,(1 + sin^2x) = cos^2x$
$\sin x\,(2 - cos^2x) = cos^2x \Longrightarrow sin^2x(2 - cos^2x)^2 = cos^4x \Longrightarrow sin^2x(2 - cos^2x)^2 = cos^4x$
$(1 - cos^2x)(4 - 4cos^2x + cos^4x) = cos^4x \Longrightarrow cos^6x - 4\,cos^4x + 8\,cos^2x = 4$
2. $sin^3x + cos^3x = (\sin x + \cos x)\left(1 - \frac{1}{2}\sin 2x\right)$
$$= \sqrt{1 + \sin 2x}\left(1 - \frac{1}{2}\sin 2x\right) = \sqrt{1 + \frac{3}{4}}\left(1 - \frac{3}{8}\right) = \frac{5\sqrt{7}}{16}$$
$\frac{a}{b} = \frac{5\sqrt{7}}{16}$ Hence $b = 16$
3. $\frac{5}{2}sin^4x + \frac{5}{3}cos^4x = 1 \Longrightarrow \left(1 + \frac{3}{2}\right)sin^4x + \left(1 + \frac{2}{3}\right)cos^4x = (sin^2x + cos^2x)^2$
$\left(\frac{\sqrt{3}}{2}sin^2x\right)^2 + \left(\frac{\sqrt{2}}{3}cos^2x\right)^2 - 2sin^2x\,cos^2x = 0 \Longrightarrow \left(\frac{\sqrt{3}}{2}sin^2x - \frac{\sqrt{3}}{2}sin^2x\right)^2 = 0$
$\frac{\sqrt{3}}{2}\,sin^2x = \frac{\sqrt{2}}{3}cos^2x \Longrightarrow tan^2x = \frac{2}{3}$
$\frac{sin^2x}{cos^2x} = \frac{2}{3} \Longrightarrow \frac{sin^2x}{2} = \frac{cos^2x}{3} = \frac{sin^2x+cos^2x}{2+3} = \frac{1}{5} \Longrightarrow sin^2x = \frac{2}{5},\ \ cos^2x = \frac{3}{5}$
$\frac{sin^8x}{8} + \frac{cos^8x}{8} = \frac{2+3}{526} = \frac{1}{125} = \frac{1}{abc} \Longrightarrow (a + b).c = (1 + 2).5 = 15$
4. As $\cos C \le 1$
$\sin A.\sin B\sin C + \cos A\cos B \le \sin A\sin B + \cos A\cos B$
$\sin A.\sin B\sin C + \cos A\cos B \le \cos(A - B)$
$1 \le cos(A - B) \Longrightarrow cos(A - B) = 1 \Longrightarrow A = B$
$\sin A.\sin B\sin C + \cos A\cos B = 1$
$sin^2A\sin C + cos^2A = 1$ Hence $\sin C = 1$ Hence $23\sin C = 23.1 = 23$
5. $\cos(\alpha + \beta) = \frac{4}{5}, \sin(\alpha - \beta) = \frac{5}{13}$
$\tan(\alpha + \beta) = \frac{3}{4}, \tan(\alpha - \beta) = \frac{5}{12}$
$$\tan 2\alpha = tan\big((\alpha + \beta) + (\alpha - \beta)\big) = \frac{tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta).\tan(\alpha - \beta)} = \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4}.\frac{5}{12}} = \frac{56}{33}$$
$\frac{a}{b} = \frac{56}{33}$ Hence $a + b = 89$
6. $tan1.tan2 \ldots\ldots\ldots\ldots.tan45 \ldots\ldots\ldots\ldots\ldots tan89$
$(\tan 1.\tan 89).(\tan 2\tan 88) \ldots\ldots\ldots\ldots.(\tan 44.\tan 46).\tan 45$
$(1).(1) \ldots\ldots\ldots\ldots..(1).1 = 1$
7. $\cos 1.\cos 2.\cos 3 \ldots\ldots\ldots\ldots\ldots \cos 179 + 34$
$= \cos 1.\cos 2.\cos 3 \ldots\ldots\ldots \cos 90 \ldots\ldots\ldots \cos 179 + 34$
$= \cos 1.\cos 2.\cos 3 \ldots\ldots\ldots (0) \ldots\ldots\ldots \cos 179 + 34$
$= 0 + 34 = 34$
8. $\sin A = \cos B + \cos C$ implies $2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\left(\frac{B+C}{2}\right).\cos\left(\frac{B-C}{2}\right)$
$\sin\frac{A}{2}\cos\frac{A}{2} = \cos\left(\frac{\pi}{2} - \frac{A}{2}\right).\cos\left(\frac{B-C}{2}\right)$ implies $\sin\frac{A}{2}\cos\frac{A}{2} = \sin\left(\frac{A}{2}\right).\cos\left(\frac{B-C}{2}\right)$
Implies $A = 0$ In Triangle ABC , Not Possible
Hence $\cos\frac{A}{2} = cos\left(\frac{B-C}{2}\right)$ Implies $A = B - C$ Implies $A + C = B$
Hence $A + B + C = 180$ Implies $B = 90$
Hence $28\sin B - 2\cos B = 28 - 0 = 28$
9. $\frac{\sin\theta}{\frac{1}{x}} = \frac{\sin\left(\theta+\frac{2\pi}{3}\right)}{\frac{1}{y}} = \frac{sin\left(\theta+\frac{4\pi}{3}\right)}{\frac{1}{z}} = \frac{sin\theta+\sin\left(\theta+\frac{2\pi}{3}\right)+sin\left(\theta+\frac{4\pi}{3}\right)}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$
$$= \frac{sin\theta+2sin(\pi+\theta).cos\frac{\pi}{3}}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}$$
$$= \frac{0}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} = 0$$

$sin\theta.\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0.\frac{1}{x}=0$ Implies $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=0$ Implies $\left(\frac{xy+yz+zx}{xyz}\right)=0$

Hence $xy+yz+zx=0$

10. $3sin\theta+4cos\theta=5$ implies $9sin^2\theta+16\,cos^2\theta+24\sin\theta\cos\theta=25$

$9(1-cos^2\theta)+16(1-sin^2\theta)+24\sin\theta.\cos\theta=25$

$9cos^2\theta+16\,sin^2\theta-24\sin\theta\cos\theta=0$ implies $(3cos\theta-4sin\theta\ \ )^2=0$

Hence $3\cos\theta-4\sin\theta=0$

SOLUTIONS PROBLEM SET 2

1. $5\cos x+3\,cos\left(x+\frac{\pi}{3}\right)=5cosx+3\ \left(cosx.cos\frac{\pi}{3}-sin\,xsin\frac{\pi}{3}\right)$

$$=5cosx+\frac{3}{2}cosx-\frac{3\sqrt{3}}{2}sinx=\frac{13}{2}cosx-\frac{3\sqrt{3}}{2}sinx$$

$$-\sqrt{\frac{169}{4}+\frac{27}{4}}\le\frac{13}{2}cosx-\frac{3\sqrt{3}}{2}sinx\le\sqrt{\frac{169}{4}+\frac{27}{4}}$$

$$-7\le\frac{13}{2}cosx-\frac{3\sqrt{3}}{2}sinx\le7$$

$$-7+15\le\frac{13}{2}cosx-\frac{3\sqrt{3}}{2}sinx+15\le7+15$$

$$8\le\frac{13}{2}cosx-\frac{3\sqrt{3}}{2}sinx+15\le22$$

Minimum value = m = 8, 3m – 2 = 22

2. $$-\sqrt{4^2+3^2}\le4sinx-3cosx\le\sqrt{4^2+3^2}$$

$$-5\le4sinx-3cosx\le5$$

$$2\le4sinx-3cosx+7\le12$$

Minimum value = 2 and maximum value = 12

Difference in maximum value and minimum value = 10

3. Let $y=2-\cos x\,.+sin^2x=3-(cosx+cos^2x)=\frac{13}{4}-\left(cosx+\frac{1}{2}\right)^2$

Now $-1\le cosx\le1$ implies $-\frac{1}{2}\le cosx+\frac{1}{2}\le\frac{3}{2}$

Hence $0\le\left(cosx+\frac{1}{2}\right)^2\le\frac{9}{4}$ implies $-\frac{9}{4}\le-\left(cosx+\frac{1}{2}\right)^2\le0$

Hence $1\le\frac{13}{4}-\left(cosx+\frac{1}{2}\right)^2\le\frac{13}{4}$ implies $1\le y\le\frac{13}{4}$

Hence $y_{max}=\frac{13}{4},y_{min}=1$ Hence $\frac{a}{b}=\frac{13}{4}$ Hence $b^2+3a=16+39=55$

4. $cos^2x-6sinx\,cosx+3sin^2x+2$

$=\left(\frac{1+cos2x}{2}\right)-3sin2x+3\left(\frac{1-cos2x}{2}\right)+2=-cos2x-3sin2x+4$

$-\sqrt{1+9}\le-cos2x-3sin2x\le\sqrt{1+9}$ Implies

$-\sqrt{10}+4\le-cos2x-3sin2x+4\le\sqrt{10}+4$

$Minimum\ Value=-\sqrt{10}+4$, Maximum Value = $\sqrt{10}+4$

Hence Sum of maximum value and minimum value = 8

5. $sin^2\theta+3sin\theta cos\theta+5cos^2\theta=\frac{1-cos2\theta}{2}+3sin\theta\,cos\theta+5\left(\frac{1+cos2\theta}{2}\right)$

$$=3+2cos2\theta+\frac{3}{2}sin2\theta$$

$$-\sqrt{4+\frac{9}{4}}\le2cos2\theta+\frac{3}{2}sin2\theta\le\sqrt{4+\frac{9}{4}}$$

$$-\frac{5}{2}+3\le3+2cos2\theta+\frac{3}{2}sin2\theta\le\frac{5}{2}+3$$

$$\frac{1}{2}\le3+2cos2\theta+\frac{3}{2}sin2\theta\le\frac{11}{2}$$

$\frac{2}{11}\le\frac{1}{3+2cos2\theta+\frac{3}{2}sin2\theta}\le2$ Hence Maximum value = 2

6. $cos^2\left(\frac{\pi}{3}-x\right)-cos^2\left(\frac{\pi}{3}+x\right)=\left(cos\frac{\pi}{3}cosx+sin\frac{\pi}{3}sinx\right)^2-\left(cos\frac{\pi}{3}cosx-sin\frac{\pi}{3}sinx\right)^2$

$=4\left(cos\frac{\pi}{3}cosx\right)\left(sin\frac{\pi}{3}sinx\right)=\frac{\sqrt{3}}{2}sin2x$

$$-1 \le sin2x \le 1$$
$$-\frac{\sqrt{3}}{2} \le \frac{\sqrt{3}}{2}sin2x \le \frac{\sqrt{3}}{2}$$
Maximum Value = $\frac{a}{b} = \frac{\sqrt{3}}{2}$
$$a^4 + 10b = 9 + 20 = 29$$

7. $9tan^2\theta - 4cot^2\theta = (3tan\theta - 2cot\theta)^2 + 2.3tan\theta.2cot\theta = (3tan\theta - 2cot\theta)^2 + 12$
   $(3tan\theta - 2cot\theta)^2 \ge 0$ implies $(3tan\theta - 2cot\theta)^2 + 12 \ge 12$
   Hence Minimum value = 12
8. $12sin\theta - 9sin^2\theta = -(2 - 3sin\theta)^2 + 4$
   $-1 \le sin\theta \le 1$ implies $-3 \le -3sin\theta \le 3$ implies $-1 \le 2 - 3sin\theta \le 5$
   Hence $0 \le (2 - 3sin\theta)^2 \le 25$ Implies $-25 \le -(2 - 3sin\theta)^2 \le 0$
   Hence $-21 \le 4 - (2 - 3sin\theta)^2 \le 4$
   Hence Maximum value = 4
9. Let $u = \sqrt{a^2cos^2\theta + b^2sin^2\theta} + \sqrt{a^2sin^2\theta + b^2cos^2\theta}$
   $u^2 = (a^2 + b^2) + 2\sqrt{a^2cos^2\theta + b^2sin^2\theta}\ \sqrt{a^2sin^2\theta + b^2cos^2\theta}$
   $u^2 - (a^2 + b^2) = 2\sqrt{a^2cos^2\theta + b^2sin^2\theta}\ \sqrt{a^2sin^2\theta + b^2cos^2\theta}$
   $[u^2 - (a^2 + b^2)]^2 = 4[(a^4 + b^4)sin^2\theta cos^2\theta + a^2b^2(cos^4\theta sin^4\theta)]$
   $[u^2 - (a^2 + b^2)]^2 = \left((a^4 + b^4 - 2a^2b^2)\right)^2 sin^2 2\theta + 4a^2b^2$
   $u^4 - 2u^2(a^2 + b^2) + (a^2 - b^2)^2 = (a^2 - b^2)^2 sin^2 2\theta$
   $u^4 - 2u^2(a^2 + b^2) + (a^2 - b^2)^2 \le (a^2 - b^2)^2$ as $0 \le sin^2 2\theta \le 1$
   $u^4 - 2u^2(a^2 + b^2) \le 0$
   $u^2[u^2 - 2(a^2 + b^2)] \le 0$ implies $0 \le u^2 \le 2(a^2 + b^2)$
   Hence Maximum value of $u^2 = 2(a^2 + b^2)$
   For $u = \sqrt{16cos^2\theta + 25sin^2\theta} + \sqrt{16\,sin^2\theta + 25\,cos^2\theta}$
   Maximum value of $u^2 = 2.16 + 2.25 = 82$

SOLUTIONS PROBLEM SET 3

1. $1 + sinx\ \ sin^2\frac{x}{2} = 0$
   For $0 \le x \le \pi$, $sinx\ \ sin^2\frac{x}{2} \ge 0$ implies $1 + sinx\ \ sin^2\frac{x}{2} \ge 1$
   Hence $1 + sinx\ \ sin^2\frac{x}{2} = 0$ has no solution for $0 \le x \le \pi$
   For $-\pi \le x < 0$, $-1 < sinx\ \ sin^2\frac{x}{2} \le 0$
   $0 < 1 + sinx\ \ sin^2\frac{x}{2} \le 1$ implies $1 + sinx\ \ sin^2\frac{x}{2} \ne 0$
   Hence $1 + sinx\ \ sin^2\frac{x}{2} = 0$ has no solution for $-\pi \le x < 0$
   Hence number of solutions are 00
2. We know that equation $a\ cosx + b\ sinx = c$ has a solution if $|c| \le \sqrt{a^2 + b^2}$
   Hence $7\cos x + 5\sin x = 2k + 1$ will have solution if $|2k + 1| \le \sqrt{49 + 25} = \sqrt{74}$
   Hence $-\sqrt{74} \le (2k + 1) \le \sqrt{74}$
   For integral values $-8 \le (2k + 1) \le 8$ implies $k = -4, -3, -2, -1, 0, 1, 2, 3$
   Hence $k = 8$
3. $3sin^2x - 7sinx + 2 = 0$ implies $(3sinx - 1)(sinx - 2) = 0$
   $sinx = \frac{1}{3}$ as $sinx \ne 2$ Implies
   $x = sin^{-1}\left(\frac{1}{3}\right), \pi - sin^{-1}\left(\frac{1}{3}\right), 2\pi + sin^{-1}\left(\frac{1}{3}\right), 3\pi - sin^{-1}\left(\frac{1}{3}\right), 4\pi + sin^{-1}\left(\frac{1}{3}\right), 5\pi - sin^{-1}\left(\frac{1}{3}\right)$
   $m = 6$ Hence $2\,m^2 + 3m + 7 = 72 + 18 + 7 = 97$
4. $2\,tan^2x - 5secx = 1$ implies $2\,sec^2x - 5\,sec\,x - 3 = 0$
   $(2\sec x + 1).(secx - 3) = 0$
   As $|\sec x| \ge 1$ Hence $2\sec x + 1 \ne 0$
   Hence $\sec x = 3$
   Solutions are points of intersection of curves $y = \sec x$ and $y = 3$

The 7th solution is between $\left[0, \frac{13\pi}{2}\right]$ and $\left[0, \frac{15\pi}{2}\right]$

Hence sum of values of max. and min. values = 28

5. $16^{sin^2x} + \frac{16}{16^{sin^2x}} = 10$ implies $y^2 - 10y + 16 = 0$ for $y = 16^{sin^2x}$

$y = 2,8$ hence $y = 16^{sin^2x} = 2, 8$

$2^{4sin^2x} = 2$ and $2^{4sin^2x} = 8 = 2^3$

Hence $4sin^2x = 1$ and $4sin^2x = 3$

$sin^2x = sin^2\frac{\pi}{6}$ and $sin^2x = sin^2\frac{\pi}{3}$

$x = n\pi \pm \frac{\pi}{6}$ and $x = n\pi \pm \frac{\pi}{3}$

Hence $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Hence Number of solutions = 08

- There are 30 questions in this question paper. Each question carries 5 marks.
- Answer all questions.
- Time allotted: 2.5 hours.

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1. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3 ?

Ans. 28

Sol. coefficient of x^{21} : $(x^0 + x^1 + x^2 + ... + x^9)^3$

$: \dfrac{(1-x^{10})^3}{(1-x)^3}$

$(1 - 3x^{10} + 3x^{20} + ...)(1 - x)^{-3}$

$:\ {}^{3+21-1}C_{21} - 3.{}^{3+11-1}C_{11} + 3.3$

$= {}^{23}C_{21} - 3.{}^{13}C_{11} + 9 = \dfrac{23 \times 22}{2} - \dfrac{3 \times 13 \times 12}{2} + 9$

$= 23 \times 11 - 39 \times 6 + 9$

$= 253 - 234 + 9 = 28$ Ans.

2. Suppose a, b are positive real numbers such that $a\sqrt{a} + b\sqrt{b} = 183$, $a\sqrt{b} + b\sqrt{a} = 182$. Find $\dfrac{9}{5}(a+b)$.

Ans. 73

Sol. $a\sqrt{a} + b\sqrt{b} = 183$ and $a\sqrt{b} + b\sqrt{a} = 182$ we have to find $\dfrac{9}{5}(a+b)$

Let $a = A^2$, $b = B^2$

$A^3 + B^3 = 183$ ....(i)

$A^2B + B^2A = 182$..........(ii)

(1) + (2)

$A^3 + B^3 + 3AB(A + B) = 183 + 3 \times 182$

$(A + B)^3 = 183 + 546$

$(A + B)^3 = 729$

$\Rightarrow A + B = 9$

from (2) $\rightarrow AB(A + B) = 182$

$AB = \dfrac{182}{9}$

$a + b = A^2 + B^2 = (A + B)^2 - 2AB$

$= 81 - \dfrac{364}{9} = \dfrac{365}{9}$

$\dfrac{9}{5} \times \dfrac{365}{9} = 73$

3. A contractor has two teams of workers : team A and team B. Team A can complete a job in 12 days and team B can do the same job in 36 days. Team A starts working on the job and team B joins team A after four days. The team A withdraws after two more days. For how many more days should team B work to complete the job ?

Ans. 16

Sol. Let total worker in team A be x

Let total worker in team B be y

Given 12 x = 36 y

x = 3y

Per worker work done per day be W.

total work 36 × W × y

36Wy = 6 × Wx + MWy

M = 2 + Number of more days.

36 y = 18 y + My

M = 18y + My

M = 18

Hence 16 more days

4. Let a, b be integers such that all the roots of the equation $(x^2 + ax + 20)(x^2 + 17x + b) = 0$ are negative integers. What is the smallest possible value of a + b ?

Ans. 25

Sol. $(x^2 + ax + 20)(x^2 + 17x + b) = 0$

$z^- \quad z^- \qquad z^- \quad z^-$

so a > 0 and b > 0 since sum of roots < 0 and product > 0

(since 20 = (1 × 20) × (2 × 10) or (4 × 5)

min a = 9

$-17 = \alpha + \beta \Rightarrow (\alpha\beta) \equiv (-1, -16), (-2, -15), (-8, -9)$

min b = 16

$(a+b)_{min} = a_{min} + b_{min} = 9 + 16 = 25$

5. Let u, v, w be real numbers in geometric progression such that u > v > w. Suppose $u^{40} = v^n = w^{60}$. Find the value of n.

Ans. 48

Sol. u = a

v = ar

$w = ar^2$

$a^{40} = (ar)^n = (ar^2)^{60}$

$a^{20} = r^{-120} \quad \Rightarrow \quad a = r^{-6}$

$r^{-240} = r^{-5n} \quad \Rightarrow \quad 5n = 240$

$n = \frac{240}{5} = 48$

6. Let the sum $\sum_{n=1}^{9}\frac{1}{n(n+1)(n+2)}$ written in its lowest terms be $\frac{p}{q}$. Find the value of q – p.

Ans. 83

Sol. $\sum_{n=1}^{9}\frac{1}{2}\ \frac{n+2-n}{n(n+1)(n+2)}$

$= \frac{1}{2}\sum_{n=1}^{9}\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right)$

$= \frac{1}{2}\left(\left(\frac{1}{1.2}-\frac{1}{2.3}\right)+\left(\frac{1}{2.3}-\frac{1}{3.4}\right)+.........+\left(\frac{1}{9\times 10}-\frac{1}{10\times 11}\right)\right)$

$= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{110}\right)$

$= \frac{1}{2}\left(\frac{55-1}{110}\right) \Rightarrow \frac{27}{110} \Rightarrow$ q – p = 110 – 27 = 83

7. Find the number of positive integers n such that $\sqrt{n}+\sqrt{n+1}<11$.

Ans. 29

Sol. $\sqrt{N}$ + $\sqrt{N+1}$ < 11

N = 1, 2, 3, 4, 5, 6, 7,8 .... 16...... 25

$\sqrt{N+1}$ + $\sqrt{N}$ = 11 .... (1)

$\frac{1}{\sqrt{N+1}-\sqrt{N}} = 11$

$\sqrt{N+1} - \sqrt{N} = \frac{1}{11}$ .... (2)

$2\sqrt{N+1} = \frac{122}{11} \Rightarrow \sqrt{N+1} = \frac{61}{11} \Rightarrow N + 1 = \frac{3721}{121}$

N = $\frac{3600}{121}$ = 29.75

So 29 values

8. A pen costs ₹ 11 and a notebook costs ₹13. Find the number of ways in which a person can spend exactly ₹ 1000 to buy pens and notebooks.

Ans. 7

Sol. Cost of a pen is ₹11

Cost of a notebook is ₹13

11x + 13y = 1000

$y = \frac{1000-11x}{13} \in I$

$\frac{12-11x}{13} \in I \Rightarrow \frac{(13-1)-(13x-2x)}{13} \in I$

$\frac{2x-1}{13} \in I \Rightarrow \frac{12x-6}{13} \in I$

$\frac{13x-(x+6)}{13} \in I \Rightarrow \frac{x+6}{13} \in I$

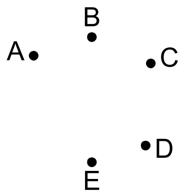
$x = 13\lambda - 6 \quad (1 \le x \le 90)$

$x \in \{7,20,.......,85\}$ $\{\lambda = \{1,2,....,3\}\}$

9. There are five cities A,B,C,D,E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once ? (The order in which he visits the cities also matters. e.g., the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.)

Ans. 60

Sol.



ways = ${}^4C_2 2! + {}^4C_3 3! + {}^4C_4 4!$

= 12 + 24 + 24 = 12 + 48 = 60

10. There are eight rooms on the first floor of a hotel, with four rooms on each side of the corridor, symmetrically situated (that is each room is exactly opposite to one other room). Four guests have to be accommodated in four of the eight rooms (that is, one in each) such that no two guests are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated?

Ans. 48

Sol. ✓ __ ✓ __

__ ✓ __ ✓

2× 4! = 48

11. Let f(x) = $\sin\frac{x}{3} + \cos\frac{3x}{10}$ for all real x. Find the least natural number n such that $f(n\pi + x) = f(x)$ for all real x.

Ans. 60

Sol. $f(x) = \sin\frac{x}{3} + \cos\frac{3x}{10}$

period of $\sin\frac{x}{3}$ is $\frac{6\pi}{1}$ and period of $\cos\frac{3x}{10}$ is $\frac{2\pi}{3/10} = \frac{20\pi}{3}$

LCM is $\frac{60\pi}{1}$ $\Rightarrow$ n = 60

12. In a class, the total numbers of boys and girls are in the ratio 4 : 3. On one day it was found that 8 boys and 14 girls were absent from the class and that the number of boys was the square of the number of girls. What is the total number of students in the class ?

Ans. 42

Sol. Ratio is 4 : 3 therefore

Boys are 4x

Girls are 3x

given $(4x - 8) = (3x - 14)^2$

$9x^2 + 196 - 84x = 4x - 8$

$9x^2 - 88x + 204 = 0$

$x = \frac{88 \pm \sqrt{88^2 - 4 \times 9 \times 204}}{18}$

$= \frac{2(22 \pm \sqrt{22^2 - 9 \times 51})}{9}$

$= \frac{2(22 \pm 5)}{9} = 2 \times \frac{27}{9}$ or $2 \times \frac{17}{9}$

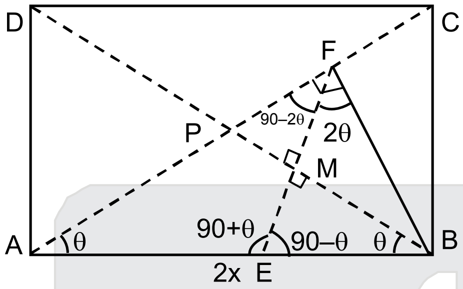
x = 6 or $\frac{34}{9}$

7x = 42 or non integer $\Rightarrow$ 42 students

13. In a rectangle ABCD, E is the midpoint of AB : F is point on AC such that BF is perpendicular to AC and FE perpendicular to BD. Suppose BC = $8\sqrt{3}$. Find AB.

Ans. 24

Sol. $\sin\theta = \frac{8\sqrt{3}}{2x} = \frac{4\sqrt{3}}{x}$



FA = 2x $\cos\theta$

FB = 2x $\sin\theta$

MB = FBx sin 2θ = 2x $\sin\theta \sin 2\theta$

$ME^2 + MB^2 = BE^2 = x^2$

$x^2\sin^2\theta + 4x^2 \sin^2\theta \sin^2 2\theta = x^2$

$\sin^2\theta + 4\sin^2\theta \sin^2 2\theta = 1$

$4\sin^2\theta \sin^2 2\theta = \cos^2\theta$

$16 \sin^4\theta = 1$

$\sin^2\theta = \frac{1}{4} \quad \Rightarrow \quad \sin\theta = \frac{1}{2}$

$\frac{4\sqrt{3}}{x} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad x = 12 \Rightarrow 2x = AB = 24$ Ans.

14. Suppose x is a positive real number such that {x}.[x] and x are in the geometric progression. Find the least positive integer n such that $x^n > 100$. (Here [x] denotes the integer part of x and {x} = x – [x])

Ans. 10

Sol. $[x]^2 = x\{x\}$

$\{x\} = a$

$[x] = ar$

$x = ar^2$

$a + ar = ar^2$

$r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{5}}{2} \Rightarrow r = \frac{1+\sqrt{5}}{2}$

$ar = I$

$a = \frac{2I}{(1+\sqrt{5})}$

$a = \frac{I(\sqrt{5}-1)}{2}$

$0 < a < 1$

$0 < \frac{I(\sqrt{5}-1)}{2} < 1$

$0 < I < \frac{2}{\sqrt{5}-1}$

$0 < I < \frac{\sqrt{5}+1}{2}$

$I = 1$

$ar = 1 \Rightarrow a = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}$

$x = ar^2 = r = \frac{\sqrt{5}+1}{2}$

$\left(\frac{\sqrt{5}+1}{2}\right)^N > 100 \Rightarrow N \log_{10}\left(\frac{\sqrt{5}+1}{2}\right) > 2$

$N > 9.5 \Rightarrow N_{min} = 10$

15. Integers 1,2,3……..n where n > 2, are written on a board. Two numbers m, k such that 1 < m < n. 1 < k < n are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers ?

Ans. 51

Sol. $\frac{\frac{n(n+1)}{2} - (2n-1)}{n-2} < 17 < \frac{\frac{n(n+1)}{2} - 3}{n-2}$

$\frac{n^2 + n - 4n + 2}{2(n-2)} < 17 < \frac{n^2 + n - 6}{2(n-2)}$

$\frac{n^2 + 3n + 2}{2(n-2)} < 17 < \frac{(n+3)(n-2)}{2(n-2)}$

$\frac{n-1}{2} < 17 < \frac{n+3}{2}$

n < 35 and n > 31

n = 32, 33, 34

case-1, n = 32

$\frac{\frac{n(n+1)}{2} - p}{(n-2)} = 17 \quad \Rightarrow \quad \frac{n(n+1)}{2} - 17\,(n-2) = p$

p = 18

case-2, n = 33 $\Rightarrow$ p = 34

case-3, n = 34 $\Rightarrow$ p = 51

Maximum sum = 51

16. Five distinct 2-digit numbers are in a geometric progression. Find the middle term.

Ans. 36

Sol. Let the numbers be a ar ar^2 ar^3 ar^4

since all are 2 digit number $r = \frac{2}{3}$ or $\frac{3}{2}$ (as fourth power of integers greater than 3 are 3 digit numbers)

Hence the five numbers are (16, 24, 36, 54, 81)

Hence middle term is 36.

17. Suppose the altitudes of a triangle are 10,12 and 15. What is its semi-perimeter ?

Ans. Bonus

Sol. $h_a : h_b : h_c = 10 : 12 : 15$

$a, b, c = \frac{1}{10} : \frac{1}{12} : \frac{1}{15}$

$a : b : c = 6 : 5 : 4$

$(a,b,c) = (6k, 5k, 4k)$

$2s = 15k$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = \sqrt{\frac{15k}{2}\left(\frac{15k}{2}-6k\right)\left(\frac{15k}{2}-5k\right)\left(\frac{15k}{2}-4k\right)}$

$\Delta = \sqrt{\frac{15k}{2}.\frac{3k}{2}.\frac{5k}{2}.\frac{7k}{2}}$

$\Delta = \frac{1}{4}\sqrt{15^2.7k^2} = \frac{15}{4}\sqrt{7}k^2$

$h_a = 10 = \frac{2\times\frac{15}{4}\sqrt{7}k^2}{6k}$

$60k = \frac{15}{2}\sqrt{7}k^2$

$8 = \sqrt{7}k \Rightarrow k = \frac{8}{\sqrt{7}}$

$s = \frac{15}{2}\times\frac{8}{\sqrt{7}} = \frac{60}{\sqrt{7}}$

18. If the real numbers x, y, z are such that $x^2 + 4y^2 + 16z^2 = 48$ and $xy + 4yz + 2zx = 24$. what is the value of $x^2 + y^2 + z^2$?

Ans. 21

Sol. $x^2 + 4y^2 + 16z^2 = 48$

$(x)^2 + (2y)^2 + (4z)^2 = 48$

$2xy + 8yz + 4zx = 48$

now we can say that

$(x)^2 + (2y)^2 + (4z)^2 - (2xy) - (8yz) - (4zx) = 0$

$[(x - 2y)^2 + (2y - 4z)^2 + (x - 4y)^2)] = 0$

$x = 2y = 4z \Rightarrow \frac{x}{4} = \frac{y}{2} = z$

$(x, y, z) = (4\lambda, 2\lambda, \lambda)$

$x^2 + 4y^2 + 16z^2 = 48$

$16\lambda^2 + 16\lambda^2 + 16\lambda^2 = 48$

so $\lambda^2 = 1$

$x^2 + y^2 + z^2 = 21\lambda^2 = 21$

19. Suppose 1,2,3 are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of c.

Ans. 36

Sol. 1,2,3 are roots of $x^4 + ax^2 + bx - c = 0$

since sum of roots is zero and fourth root is – 6

Hence c = 36

20. What is the number of triples (a, b, c) of positive integers such that (i) $a < b < c < 10$ and (ii) a, b, c, 10 form the sides of a quadrilateral?

Ans. 73

Sol. $a + b + c > 10$

$\therefore$ (a,b,c) can be

a	b	c
1	2	8,9
1	3	7,8,9
1	4	6,7,8,9
1	5	6,7,8,9
1	6	7,8,9
1	7	8,9
1	8	9
2	3	6,7,8,9
2	4	5,6,7,8,9
2	5	6,7,8,9
2	6	7,8,9
2	7	8,9
2	8	9
3	4	5,6,7,8,9
3	5	6,7,8,9
3	6	7,8,9
3	7	8,9
3	8	9
4	5	6,7,8,9
4	6	7,8,9
4	7	8,9
4	8	9
5	6	7,8,9
5	7	8,9
5	8	9
6	7	8,9
6	8	9
7	8	9

Total 73 cases

21. Find the number of ordered triples (a, b, c) of positive integers such that abc = 108.

Ans. 60

Sol. $abc = 3^3.2^2$

$a = 3^{\alpha_1}.2^{\beta_1}$, $b = 3^{\alpha_2}.2^{\beta_2}$, $c = 3^{\alpha_3}.2^{\beta_3}$

$\alpha_1 + \alpha_2 + \alpha_3 = 3$ and $\beta_1 + \beta_2 + \beta_3 = 3$

${}^5C_2$ and ${}^4C_2$

Total = ${}^4C_2 \times {}^5C_2 = 10 \times 6 = 60$

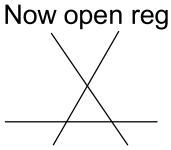
22. Suppose in the plane 10 pair wise nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed?

Ans. 46

Sol. Number of non-overlapping polygons = 56 – 20 = 46

1 line divide plane into 2 regions
2 lines divide plane into 4 regions
3 lines divide plane into 7 regions
4 lines divide plane into 11 regions
5 lines divide plane into 16 regions
6 lines divide plane into 22 regions
7 lines divide plane into 29 regions
8 lines divide plane into 37 regions
9 lines divide plane into 46 regions

10 lines divide plane into 56 regions
Now open regions for 3 lines are 6



Similarly for 10 lines are 20
Note : If we consider overlapping polygons then maximum possible number of polygons = ${}^{10}C_3 + {}^{10}C_4 + ... + {}^{10}C_{10} = 2^{10} - 1 - 10 - 45 = 968$

23. Suppose an integer r, a natural number n and a prime number p satisfy the equation $7x^2 - 44x + 12 = p^n$. Find the largest value of p.

Ans. 47

Sol. $7x^2 - 44x + 12 = p^n$

$7x^2 - 42x - 2x + 12 = p^n$

$(7x - 2)(x - 6) = p^n$

$7x - 2 = p^\alpha$ and $x - 6 = p^\beta$

$(7x - 2) - 7(x - 6) = p^\alpha - 7p^\beta$

$40 = p^\alpha - 7p^\beta$

If $\alpha,\beta \in N$, p is divisors of 40 $\Rightarrow p = 2$ or 5

If $p = 2$, $40 = 2^\alpha - 7.2^\beta$ $\Rightarrow 2^3.5 = 2^\alpha - 7.2^\beta$

$\Rightarrow \beta = 3$ and $2^\alpha = 40 + 56$ $\Rightarrow \alpha \notin Z$ hence not possible

If $p = 5$ then $40 = 5^\alpha - 7.5^\beta$ $\Rightarrow 2^3.5 = 5^\alpha - 7.5^\beta$

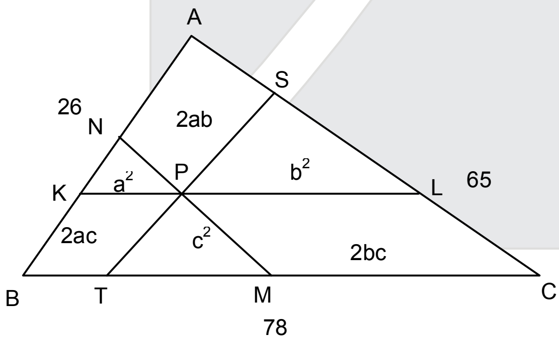
$\Rightarrow \beta = 1$ and $5^\alpha = 40 + 35$ $\Rightarrow \alpha \notin Z$ hence not possible

so $\beta = 0 \Rightarrow p^\alpha = 47$ $\Rightarrow p = 47$ and $\alpha = 1$

24. Let P be an interior point of a triangle ABC whose side lengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N. The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length.

Ans. Bonus

Sol.



Let MN = ST = KL = ℓ

$\frac{\ell}{26} = \sqrt{\frac{(b+c)^2}{(a+b+c)^2}}$

$\frac{\ell}{26} = \frac{b+c}{a+b+c}$

$\frac{\ell}{65} = \frac{a+c}{a+b+c}$

$\frac{\ell}{78} = \frac{\ell}{26} = 2$

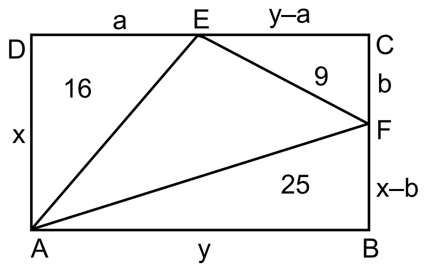
$\frac{\ell}{26} + \frac{\ell}{65} + \frac{\ell}{78} = 2$

$\ell = 30$ which is not possible as ℓ has to be less than 26

25. Let ABCD be a rectangle and let E and F be points on CD and BC respectively such that area (ADE) = 16, area (CEF) = 9 and area (ABF) = 25. What is the area of triangle AEF ?

Ans. 30

Sol.



$xa = 32 \quad \Rightarrow \quad xa = 32$

$b(y - a) = 18 \quad \Rightarrow \quad by - ab = 18$

$y(x - b) = 50 \quad \Rightarrow \quad xy - by = 50$

$by - \dfrac{32b}{x} = 18 \Rightarrow \quad b = \dfrac{18b}{xy - 32}$

$xy - \dfrac{18xy}{xy - 32} = 50$

$xy = t$

$$\frac{t^2 - 32t - 18t}{t - 32} = 50$$

$t^2 - 50t = 50t - 1600$

$t^2 - 100t + 1600 = 0$

$t = 80 , 20$

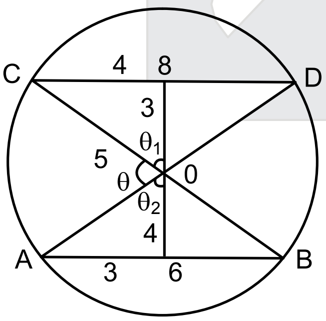
Now $xy = 80$

Area of $\Delta AEF = 80 - (16 + 9 + 25) = 30$

26. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lies between these chords. Suppose AB = 6, CD = 8. Suppose further that the area of the part of the circle lying between the chords AB and CD is $(m\pi + n) / k$, where m, n, k are positive integers with gcd(m, n, k) = 1. What is the value of m + n + k ?

Ans. 75

Sol.



$A = 2\left[\frac{1}{2} \times 25 \times \theta\right] + \frac{1}{2} \times 3 \times 8 + \frac{1}{2} \times 4 \times 6$

$\theta = [\pi - (\theta_1 + \theta_2)] = \left[\pi - \left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{3}{4}\right)\right]$

$\theta = \dfrac{\pi}{2}$

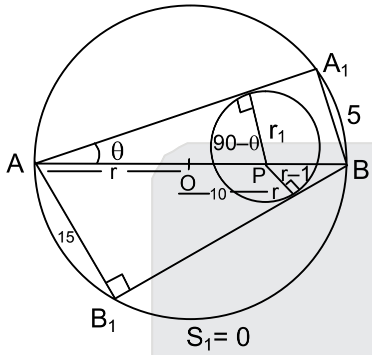
$A = 24 + \dfrac{25\pi}{2} \quad \Rightarrow \quad A = \dfrac{48 + 25\pi}{2}$

$(m + n + k) = (48 + 2 + 25) = 75$

27. Let Ω_1 be a circle with centre O and let AB be diameter of Ω_1. Let P be a point on the segment OB different from O. Suppose another circle Ω_2 with centre P lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1 respectively such that A_1 and B_1 are on the opposite sides of AB. Given that $A_1B = 5$, $AB_1 = 15$ and $OP = 10$, find the radius of Ω_1.

Ans. 20

Sol.



$$\frac{r_1}{r+10} = \frac{5}{2r} \quad ....(1)$$

$$\frac{r_1}{r-10} = \frac{15}{2r} \quad ....(2)$$

$$\frac{r-10}{r+10} = \frac{1}{3}$$

$3r - 30 = r + 10$

$2r = 40$

$r = 20$

28. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of 3pq for all positive integers n. Find the least possible value of p + q.

Ans. 28

29. For each positive integer n, consider the highest common factor h_n of the two numbers n! + 1 and (n + 1)!. For n < 100, find the largest value of h_n.

Ans. 97

Sol. n! + 1 is not divisible by 1,2,....,n

(n + 1)! is divisible by 1,2,....,n

so HCF $\geq$ n + 1

also (n + 1)! is not divisible by n +2, n + 3.....

so HCF can be n + 1 only

Let us start by taking n = 99

$\Rightarrow$ 99! + 1 and 100!

HCF = 100 is not possible as 100 divides 99!

composite number will not be able to make it

so let us take prime i.e. n = 97

now 96! + 1 and 97! are both divisible by 97

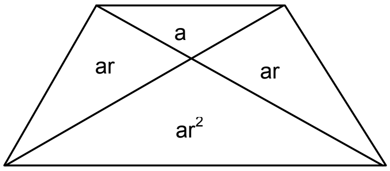
so HCF = 97

(by Wilson's theorem (p – 1)! + 1) is divisible by p)

30. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium ABCD. The product of these areas, taken two at time, are computed. If among the six products so obtained, two product are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.

Ans. 13

Sol.



Case-1 $a^2r^2 = 1296$

$ar^2 = 576$

$$r = \left(\frac{36}{24}\right)^2 = \frac{9}{4}$$

$a^2 = 24 \times 24 \times \frac{4}{9} \Rightarrow a = 24 \times \frac{2}{3}$

$a = 16$

Case-2 $a^2r = 576$

$a^2r^3 = 1296$

$$r^2 = \frac{1296}{576}$$

$r^2 = \left(\frac{3}{2}\right)^2 \Rightarrow r = \frac{3}{2}$

$a^2 = 576 \times \frac{2}{3}$

$a^2 = 192 \times 2$

$a^2 = 384$

Case-3 $a^2r^3 = 1296$

$a^2r^2 = 576$

$$r^2 = \frac{1296}{576}$$

$r = \frac{9}{4}$

$a^2 = \frac{576 \times 16}{81}$

$a = \frac{32}{3}$

area $= a(r + 1)^2$

Case-1 : area $= 16\left(1+\frac{9}{4}\right)^2 = 169 \Rightarrow$ square root is 13

Case-2 : area $= \left(1+\frac{3}{2}\right)^2 = 122.47$

Case-3 : area $= \frac{32}{3}\left(1+\frac{9}{4}\right)^2$ so maximum area is 13

INSTRUCTION

Number of Questions : 30 **Max. Marks : 102**

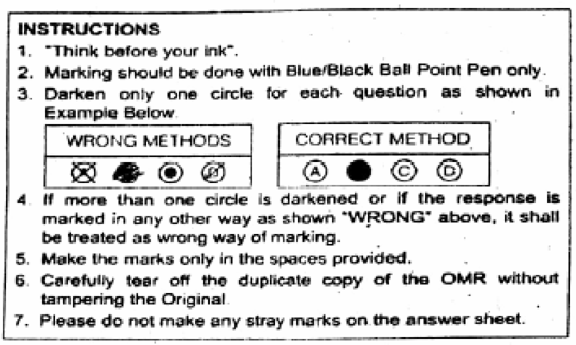
1. Use of mobile phones, smartphones, ipads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

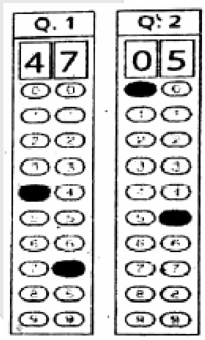
2. The cirrection is done by machine through scanning. On OMR sheet, darken bubbles completely with a black pencil or a black blue pen. Darken the bubbles completely only after you are sure of your answer : else, erasing lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name , email addredd.a and date of birth entered on the OMR sheet will be your login credentials for accessing your PROME score.

4. Incomplete /incorrectly and craelessly filled information may disqualify your candidature.

5. Each quaestion has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubble with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.





6. The answer you write on OMR sheet is irrelevant. The darken bubble will be considered as your final answer.

7. Questions 1 to 6 carry 2 marks each : Questions 7 to 21 carry 3 marks each : Questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it further for verification purposes.

13. You may take away the question paper after the examination.

1. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n ?

एक पुस्तक तीन खण्डों में प्रकाशित है, और कुल पृष्ठ संख्या 1 से शुरू होती है। दूसरे खंड की पृष्ठ संख्या पहले खंड के आगे से शुरू होती है व तीसरे की दूसरे के आगे से। दूसरे खंड में पहले खंड से 50 अधिक पृष्ठ हैं, व तीसरे खंड में दूसरे से डेढ़ गुना पृष्ठ हैं। तीनों खंडों के पृथम पृष्ठ की पृष्ठ संख्या का योग 1709 है। अगर n अंतिम पृष्ठ संख्या है, तो n को विभाजित करने वाली सबसे बड़ी अभाज्य संख्या कौनसी है ?

Sol. **(17)**

Let the number of pages in volume-1 be x

$\therefore$ Number of pages in second volume = x + 50

$\therefore$ Number of pages in third volume = $\frac{3}{2}(x + 50)$

Moreover $1 + (x + 1) + (2x + 51) = 1709$

$\Rightarrow 3x + 53 = 1709 = \Rightarrow x = 552$

So $n = 552 + 602 + 903 = 2057$

So $n = 11^2 \times 17$

Hence largest prime factor of n = 17

Hindi : माना भाग-1 में पृष्ठों की संख्या x है।

$\therefore$ भाग दो में पृष्ठों की संख्या = x + 50

$\therefore$ भाग तीन में पृष्ठों की संख्या = $\frac{3}{2}(x + 50)$

तथा $1 + (x + 1) + (2x + 51) = 1709$

$\Rightarrow 3x + 53 = 1709 = \Rightarrow x = 552$

इसलिए $n = 552 + 602 + 903 = 2057$

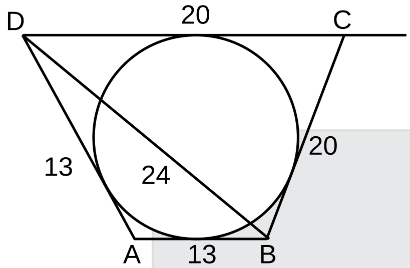
इसलिए $n = 11^2 \times 17$

अतः n का अधिकतम अभाज्य गुणनखण्ड = 17

2. In a quadrilateral ABCD, it is given that AB = AD = 13, BC = CD = 20, BD = 24. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r ?

एक चतुर्भुज ABCD में ये दिया हुआ है कि AB = AD = 13, BC = CD = 20, BD = 24 है। यदि r इस चतुर्भुज के अंदर बनाए जा सकन वाले अन्तः वृत्त की त्रिज्या है, तो r से निकटतम पूर्णांक का मान क्या होगा ?

Sol. **(8)**



Area of ABCD = Area of ΔABD + Area of ΔBCD

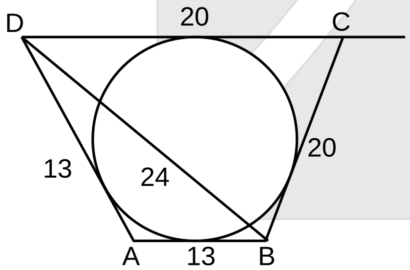
$= \sqrt{25\times12\times12\times1}+\sqrt{32\times8\times12\times12}$

= 60 + 192 = 252

Indadius(r) $= \frac{\text{Area}}{\text{semi}-\text{perimeter}} = \frac{252}{33} = \frac{84}{11} = 7.\overline{64}$

Hence integer nearest to r is 8

Hindi :



ABCD का क्षेत्रफल = ΔABD का क्षेत्रफल + ΔBCD का क्षेत्रफल

$= \sqrt{25\times12\times12\times1}+\sqrt{32\times8\times12\times12}$

= 60 + 192 = 252

अन्तः त्रिज्या (r) $= \frac{\text{क्षेत्रफल}}{\text{अर्द्ध परिमाप}} = \frac{252}{33} = \frac{84}{11} = 7.\overline{64}$

अतः r के निकटतम पूर्णांक 8 है।

3. Consider all 6-digit numbers of the form **abccba** where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7.

ऐसी 6-अंकों की संख्या **abccba** के बारे में सोचो जिनमें b विषम है। ऐसी कितनी 6-अंकों की संख्याएं होंगी जो कि 7 से विभाजित हो जाती हैं ?

Sol. **(70)**

abccba (b is odd)

$= a(10^5 + 1) + b(10^4 + 10) + c(10^3 + 10^2)$

$= a\,(1001 - 1)\,100 + a + 10b(1001) + (100)\,(11)\,c$

$= (7.11.13.100)a - 99a + 10b(7.11.13) + (98 + 2)(11)c$

$= 7p + (c - a)$ where p is an integer

Now if $c - a$ is a multiple of 7

$c - a = 7, 0, -7$

Hence number of ordered pairs of (a, c) is 14

since b is odd

Number of such number $= 14 \times 5 = 70$

Hindi : abccba (b विषम है)

$= a(10^5 + 1) + b(10^4 + 10) + c(10^3 + 10^2)$

$= a\,(1001 - 1)\,100 + a + 10b(1001) + (100)\,(11)\,c$

$= (7.11.13.100)a - 99a + 10b(7.11.13) + (98 + 2)(11)c$

$= 7p + (c - a)$ जहाँ p एक पुर्णांक है

अब यदि $c - a$, 7 का गुणज है

$c - a = 7, 0, -7$

अतः (a, c) के क्रमित युग्मों की संख्या = 14

चूकिं b विषम है।

इस प्रकार की संख्या $= 14 \times 5 = 70$

4. The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is 1,6,5,8,9,0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.

समीकरण $166 \times 56 = 8590$ किसी आधार (base) $b \geq 10$ में सही है (मतलब कि 1,6,5,8,9,0 आधार (base) b में अंक हैं) ऐसी सभी सम्भव संख्याओं $b \geq 10$ का योग क्या होगा ?

Sol. **(12)**

$166 = b^2 + 6b + 6$

$56 = 5b + 6$

$8590 = 8b^3 + 5b^2 + 9b$

Now $(b^2 + 6b + 6)(5b + 6) = 8b^3 + 5b^2 + 9b$

$\Rightarrow \quad 5b^3 + 36b^2 + 66b + 36 = 8b^3 + 5b^2 + 9b$

$\Rightarrow \quad 3b^3 - 31b^2 - 57b - 36 = 0$

$\Rightarrow \quad (b - 12)(3b^2 + 5b + 3) = 0$

$\Rightarrow \quad b = 12$

We have only one b which is 12

So sum = 12

Hindi : $166 = b^2 + 6b + 6$

$56 = 5b + 6$

$8590 = 8b^3 + 5b^2 + 9b$

अब $(b^2 + 6b + 6)(5b + 6) = 8b^3 + 5b^2 + 9b$

$\Rightarrow \quad 5b^3 + 36b^2 + 66b + 36 = 8b^3 + 5b^2 + 9b$

$\Rightarrow \quad 3b^3 - 31b^2 - 57b - 36 = 0$

$\Rightarrow \quad (b - 12)(3b^2 + 5b + 3) = 0$

$\Rightarrow \quad b = 12$

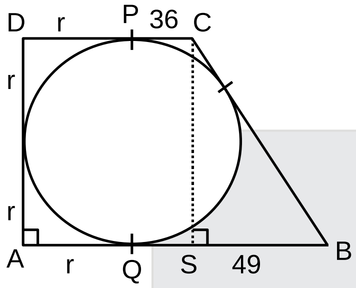
b का केवल एक मान 12 है।

अतः योगफल = 12

5. Let ABCD be a trapezium in which AB || CD and AD $\perp$ AB. Suppose ABCD has an incircle which touches AB at Q and CD at P. Given that PC = 36 and QB = 49, Find PQ.

ABCD एक समलंब चतुर्भुज है जिसमें कि AB || CD व AD $\perp$ AB मान लो कि इस चतुर्भुज का एक अंतः वृत्त AB से Q में व CD से P में मिलता है। अंगर PC = 36 व QB = 49 तो PQ का मान क्या होगा ?

Sol. (84)



Let incircle touch BC at R

So CR = 36, BR = 49

Further let inradius = r

So AQ = PD = r & AD = 2r

Let perpendicular from C meet AD at S

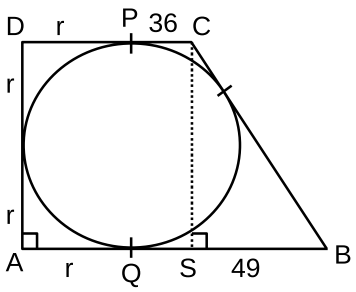
So BS = 13, BC = 85

Now $(CS)^2 = 85^2 - 13^2 = 98 \times 72 = 49 \times 144$

So CS = 7 × 12 = 84

Hence PQ = 84

Hindi :



माना अर्द्धवृत्त BC को R पर स्पर्श करता है।

इसलिए CR = 36, BR = 49

पुनः माना अन्तः त्रिज्या = r

इसलिए AQ = PD = r & AD = 2r

माना C से लम्ब AD को S पर मिलता है।

इसलिए BS = 13, BC = 85

अब $(CS)^2 = 85^2 - 13^2 = 98 \times 72 = 49 \times 144$

So CS = 7 × 12 = 84

अतः PQ = 84

6. Integers a, b, c satisfy a + b – c = 1 and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$?

a, b, c ऐसे पूर्णांक हैं जिनके लिए a + b – c = 1 व $a^2 + b^2 - c^2 = -1$. $a^2 + b^2 + c^2$ के जो भी मान संभव हैं। उनका योग क्या होगा ?

Sol. **(18)**

$a + b - c = 1, \quad a^2 + b^2 - c^2 = -1$

$a + b - 1 = c$

$\Rightarrow a^2 + b^2 + 1 + 2ab - 2(a + b) = c^2 \quad \Rightarrow \quad ab = a + b \quad \Rightarrow (a - 1)(b - 1) = 1$

So $a - 1 = b - 1 = \pm 1 \quad \Rightarrow a = b = 2$ or $a = b = 0$

So c = 3 (when a = b = 2) or c = –1 (when a = b = 0)

Hence $a^2 + b^2 + c^2 = 17$ or 1 ∴ Sum = 18

Hindi : $a + b - c = 1, \quad a^2 + b^2 - c^2 = -1$

$a + b - 1 = c$

$\Rightarrow a^2 + b^2 + 1 + 2ab - 2(a + b) = c^2 \quad \Rightarrow \quad ab = a + b \quad \Rightarrow (a - 1)(b - 1) = 1$

इसलिए $a - 1 = b - 1 = \pm 1 \quad \Rightarrow a = b = 2$ or $a = b = 0$

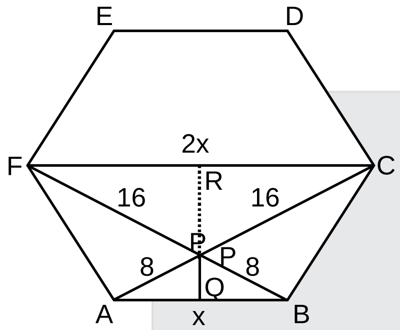
इसलिए c = 3 (जब a = b = 2) या c = –1 (जब a = b = 0)

अतः $a^2 + b^2 + c^2 = 17$ or 1 ∴ योग = 18

7. A point P in the interior of a regular hexagon is at distance 8,8,16 units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r ?

बिन्दु P एक सम–षट्भुज का भीतरी बिन्दु है और षट्भुज के तीन क्रमानुगत कोनों से उसकी दूरी क्रमशः 8,8 व 16 है। अगर r षट्भुज के परिवृत्त की त्रिज्या का मान है तो r के सबसे करीब कौनसा पूर्णांक होगा।

Sol. **(14)**



Note that CF = 2AB, PA = 2PC & PB = 2PF

Hence ΔPAB is similar to ΔPFC, hence A_1P_1C & B_1P_1F are collinear. Let each side of hexagon be equal to x .

Let Q & R be foot of altitudes from P to base AB & CF respectively. So R is centre of hexagon

Now $\frac{1}{3} \times \frac{\sqrt{3}x}{2} = \sqrt{64 - \frac{x^2}{4}}$

$\Rightarrow \quad \frac{x^2}{12} = 64 - \frac{x^2}{4}$

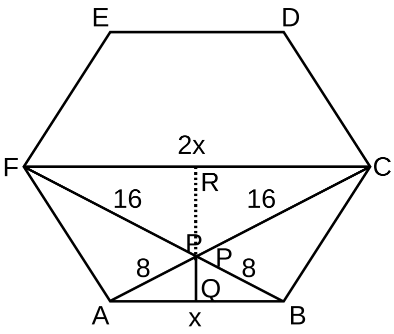
$\Rightarrow \quad \frac{4x^2}{12} = 64 \quad \Rightarrow \quad x = 8\sqrt{3}$

Note that circumradius of a regular hexagon = side of regular hexagon

Hence $r = 8\sqrt{3} \approx 13.856$

Hence nearest integer = 14

Hindi :



दिया गया है CF = 2AB, PA = 2PC & PB = 2PF

अतः ΔPAB , ΔPFC के समरूप है अतः A_1P_1C & B_1P_1F संरेख है। माना षटभुज की प्रत्येक भुजा x के बराबर है।

माना Q एवं R, P से AB एवं CF पर क्रमशः शीर्षलम्ब का पाद है, इसलिए R षटभुज का केन्द्र है।

अब $$\frac{1}{3} \times \frac{\sqrt{3}x}{2} = \sqrt{64 - \frac{x^2}{4}}$$

$$\Rightarrow \quad \frac{x^2}{12} = 64 - \frac{x^2}{4}$$

$$\Rightarrow \quad \frac{4x^2}{12} = 64$$

$$\Rightarrow \quad x = 8\sqrt{3}$$

दिया गया है सम्षट्भुज की परित्रिज्या = समषट्भुज की भुजा

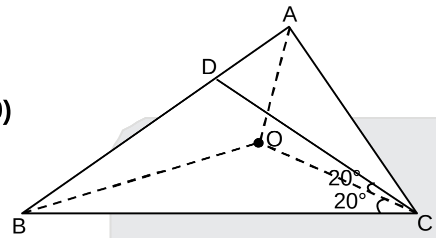
अतः $r = 8\sqrt{3} \approx 13.856$

अतः निकटतम पूर्णांक = 14

8. Let AB be a chord of circle with centre O. Let C be a point on the circle such that ∠ABC = 30° and O lies inside the triangle ABC. Let D be a point on AB such that ∠DCO = ∠OCB = 20°. Find the measure of ∠CDO in degrees.

AB एक वृत्त की जीवा है जिसका केन्द्र O है। मानलो कि C वृत्त पर एक ऐसा बिन्दु है जिससे कि ∠ABC = 30° व O त्रिभुज ABC के अंदर है। मानलो कि D वृत्त AB पर एक ऐसा बिन्दु है जिससे कि ∠DCO = ∠OCB = 20° । ∠CDO का मान डिग्री में पता करो।

Sol. **(80)**



∠OCB = 20°

&

∠OBC = 20°

&

∠OBA = 10°

&

∠OAB = 10°

Since ∠BOC = 140 ⇒ ∠A = 70°

&

∠OAC = 60°

&

∠ACD = 40°

Now C is circumcenter of ΔAOD

as ∠OCD = 2∠OAD

&

∠AOD = $\frac{1}{2}$ ∠OAD = 20°

&

$\angle DOC = \angle AOD + \angle AOC$

$= 20 + 60$

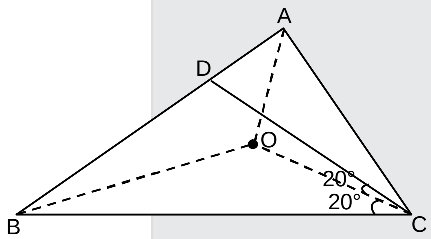
$= 80$

$\Rightarrow \quad \angle ODC = 180 - (\angle DOC + \angle OCD)$

$= 180 - (80 + 20)$

$= 80°$

Hindi.



$\angle OCB = 20°$

तथा

$\angle OBC = 20°$

तथा

$\angle OBA = 10°$

तथा

$\angle OAB = 10°$

चूंकि $\angle BOC = 140 \Rightarrow \angle A = 70°$

तथा

$\angle OAC = 60°$

तथा

$\angle ACD = 40°$

अब C , ΔAOD का परिकेन्द्र है

चूकि $\angle OCD = 2\angle OAD$

तथा

$\angle AOD = \frac{1}{2} \angle OAD = 20°$

तथा

$\angle DOC = \angle AOD + \angle AOC$

$= 20 + 60$

$= 80$

$\Rightarrow$ $\angle ODC = 180 - (\angle DOC + \angle OCD)$

$= 180 - (80 + 20)$

$= 80°$

9. Suppose a,b are integers and a + b is a root of $x^2 + ax + b = 0$. What is the maximum possible values of b^2 ?

माना कि a,b पूर्णाक है तथा a + b समीकरण $x^2 + ax + b = 0$ का एक हल है। b^2 का अधिकतम सम्भव मान क्या है ?

Sol. **(81)**

If "a + b" is a root it satisfies the equation

Hence $(a + b)^2 + a(a + b) + b = 0$

$\Rightarrow$ $2a^2 + 3ba + (b^2 + b) = 0$

Now since "a" is an integer Discriminant is a perfect square

$\Rightarrow$ $9b^2 - 8(b^2 + b) = p^2$ for same $p \in Z$

$\Rightarrow$ $(b - 4)^2 - 16 = p^2$

$\Rightarrow$ $(b - 4 + b)(b - 4 - p) = 16$

$b - 4 + p = \pm 8$, $b - 4 - p = \pm 2$, $b - 4 + p = b - 4 - p = \pm 4$

So $b - 4 = 5, -5, 4, -4$

$\Rightarrow$ $b = 9, -1, 8, 0 \Rightarrow (b^2)_{max} = 81$

Hindi : यदि "a + b" मूल है, यह दी गई समीकरण को संतुष्ट करता है।

अतः $(a + b)^2 + a(a + b) + b = 0$

$\Rightarrow$ $2a^2 + 3ba + (b^2 + b) = 0$

अब "a" पूर्णांक है विवेचक, पूर्ण वर्ग होगा।

$\Rightarrow$ $9b^2 - 8(b^2 + b) = p^2$ किसी $p \in Z$ के लिए

$\Rightarrow$ $(b - 4)^2 - 16 = p^2$

$\Rightarrow$ $(b - 4 + b)(b - 4 - p) = 16$

$b - 4 + p = \pm 8$, $b - 4 - p = \pm 2$, $b - 4 + p = b - 4 - p = \pm 4$

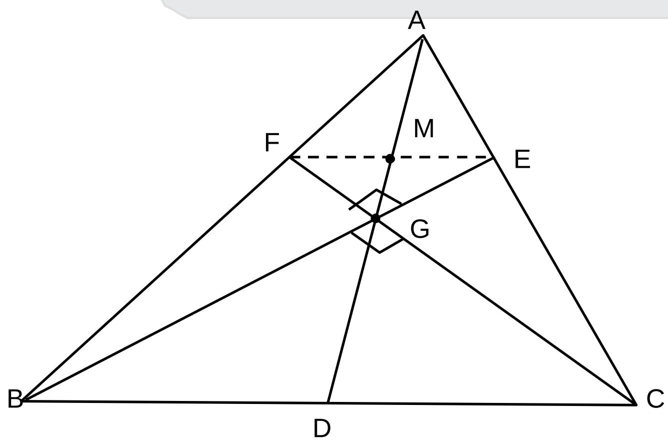
इसलिए $b - 4 = 5, -5, 4, -4$

$\Rightarrow$ $b = 9, -1, 8, 0 \Rightarrow (b^2)_{max} = 81$

10. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.

एक त्रिभुज ABC में B से CA तक की माध्यिका C से AB तक की माध्यिका से लम्ब है। अगर A से BC तक की माध्यिका की लम्बाई 30 है तो $(BC^2 + CA^2 + AB^2)/100$ का मान ज्ञात करो।

Sol. **(24)**



Let G be centroid, D,E,F be mid-points of BC, CA, AB & M be mid-point of FE.

Let BE = 3x, CF = 3y, given AD = 30

Hence AM = 5, GM = 5, GD = 10, BG = 2x, GE = x, CG = 2y, GF = y

Now D is mid-point of hypotenuse of right angle triangle BGC
So D is circum centre of the triangle
SO BD = GD = 10 $\Rightarrow$ BC = 20
Hence $4(x^2 + y^2) = 400 \Rightarrow x^2 + y^2 = 100$

Now $9(x^2 + y^2) = \frac{1}{4}\{2BC^2 + 2AB^2 - AC^2 + 2BC^2 + 2AC^2 - AB^2\}$

$\Rightarrow 900 \times 4 = 4BC^2 + AB^2 + AC^2$

$\Rightarrow AB^2 + AC^2 = 3600 - 1600 = 2000$

Hence $\frac{AB^2 + BC^2 + CA^2}{100} = \frac{2400}{100} = 24$

11. There are several tea cups in the kitchen, some with handle and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen ?
किचन में कई चाय के कप है, कुछ में हैडल है, और कुछ में नहीं है। अगर इनमें से दो कप बिना हैड़ल के व तीन कप हैंडल के चुनने हो तो यह 1200 तरीकों से किया जा सकता है। किचन में कितने कप है ?

Sol. **(29)**

Let cups without handle equals to x & cups with handle equals to y

$\Rightarrow \quad {}^{x}C_2 \times {}^{y}C_3 = 1200 = 2^4 \times 3 \times 5^2$

$\frac{x(x-1)}{2} \times \frac{y(y-1)(y-2)}{6} = 2^4 \times 3 \times 5^2$

x = 25, y = 4 and x = 16, y = 5

x + y is maximum when x = 25, y = 4

maximum possible cups equals to 29

Hindi. माना बिना हैन्डल वाले कपों की संख्या x तथा हैन्डल वाले कपों की संख्या y है

$\Rightarrow \quad {}^{x}C_2 \times {}^{y}C_3 = 1200 = 2^4 \times 3 \times 5^2$

$\frac{x(x-1)}{2} \times \frac{y(y-1)(y-2)}{6} = 2^4 \times 3 \times 5^2$

x = 25, y = 4 तथा x = 16, y = 5

x + y अधिकतम होगा जब x = 25, y = 4

अधिकतम सम्भावित कप 29 के बराबर है

12. Determine the number of 8-tuples $(\epsilon_1, \epsilon_2, ..., \epsilon_8)$ such that $\epsilon_1, \epsilon_2, ..., \epsilon_8 \in \{1, -1\}$ and

$$\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + ... + 8\epsilon_8$$

is a multiple of 3.

ऐसे कितने क्रमवार–समूच्चय $(\epsilon_1, \epsilon_2, ..., \epsilon_8)$ है जिसमें $\epsilon_1, \epsilon_2, ..., \epsilon_8 \in \{1, -1\}$ व $\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + ... + 8\epsilon_8$ का मान 3 से भाज्य हो ?

Sol. **(88)**

$\epsilon_1, \epsilon_2, \epsilon_3, \ldots\ldots\ldots\ldots \epsilon_8 \in \{-1, 1\}$.

$\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + \ldots\ldots + 8\epsilon_8$

$\Rightarrow \quad (\epsilon_1 + 4\epsilon_4 + 7\epsilon_7) + (2\epsilon_2 + 5\epsilon_5 + 8\epsilon_8) + 3(\epsilon_3 + 2\epsilon_6)$

$\Rightarrow \quad (\epsilon_1 + \epsilon_4 + \epsilon_7) + 2(\epsilon_2 + \epsilon_5 + \epsilon_8) + 3m$ (m is an integer)

$\Rightarrow \quad (\epsilon_1 - \epsilon_2) + (\epsilon_4 - \epsilon_5) + (\epsilon_7 - \epsilon_8) + 3q$ (q is an integer)

$\epsilon_1 - \epsilon_2 = 2, 0, -2$ & similarly others

$p_{12} + p_{45} + p_{78} + 3q$ (where $p_{12} = \epsilon_1 - \epsilon_2$

$p_{45} = \epsilon_4 - \epsilon_5$

$p_{78} = \epsilon_7 - \epsilon_8$)

$p_{12} = p_{45} = p_{78} = 0 \quad \Rightarrow \quad$ 8 cases

$p_{12} = p_{45} = p_{78} = 2$ (or -2) $\quad \Rightarrow \quad$ 2 cases

$p_{12} = 2, \; p_{45} = -2, \; p_{78} = 0 \quad \Rightarrow \quad (2 \times 6)$ cases

Hence number of tuples = 22 × 4 = 88

↓

(ϵ_3 & ϵ_6 can be any one of 1 or –1)

Hindi.

$\epsilon_1, \epsilon_2, \epsilon_3, \ldots\ldots\ldots\ldots \epsilon_8 \in \{-1, 1\}$.

$\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + \ldots\ldots + 8\epsilon_8$

$\Rightarrow \quad (\epsilon_1 + 4\epsilon_4 + 7\epsilon_7) + (2\epsilon_2 + 5\epsilon_5 + 8\epsilon_8) + 3(\epsilon_3 + 2\epsilon_6)$

$\Rightarrow \quad (\in_1 + \in_4 + \in_7) + 2(\in_2 + \in_5 + \in_8) + 3m$ (m पूर्णांक है)

$\Rightarrow \quad (\in_1 - \in_2) + (\in_4 - \in_5) + (\in_7 - \in_8) + 3q$ (q एक पूर्णांक है)

$\in_1 - \in_2 = 2, 0, -2$ तथा इसी प्रकार अन्य similarly others

$p_{12} + p_{45} + p_{78} + 3q$ (जहाँ $p_{12} = \in_1 - \in_2$

$p_{45} = \in_4 - \in_5$

$p_{78} = \in_7 - \in_8$)

$p_{12} = p_{45} = p_{78} = 0 \quad \Rightarrow$ 8 स्थितियाँ

$p_{12} = p_{45} = p_{78} = 2$ (or -2) $\quad \Rightarrow$ 2 स्थितियाँ

$p_{12} = 2,\ p_{45} = -2,\ p_{78} = 0 \quad \Rightarrow$ (2×6) स्थितियाँ

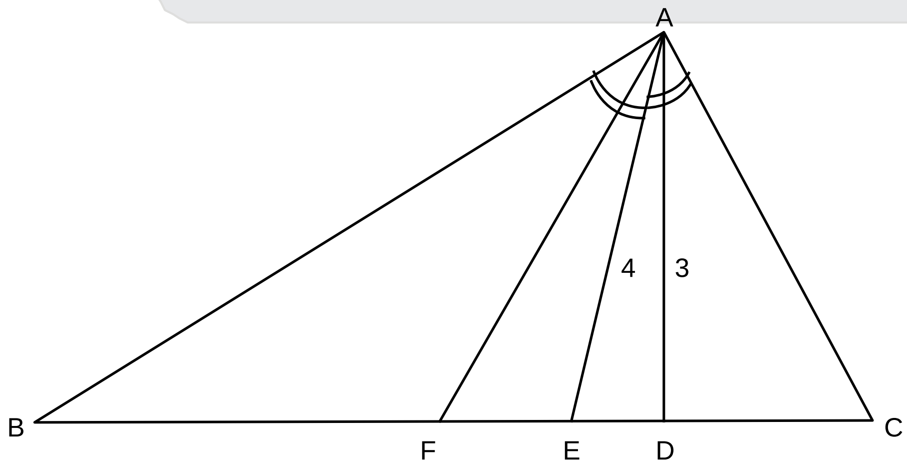
अतः क्रमवार–समुच्चय की संख्या = $22 \times 4 = 88$

$\downarrow$

($\in_3$ & $\in_6$ संख्या 1 या –1 में से कोई एक हो सकता है)

13. In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the median through A.
एक त्रिभुज ABC, जिसमें कोण A समकोण है, ऐसा है कि A से लम्ब की लम्बाई व $\angle A$ कोण–समद्विभाजक की लम्बाई क्रमशः 3 व 4 है। A से मध्यिका की लम्बाई कितनी होगी?

Sol. **(24)**



$\angle CAE = 45° = \angle BAE$
AD = 3

Let BC = a, CA = b, AB = c

$\frac{1}{2}bc = \frac{1}{2}a \cdot 3 \Rightarrow bc = 3a$

$\frac{2bc}{b+c}\cos\frac{A}{2} = 4 \Rightarrow \frac{6a}{b+c} \cdot \frac{1}{\sqrt{2}} = 4$

$\Rightarrow 2\sqrt{2}\left(b+c\right) = 3a$

$\Rightarrow 8(b^2 + c^2 + 2bc) = 9a^2$

$\Rightarrow 8(a^2 + 6a) = 9a^2$

$\Rightarrow 48a = a^2 \Rightarrow a = 48$

So AF = $\frac{a}{2} = 24$

14. If x = cos1°cos2°cos3°…..cos89° and y = cos2°cos6°cos10°…..cos86°, then what is the integer nearest to $\frac{2}{7}\log_2(y/x)$?

अगर x = cos1°cos2°cos3°…..cos89° व y = cos2°cos6°cos10°…..cos86°, तो संख्या $\frac{2}{7}\log_2(y/x)$ के सबसे करीब कौनसा पूर्णांक होगा ?

Sol. **(19)**

$\frac{y}{x} = \frac{\cos 2°\cos 6°\cos 10°.........\cos 86°}{\cos 1°\cos 2°\cos 3°.........\cos 89°}$

$= 2^{44} \times \sqrt{2}\ \frac{\cos 2°\cos 6°\cos 10°.........\cos 86°}{\sin 2°\sin 4°.........\cos 88°}$

$= \frac{2^{89/2}\sin 4°\sin 8°\sin 12°..........\sin 88°}{\sin 2°\sin 4°\sin 6°..........\sin 88°}$

$= \frac{2^{89/2}}{\cos 4°\cos 8°\cos 12°............\cos 88°}$

$= \frac{2^{89/2}}{\left(\frac{1}{2^{22}}\right)} = 2^{\frac{89}{2}+22}$

$= 2^{\frac{133}{2}}$

$\Rightarrow$ $\frac{2}{7}\log_2(y/x) = \frac{2}{7}\log_2 2^{\frac{133}{2}} = \frac{2}{7} \times \frac{133}{2} = 19$

15. Let a and b natural numbers such that 2a – b, a – 2b and a + b are all distinct squares. What is the smallest possible value of b ?

मानलो कि a व b ऐसी प्राकृतिक संख्या है जिससे कि 2a – b, a – 2b व a + b सभी अलग–अलग पूर्णाकों के वर्ग है। b का न्यूनतम सम्भव मान क्या होगा ?

Sol. **(21)**

$2a - b = k_1^2$ ……(1)

$a - 2b = k_2^2$ ……(2)

$a + b = k_3^2$ ……(3)

Add (2) & (3) we get

$2a - b = k_2^2 + k_3^2$

$\Rightarrow$ $k_2^2 + k_3^2 = k_1^2$ $(k_2 < k_3)$

For least 'b' difference of k_3^2 & k_2^2 is also least and must be multiple of 3

$\Rightarrow$ $k_2^2 = a - 2b = a^2$ & $k_3^2 = a + b = 12^2$

$\Rightarrow$ $k_3^2 - k_2^2 = 3b = 144 - 81 = 63 \Rightarrow b = 21$

$\Rightarrow$ least b is 21

Hindi. $2a - b = k_1^2$ ……(1)

$a - 2b = k_2^2$ ……(2)

$a + b = k_3^2$ ……(3)

(2) व (3) को जोड़ने पर

$2a - b = k_2^2 + k_3^2$

$\Rightarrow$ $k_2^2 + k_3^2 = k_1^2$ $(k_2 < k_3)$

'b' के न्युनतम मान के लिए k_3^2 & k_2^2 के न्युनतम अन्तर न्युनतम होगा तथा 3 का गुणज होगा

$\Rightarrow$ $k_2^2 = a - 2b = a^2$ & $k_3^2 = a + b = 12^2$

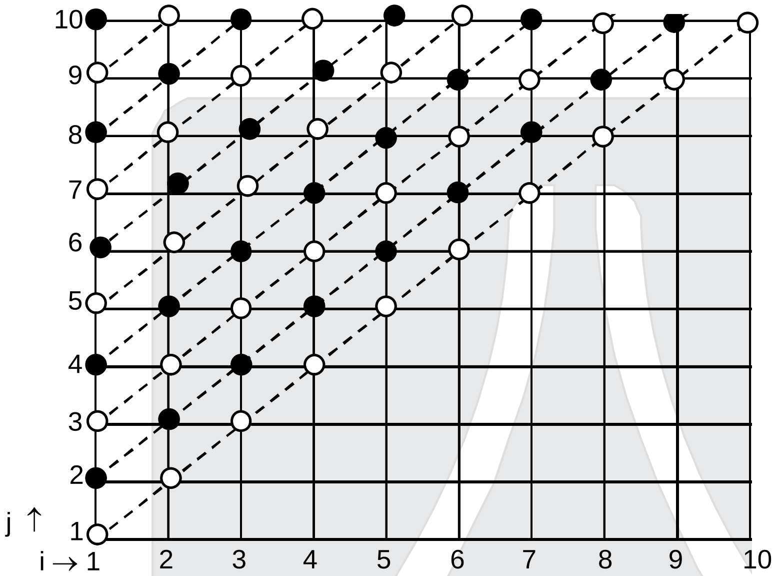
$\Rightarrow$ $k_3^2 - k_2^2 = 3b = 144 - 81 = 63 \Rightarrow b = 21$

$\Rightarrow$ b का न्युनतम मान 21 है

16. What is the value of $\sum\limits_{\substack{1\le i<j\le 10\\ i+j=odd}} (i+j) - \sum\limits_{\substack{1\le i<j\le 10\\ i+j=even}} (i+j)$?

निम्न का मान पता करो। $\sum\limits_{\substack{1\le i<j\le 10\\ i+j=odd}} (i+j) - \sum\limits_{\substack{1\le i<j\le 10\\ i+j=even}} (i+j)$?

Sol. **(55)**



Sum of odd = $\underbrace{3+5+7+9+.......+19}_{9}+\underbrace{5+7+........+17}_{7}+\underbrace{7+9+........+15}_{5}+\underbrace{9+11+13}_{3}+11$

Sum of even = $\underbrace{4+6+.......+18}_{8}+\underbrace{6+8+.......+16}_{6}+\underbrace{8+10+12+14}_{4}+\underbrace{10+12}_{2}$

Difference = (–1 –1 –1……….8 times) + 19 + (–1 –1 –1……….6 times) + 17 + (–1 –1 –1 –1) + 15

+ (–1 –1) + 13 + 11

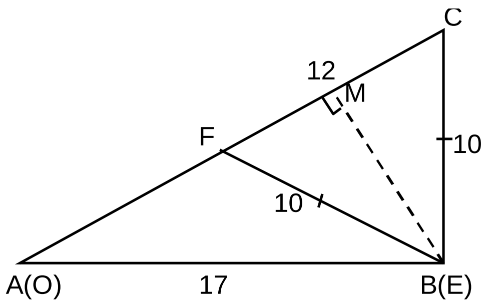
= –8 – 6 –4 –2 + 19 + 17 + 15 + 13 + 11 = 75 – 20 = 55

17. Triangles ABC and DEF are such that $\angle A = \angle D$, AB = DE = 17, BC = EF = 10 and AC – DF = 12. What is AC + DF ?

ABC व DEF ऐसे त्रिभुज है कि $\angle A = \angle D$, AB = DE = 17, BC = EF = 10 और AC – DF = 12 है। AC + DF का मान क्या है ?

Sol. **(30)**

Let A coincides with D, B coincides with E. With E(B) as centre draw a circle with radius 10 intersecting the line, making angle $\theta = \angle A$ with AB, at F & C.



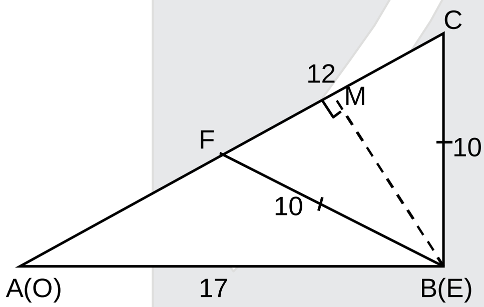
Let N be the foot of perpendicular from B(E) to CF

So BM = 8 Hence AM = $\sqrt{17^2 - 8^2} = \sqrt{(25)(9)} = 15$

Hence AF = 15 – 6 = 9 & AC = 15 + 6 = 21

So AC + DF = 30

Hindi. माना A , D के साथ सम्पाती है तथा B, E के साथ सम्पाती है। E(B) को केन्द्र मानते हुए 10 त्रिज्या का एक वृत्त खीचा जाता है जो रेखा को प्रतिच्छेद करता है तथा AB के साथ F और C पर $\theta = \angle A$ कोण बनाता है



माना N, B(E) से CF पर लम्ब का लम्बपाद है

इसलिए BM = 8 अतः AM = $\sqrt{17^2 - 8^2} = \sqrt{(25)(9)} = 15$

अतः AF = 15 – 6 = 9 & AC = 15 + 6 = 21

इसलिए AC + DF = 30

18. If a,b,c $\geq$ 4 are integers, not all equal, and 4abc = (a + 3) (b + 3) (c + 3), then what is the value of a + b + c ?

अगर a,b,c $\geq$ 4 पूर्णांक है, सभी बराबर नहीं है, और 4abc = (a + 3) (b + 3) (c + 3) तो a + b + c का मान क्या है ?

Sol. **(16)**

4abc = 27 + 3(ab + bc + ca) + 9 (a + b + c) + abc

$\Rightarrow$ 3abc = 27 + 3(ab + bc + ca) + 9 (a + b + c)

$\Rightarrow$ abc = 9 + (ab + bc + ca) + 3 (a + b + c)

$\Rightarrow$ abc – (ab + bc + ca) + (a + b + c) –1 = 8 + 4 (a + b + c)

$\Rightarrow$ (a – 1) (b – 1) (c – 1) = 8 + 4 (a + b + c)

Put a – 1 = A, b – 1 = B, c – 1 = C,

$\Rightarrow$ ABC = 20 + 4 (A + B + C)

$\Rightarrow$ $A = \frac{4(5+B+C)}{BC-4}$

$\Rightarrow$ B = 3, C = 4, A = 6

or they can be interchanged

$\Rightarrow$ (a, b, c) are anangemets of (4, 5, 7)

$\Rightarrow$ a + b + c = 16

19. Let N = 6 + 66 + 666 + ….. + 666….66, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N ?

मानलो कि N = 6 + 66 + 666 + ….. + 666….66 जहाँ आखिरी संख्या में सौ 6' के अंक है। N में अंक 7 कितनी बार आएगा ?

Sol. **(33)**

$$N = 6 + 66 + 666 + \ldots\ldots \underbrace{6666\ldots\ldots66}_{100\,\text{times}}$$

$$= \frac{6}{9}\left[9+99+\ldots\ldots+\underbrace{999\ldots\ldots99}_{100\,\text{times}}\right]$$

$$= \frac{6}{9}[(10-1)+(10^2-1)+\ldots\ldots+(10^{100}-1)]$$

$$= \frac{6}{9}[(10+10^2+\ldots\ldots+10^{100})-100]$$

$= \frac{6}{9}[(10^2 + 10^3 + \ldots\ldots + 10^{100}) - 90]$

$= \frac{6}{9}\left(10^2\frac{(10^{99}-1)}{9}\right) - 60$

$= \frac{200}{27}(10^{99} - 1) - 60$

$= \frac{200}{27}\left(\underbrace{999.........99}_{99\text{ times}}\right) - 60$

$= \frac{1}{3}\left(\underbrace{222.........2}_{99\text{ times}}00\right) - 60$

$= \underbrace{740\ 740.........740}_{740\text{ comes } 33\text{ times}}0 - 60$

$= \underbrace{740\ 740.........740}_{32\text{ times}} + 340$

$\Rightarrow$ 7 comes 33 times

$\Rightarrow$ 7, 33 बार आता है

20. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.
ऐसे सभी धनात्मक पूर्णाकों का योग पता करो जिनके अंकों का गुणनफल $n^2 - 15n - 27$ है।

Sol. **(17)**

$n^2 - 15n - 27$ is always odd number for all $n \in I$

n must be maximum of two digit number.

because maximum product of three digit number is 729 & minimum value of $n^2 - 15n - 27$ for 3 digits number is 10000 – 1500 – 27 which is greater than 729.

$n^2 - 15n - 27$ is increasing function for all $n \in \{8, 9, 10, \ldots\ldots\}$

at n = 17, $n^2 - 15n - 27$ is equal to 17

at n = 19, $n^2 - 15n - 27$ is equal to 49

at n = 21, $n^2 - 15n - 27$ is equal to 99

And maximum product of digits of two digit number is 81

$\Rightarrow$ So n must be less than 21

$\Rightarrow$ Between 1 to 15, $n^2 - 15n - 27$ is negative

$\Rightarrow$ So n = 17 only

Sum of possible number equal to 17

Hindi. $n^2 - 15n - 27$ सदैव सभी $n \in I$ के लिए विषम संख्या है

n दो अंक की संख्या का अधिकतम मान होगा

क्योकि तीन अंक संख्या का अधिकतम गुणनफल 729 तथा $n^2 - 15n - 27$ का न्यूनतम मान , तीन अंक की संख्या होने के लिए 10000 – 1500 – 27 जो 729 से बड़ा है।

$n^2 - 15n - 27$ सभी $n \in \{8, 9, 10, \ldots\}$ के लिए वर्धमान है

n = 17 पर, $n^2 - 15n - 27$ का मान 17 है

n = 19 पर, $n^2 - 15n - 27$ का मान 49 है

n = 21 पर, $n^2 - 15n - 27$ का मान 99 है

तथा दो अंक की संख्या का अधिकतम गुणनफल 81 है

$\Rightarrow$ n, 21 से कम होगा।

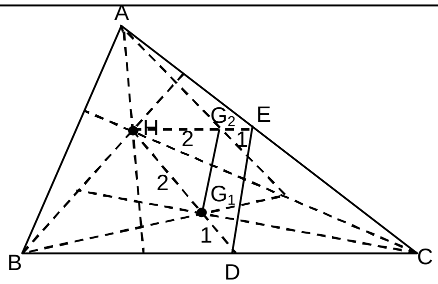
$\Rightarrow$ 1 से 15 के मध्य , $n^2 - 15n - 27$ ऋणात्मक है

$\Rightarrow$ इसलिए n = 17 केवल

सम्भावित संख्या का योगफल 17 के बराबर है

21. Let ABC be an acute-angled triangle and let H be its orthocentre. Let G_1, G_2 and G_3 be the centroids of the triangles HBC , HCA and HAB respectively. If the area of triangle $G_1G_2G_3$ is 7 units, what is the area of triangle ABC ?
मानलो कि ABC एक न्यूनकोण त्रिभुज है और H उसका लंबकेन्द्र है। मानलो कि G_1, G_2 व G_3 क्रमशः त्रिभुज HBC , HCA व HAB के केन्द्रक है। अगर त्रिभुज $G_1G_2G_3$ का क्षेत्रफल 7 है, तो त्रिभुज ABC का क्षेत्रफल कितना होगा ?

Sol. **(63)**



AB = 2DE .....(1)

In $\Delta H G_1 G_2$ & $\Delta H D E$

$$\frac{HG_1}{HD} = \frac{G_1G_2}{DE} = \frac{2}{3}$$

$$G_1G_2 = \frac{2}{3} DE = \frac{2}{3}\left(\frac{AB}{2}\right) = \frac{AB}{3}$$

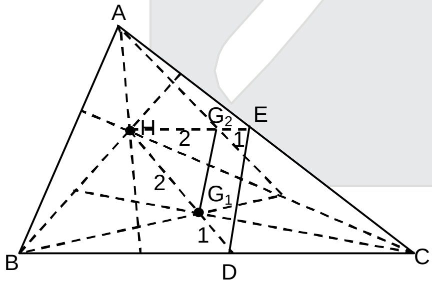
Becomes $\Delta G_1 G_2 G_3$ ~ ΔABC

$$\Rightarrow \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta G_1G_2G_3} = \frac{(AB)^2}{(G_1G_2)^2} = \left(\frac{AB}{G_1G_2}\right)^2 = \left(\frac{3}{1}\right)^2$$

$\Rightarrow$ Area of $\Delta ABC = 9 \times$ (Area of $\Delta G_1 G_2 G_3$)

$\Rightarrow$ Area of $\Delta ABC = 9 \times 7 = 63$

Hindi.



AB = 2DE .....(1)

$\Delta H G_1 G_2$ और $\Delta H D E$ में

$$\frac{HG_1}{HD} = \frac{G_1G_2}{DE} = \frac{2}{3}$$

$$G_1G_2 = \frac{2}{3} DE = \frac{2}{3}\left(\frac{AB}{2}\right) = \frac{AB}{3}$$

से $\Delta G_1 G_2 G_3 \sim \Delta ABC$

$$\Rightarrow \quad \frac{\Delta ABC \text{ क्षेत्रफल}}{\Delta G_1G_2G_3 \text{ क्षेत्रफल}} = \frac{(AB)^2}{(G_1G_2)^2} = \left(\frac{AB}{G_1G_2}\right)^2 = \left(\frac{3}{1}\right)^2$$

$\Rightarrow$ ΔABC का क्षेत्रफल = 9 × ($\Delta G_1 G_2 G_3$ का क्षेत्रफल)

$\Rightarrow$ ΔABC का क्षेत्रफल = 9 × 7 = 63

22. A positive integer k is said to be good if there exists a partition of {1,2,3,…..,20} in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there ?
एक पूर्णाक k को हम अच्छा कहेंगे अगर {1,2,3,…..,20} को हम उचित उपसमूच्चयों (proper subsets) में विभाजित कर सकते है (ऐसे कि एक संख्या एक ही उपसमूच्चय में हो) ताकि हर उपसमूच्चय में आने वाली संख्याओं का योग k हो। कितनी संख्याएँ अच्दी है ?

Sol. **(6)**

Sum of numbers equals to $\frac{20 \times 21}{2} = 210$ & 210 = 2 × 3 × 5 × 7

	Number of partition	**sum**
I	2	105
II	3	70
III	5	42
IV	7	30
V	6	35
VI	10	21

So K can be 21, 30, 35, 47, 70, 105

Good numbers equal to 6

Case-I A = {1, 2, 3, 4, 5, 16, 17, 18, 19, 20} , B = {6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

Case-II A = {20, 19, 18, 13}, B = {17, 16, 15, 12, 10}, C = {1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14}

Case-III A = {20, 10, 12}, B = {18, 11, 13}, C = {16, 15, 9, 2}, D = {19, 8, 7, 5, 3}, E = {1, 4, 6, 14, 17}

Case-IV A = {20, 10}, B = {19, 11}, C = {18, 12}, D = {17, 13}, E = {16, 14}, F = {1, 15, 5}, G = {2, 3, 4, 6, 7, 8}

Case-V A = {20, 15}, B = {19, 16}, C = {18, 17}, D = {14, 13, 8}, E = {12, 11, 10, 2}, F = {1, 3, 4, 5, 6, 7, 9}

Case-VI A = {1, 20}, B = {2, 19}, C = {3, 18}…………….., J = {10, 11}

Hindi. संख्याओं का योगफल = $\frac{20 \times 21}{2} = 210$ & 210 = 2 × 3 × 5 × 7

	भागो की संख्या	**योगफल**
I	2	105
II	3	70
III	5	42
IV	7	30
V	6	35
VI	10	21

K का मान 21, 30, 35, 47, 70, 105 हो सकता है

अच्छी संख्या 6 के बराबर है

स्थिति-I A = {1, 2, 3, 4, 5, 16, 17, 18, 19, 20} , B = {6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

स्थिति -II A = {20, 19, 18, 13}, B = {17, 16, 15, 12, 10}, C = {1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 14}

स्थिति –III A = {20, 10, 12}, B = {18, 11, 13}, C = {16, 15, 9, 2}, D = {19, 8, 7, 5, 3}, E = {1, 4, 6, 14, 17}

स्थिति -IV A = {20, 10}, B = {19, 11}, C = {18, 12}, D= {17, 13}, E = {16, 14}, F = {1, 15, 5}, G = {2, 3, 4, 6, 7, 8}

स्थिति -V A = {20, 15}, B = {19, 16}, C = {18, 17}, D = {14, 13, 8}, E = {12, 11, 10, 2},

F = {1, 3, 4, 5, 6, 7, 9}

स्थिति -VI A = {1, 20}, B = {2, 19}, C = {3, 18}…………….., J = {10, 11}

23. What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n\,(a+b+c)$$

holds for all positive real numbers a,b,c.

ऐसा सबसे बड़ा पूर्णाक n कौनसा है जिससे कि

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n\,(a+b+c)$$

सभी धनात्मक वास्तविक संख्याओं a,b,c के लिए सच हो ?

Sol. **(14)**

Since $\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}$

So $$\frac{a^2}{\frac{b}{39}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq \frac{(a+b+c)^2}{a\left(\frac{1}{29}+\frac{1}{31}\right)+b\left(\frac{1}{29}+\frac{1}{31}\right)+c\left(\frac{1}{29}+\frac{1}{31}\right)}$$

$$\geq \frac{(a+b+c)}{\left(\frac{1}{29}+\frac{1}{31}\right)}$$

$$\geq \frac{a+b+c}{\frac{60}{29\times 31}}$$

$$\geq \frac{29\times 31}{60}(a+b+c)$$

$$\geq 14.98\,(a+b+c)$$

So n = 14

Hindi. माना कि $\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\geq\frac{(a+b+c)^2}{x+y+z}$

इसलिए $\frac{a^2}{\frac{b}{39}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}}\geq\frac{(a+b+c)^2}{a\left(\frac{1}{29}+\frac{1}{31}\right)+b\left(\frac{1}{29}+\frac{1}{31}\right)+c\left(\frac{1}{29}+\frac{1}{31}\right)}$

$$\geq\frac{(a+b+c)}{\left(\frac{1}{29}+\frac{1}{31}\right)}$$

$$\geq\frac{a+b+c}{\frac{60}{29\times31}}$$

$$\geq\frac{29\times31}{60}(a+b+c)$$

$$\geq 14.98\ (a+b+c)$$

इसलिए n = 14

24. If N is the number of triangles of different shapes (i.e. not similar) whose angles are all integers (in degrees), what is N/100 ?
अगर N अलग अलग आकार के त्रिभुज (मतलब असमरूप त्रिभुज) है जिनके सभी कोण (डिग्री में) पूर्णांक हैं तो [N/100] का मान क्या होगा ?

Sol. **(27)**

x + y + z = 180

$x^1 + y^1 + z^1 = 177$

Total = ${}^{177+2}C_2 \Rightarrow$ Total = $\frac{179\times178}{2}=179\times89$

Total = 3! (α β γ) + 3 (α α β) + α α α

= 6(α β γ) + 3(α α β) + 1

For (α α α), number of ways = 1

For (α α β), 2α + β = 177, number of ways = (88 cases)

For $\alpha \beta \gamma$, number of ways = $\frac{179 \times 89 - 3 \times 88 - 1}{6}$

$= \frac{15931 - 265}{6}$

= 2611

So total way 2611 + 88 + 1 = 2700

Ans. 27

Hindi. x + y + z = 180

$x^1 + y^1 + z^1 = 177$

कुल तरीके = ${}^{177+2}C_2$ $\Rightarrow$ कुल तरीके = $\frac{179 \times 178}{2} = 179 \times 89$

कुल तरीके = 3! (α β γ) + 3 (α α β) + α α α

= 6(α β γ) + 3(α α β) + 1

(α α α)के लिए, तरीकों की संख्या = 1

(α α β)के लिए, 2α + β = 177, तरीकों की संख्या = 88

α β γ के लिए , तरीकों की संख्या = $\frac{179 \times 89 - 3 \times 88 - 1}{6}$

$= \frac{15931 - 265}{6} = 2611$

कुल तरीके 2611 + 88 + 1 = 2700

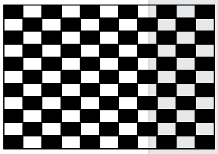
Ans. 27

25. Let T be the smallest positive integers which, when divided by 11,13,15 leaves remainders in the sets {7,8,9}, {1,2,3}, {4,5,6} respectively. What is the sum of the square of the digits of T ?
मान लो कि T सबसे छोटा धनात्मक पूर्णांक है जिसका 11,13 व 15 से विभाजन करने पर शेष क्रमशः समूच्चय {7,8,9}, {1,2,3} व {4,5,6} में है। तो फिर T के अंकों के वर्ग का योग क्या होगा ?

Sol. **(81)**

26. What is the number of ways in which one can choose 60 units square from a 11×11 chessboard such that no two chosen square have a side in common ?
11×11 की शतरंज की बिसात से 60 इकाई वर्ग कितनी तरह से चुन सकते है कि चुने हुए वर्गो की कोई भी भुजा साझा ना हो ?

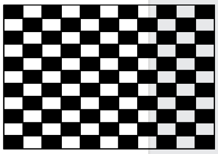
Sol. **(62)**



Either select 60 black squares from 61 black square or select all 60 white squres

$\Rightarrow$ Total equal to ${}^{61}C_{60} + {}^{60}C_{60} = 61 + 1 = 62$

Hindi.



या तो 61 काले वर्गो से 60 काले वर्ग चुनना या सभी 60 सभी सफेद वर्ग चुनना

कुल तरीके ${}^{61}C_{60} + {}^{60}C_{60} = 61 + 1 = 62$

27. What is the number of ways in which one can colour the square of a 4×4 chessboard with colours red and blue such that each row as well as each column has exactly two red squares and blue squares ?
4×4 की शतरंज की बिसात के हर एक वर्ग को लाल या नीले में रंगना है, ऐसे कितने तरीके होंगे कि हर एक बेड़ी पंक्ति और हर एक खड़ी पक्ति में दो नीले व दो लाल वर्ग हो?

Sol. **(90)**

First row can be filled by ${}^{4}C_{2}$ ways = 6 ways.

Case-I Second row is filled same as first row

$\Rightarrow$ here second row is filled by one way

3rd row is filled by one way

4th row is filled by one way

Total ways in Case-**I** equals to $^4C_1 \times 1 \times 1 \times 1 = 6$ ways

R	R	B	B
R	R	B	B

Case-II Exactly 1 R & 1 B is interchanged in second row in comparision to 1st row

$\Rightarrow$ here second row is filled by 2×2 way

3rd row is filled by two way

4th row is filled by one way

$\Rightarrow$ Total ways in Case-**II** equals to $^4C_1 \times 2 \times 2 \times 2 \times 1 = 48$ ways

R	R	B	B
R	B	B	R

Case-III Both R and B is replaces by other in second row as compared to 1st row

$\Rightarrow$ here second row is filled by 1 way

3rd row is filled by $^4C_2$ way

4th row is filled by one way

$\Rightarrow$ Total ways in 3th Case equals to $^4C_2 \times 1 \times 6 \times 1 = 36$ ways

$\Rightarrow$ Total ways of all cases equals to 90 ways

Hindi. प्रथम पक्ति को $^4C_2$ तरीकों से भरा जा सकता है = 6 तरीके

स्थिति -I दूसरी पक्ति को पहली पक्तिी के समान भरा जाता है

$\Rightarrow$ यहां दूसरी पक्ति को एक तरीके भरा जाता है

3rd पक्ति को एक तरीके से भरा जाता है

4th पक्ति को एक तरीके से भरा जाता है

स्थिति –I में कुल तरीके = $^4C_1 \times 1 \times 1 \times 1 = 6$ तरीके

R	R	B	B
R	R	B	B

स्थिति -II ठीक 1 R और 1 B को दूसरी पक्ति को आपस में बदला जाता है तथा पहली पक्ति से तुलना कि जाती है

$\Rightarrow$ दूसरी पक्ति को 2×2 तरीके से भरा जाता है

तीसरी पक्ति को दो तरीके से भरा जाता है

चौथी पक्ति को एक तरीके से भरा जाता है

$\Rightarrow$ स्थिति –II में कुल तरीके = $^4C_1 \times 2 \times 2 \times 2 \times 1 = 48$ तरीके

R	R	B	B
R	B	B	R

स्थिति -III दोनो R व B को अन्य दूसरी पक्ति में प्रथम पक्ति की तुलना में हटाया जाता है।

$\Rightarrow$ दूसरी पक्ति को एक तरीके से भरा जाता है

3^{th} पक्ति को $^4C_2$ तरीके से भरा जाता है

चौथी पक्ति को एक तरीके से भरा जाता है

$\Rightarrow$ स्थिति –III में कुल तरीके = $^4C_2 \times 1 \times 6 \times 1 = 36$ तरीके

$\Rightarrow$ सभी स्थिति में कुल तरीके 90

28. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N.
मान लो कि अलग–अलग कम्पनियों की 8 चोकलेट को तीन बच्चों में N तरीकों से बाँटा जा सकता है जिससे कि हर बच्चे को कम से कम एक चोकलेट मिले और किन्ही भी दो बच्चों को बराबर की संख्या में चोकलेट ना मिले। N के अंको का योग कितना होगा ?

Sol. **(24)**

$8 \rightarrow (1,2,5)$ or $(1, 3, 4)$

Number of ways $\frac{\lfloor 8}{\lfloor 2 \lfloor 5 \lfloor 1} \times \lfloor 3 + \frac{\lfloor 8}{\lfloor 1 \lfloor 3 \lfloor 4} \times \lfloor 3$

$= \left(\frac{8\times7\times6}{2} + \frac{8\times7\times6\times5}{6}\right)\times 6$

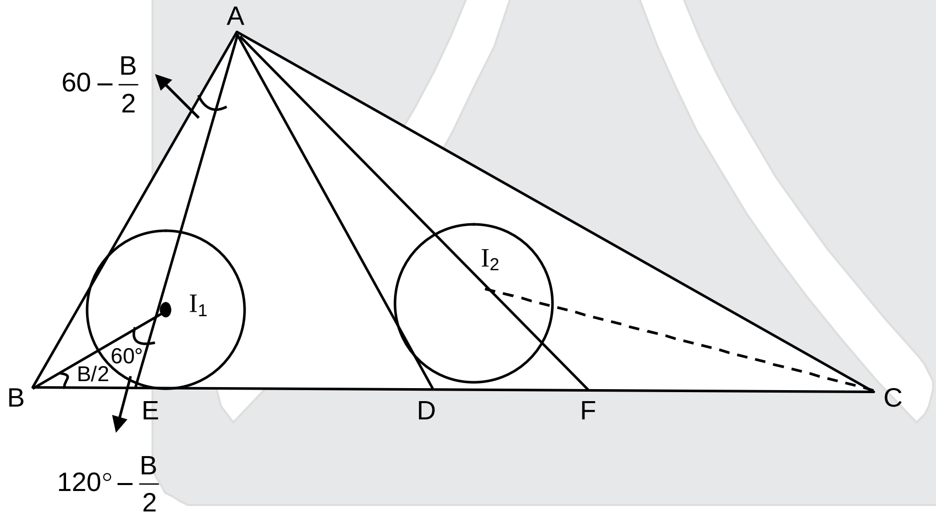
$= 56 \times 6\ (3 + 5) \Rightarrow 56 \times 48 = 2688$

Sum of digits = 24

29. Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incentres of triangles ABD and ACD respectively. Let AI_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60°$. What is the measure of $\angle CI_2F$ in degrees ?

मान लो कि D एक त्रिभुज ABC की भुजा BC का आंतरिक बिन्दु है। मान लो कि I_1 व I_2 क्रमशः त्रिभुज ABC व त्रिभुज ACD के अंतः केंन्द्र है। मानलो कि रेखा AI_1 व रेखा AI_2 रेखा BC से क्रमशः E व F में मिलती है। अगर $\angle BI_1E = 60°$. तो डिग्री में $\angle CI_2F$ का मान क्या होगा ?

Sol. **(30)**



$\angle BAD = 120° - B$

$\angle CAD = \angle A - (120° - B)$

$= A + B - 120°$

$\angle FAC = \frac{A+B}{2} - 60° \Rightarrow 90° - \frac{C}{2} - 60° \Rightarrow 30° - \frac{C}{2}$

$\angle AFC = 180° - \left(c + 30° - \frac{C}{2}\right)$

$= 150 - C/2$

$\angle CI_2F = 180°m - \left(150° - \frac{C}{2} + \frac{C}{2}\right) = 30°$

30. Let $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be a polynomial in which a_i is non-negative integer for each $i \in \{0,1,2,3,....,n\}$. If P(1) = 4 and P(5) = 136, what is the value of P(3)?

यदि $P(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ एक बहुपद है जहाँ a_i अऋणात्मक पूर्णांक हैं, व P(1) = 4 व P(5) = 136 तो P(3) का मान क्या होगा ?

Sol. **(34)**

$a_0 + a_1 + a_2 + ... + a_n = 4$

$\Rightarrow a_i \le 4$

$a_0 + 5a_1 + 5^2a_2 + ... + a\,5^na_n = 136$

$\Rightarrow a_0 = 1 + 5\lambda \Rightarrow a_0 = 1$

Hence $5a_1 + 5^2a_2 + ... + 5^na_n = 135$

$a_1 + 5a_2 + ...\ 5^{n-1}\ a_{n-1} = 27$

$\Rightarrow a_1 = 5\lambda + 2 \Rightarrow a_1 = 2$

$\Rightarrow 5a_2 + ...\ 5^{n-1}\ a_{n-1} = 25$

$a_2 + 5a_3 + ...\ 5^{n-2}\ a_{n-2} = 5$

$\Rightarrow a_2 = 5\lambda \Rightarrow a_2 = 0$

$a_3 + 5a_4 + ... + 5^{n-3}\ a_{n-3} = 1$

$a_3 = 1$

$\Rightarrow a_4 + 5a_5 + ... + 5^{n-4}a_{n-3} = 0$

$a_4 = a_5 = ...\ a_n = 0$

Hence $P(n) = x^3 + 2x + 1$

P(3) = 34

INSTRUCTION

Number of Questions: 30 **Max. Marks: 102**

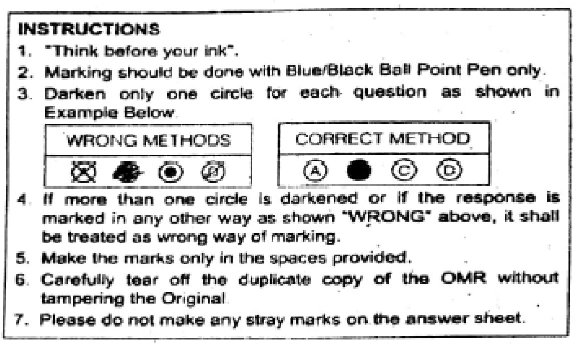
1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

2. The correction is done by machine through scanning. On OMR sheet, darken bubbles completely with a black pencil or a black blue pen. Darken the bubbles completely only after you are sure of your answer: else, erasing lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name, email address and date of birth entered on the OMR sheet will be your login credentials for accessing your PROM score.

4. Incomplete /incorrectly and carelessly filled information may disqualify your candidature.

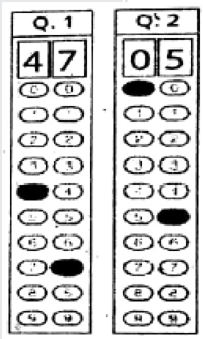
5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubble with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.



INSTRUCTIONS
1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below

WRONG METHODS	CORRECT METHOD

4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original
7. Please do not make any stray marks on the answer sheet.



6. The answer you write on OMR sheet is irrelevant. The darken bubble will be considered as your final answer.

7. Questions 1 to 6 carry 2 marks each: Questions 7 to 21 carry 3 marks each: Questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

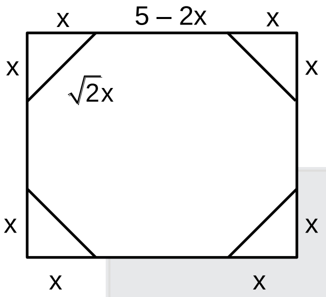
12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it further for verification purposes.

13. You may take away the question paper after the examination.

1. From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area **removed** to the nearest integer?

एक वर्ग जिसकी भुजाओं की लम्बाई 5 है, उसके चारों कोनों से त्रिभुजाकार टुकड़े काट कर एक नियमित अष्टभुज बनाया जाता है। कितना क्षेत्रफल हटाया गया है, उसका मान सबसे करीबी पूर्णांक तक ज्ञात करो।

Sol. **(4)**



$5 - 2x = \sqrt{2}\,x \Rightarrow x = \frac{5}{2+\sqrt{2}}$

Area removed हटाया गया क्षेत्रफल = $2x^2 = \frac{2 \times 25}{2\left(\sqrt{2}+1\right)} = 25\left(\sqrt{2}-1\right)^2 \approx 4.3$

So area removed to nearest integer is 4

हटाया गया क्षेत्रफल के सबसे करीबी पूर्णांक 4 है।

2. Let $f(x) = x^2 + ax + b$. If for all nonzero real x

$$f\left(x+\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

and the roots of $f(x) = 0$ are integers, what is the value of $a^2 + b^2$?

मान लो कि $f(x) = x^2 + ax + b$ है। यदि सभी अशून्य वास्तविक संख्या x के लिए

$$f\left(x+\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

और $f(x) = 0$ के सभी हल पूर्णांक है, तो $a^2 + b^2$ का मान क्या है?

Sol. **(13)**

$$f\left(x+\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2 + a\left(x+\frac{1}{x}\right) + b = x^2 + ax + b + \frac{1}{x^2} + \frac{a}{x} + b$$

$\Rightarrow b = 2$

$\Rightarrow \alpha\beta = 2 \qquad \Rightarrow (\alpha,\beta) = (1,2)$ or या $(-1, -2)$

$\Rightarrow a = 3$ or या $-3 \qquad \Rightarrow a^2 + b^2 = 9 + 4 = 13$

3. Let x_1 be a positive real number and for every integer $n \geq 1$ let $x_{n+1} = 1 + x_1x_2......x_{n-1}x_n$. If $x_5 = 43$, what is the sum of digits of the largest prime factor of x_6?

मान लो कि x_1 एक धनात्मक वास्तविक संख्या है और सभी $n \geq 1$ पूर्णाकों के लिए $x_{n+1} = 1 + x_1x_2......x_{n-1}x_n$ है। अगर $x_5 = 43$ तो x_6 के सबसे बड़े अभाज्य गुणनखंड के अंकों (digits) का योग ज्ञात करो।

Sol. **(13)**

$x_5 = 1 + x_1\, x_2\, x_3\, x_4 \qquad \Rightarrow x_1\, x_2\, x_3\, x_4 = 42$

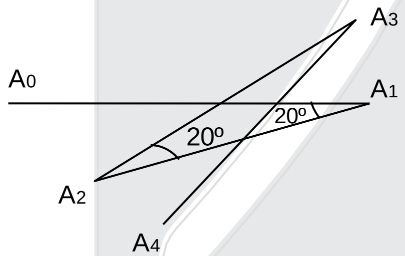
$x_6 = 1 + x_1\, x_2\, x_3\, x_4\, x_5 \qquad = 1+(42)\,(43) = 1807 = 13 \times 139$

$\Rightarrow$ largest prime factor सबसे बड़ा अभाज्य गुणनखंड = 139 $\Rightarrow$ sum of digits अंकों का योग = 13

4. An ant leaves the anthill for its morning exercise. It walks 4 feet east and then makes a 160° turn to the right and walks 4 more feet. It then makes another 160° turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again, what is the distance in feet it would have walked?

एक चींटी अपनी बांबी से सुबह के व्यायाम के लिए निकलती है। वह 4 फुट पूरब की ओर चलती है, फिर 160° दाएँ ओर मुड़ कर 4 फुट और चलती है। फिर वह एक बार और 160° दाएँ ओर मुड़ कर 4 फुट चलती है। अगर चींटी इसी क्रम में चलती रहती तो वापिस अपनी बांबी तक पहुंचने में उसने कुल कितनी दूरी (फुट में) चली होती?

Sol. **(36)**



Let A_0 (0,0)

A_1 (4cos0, 4sin0)

A_2 (4cos0 + 4cos160, 4sin0 + 4sin160)

$A_n = (0,0)$

$\Rightarrow 4(\cos 0 + \cos 160 + ... + \cos 160(n-1)) = 0$ and $4(\sin 0 + \sin 160 + ... + \sin 160(n-1)) = 0$

$\Rightarrow \sin(80n) = 0$

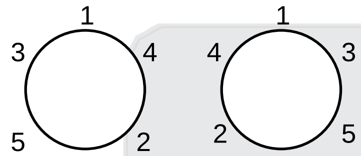
$\Rightarrow$ **n = 9**

$\Rightarrow$ **distance covered =** $4 \times 9 = 36$ feet

5. Five persons wearing badges with numbers 1, 2, 3, 4, 5 are seated on 5 chairs around a circular table. In how many ways can they be seated so that no two persons whose badges have consecutive numbers are seated next to each other? (Two arrangements obtained by rotation around the table are considered different.)

पाँच बिल्ले, जिन पर 1, 2, 3, 4, 5 लिखा है, पहने हुए पाँच लोग एक गोल मेज के चारों तरफ पाँच कुर्सियों पर बैठे है। वह ऐसी कितनी तरह से बैठ सकते है, जिससे कि कोई भी दो लोग जिनके बिल्ले पर लिखी संख्याएँ क्रमागत हों, वह अगल–बगल ना बैंठे हों? (एक बैठने का तरीका जो किसी दूसरे तरीके को घुमा देने से मिलता हो, उसे दूसरे तरीके से भिन्न माना जाएगा।)

Sol. **(10)**



two ways and 5 arrangements = 2 × 5 = 10

दो तरीके व पॉच क्रमचय = 2 × 5 = 10

6. Let $\overline{abc}$ be a three digit number with nonzero digits such that $a^2 + b^2 = c^2$. What is the largest possible prime factor of $\overline{abc}$?

मान लो कि $\overline{abc}$ तीन अंकों की ऐसी संख्या है, जिसके अंक अशून्य हैं व $a^2 + b^2 = c^2$ है। $\overline{abc}$ का सबसे बड़ा अभाज्य गुणनखण्ड क्या संभव है?

Sol. **(29)**

a,b,c form Pythagoras triplet

$\Rightarrow$ abc = 345 or 435

345 = 3 × 5 × 23 and 435 = 5 × 3 × 29

$\Rightarrow$ Largest possible prime factor = 29

a,b,c पाइथागोरस त्रिक बनाते हैं

$\Rightarrow$ abc = 345 or 435

345 = 3 × 5 × 23 और 435 = 5 × 3 × 29

$\Rightarrow$ अधिकतम संभावित अभाज्य गुणनखण्ड़ = 29

7. On a clock, there are two instants between 12 noon and 1 PM, when the hour hand and the minute hand are at right angles. The difference in minutes between these two instants is written as $a + \frac{b}{c}$, where a, b, c are positive integers, with b < c and b/c in the reduced from. What is the value of a + b + c?

एक घड़ी में दोपहर 12 बजे और दोपहर 1 बजे के बीच में दो बार ऐसा होता है जब घंटे की सूई और मिनट की सूई एक दूसरे के लम्ब होती है। इन दो समय के बीच में <u>मिनटों</u> में अंतर $a + \frac{b}{c}$ की तरह लिखें, जहाँ a, b, c धनात्मक पूर्णांक हैं, b < c व b/c भिन्न का न्यूनतम रूप (reduced from या simplest form) है। a + b + c का मान क्या है?

Sol. **(51)**

Minute hand covers 6^o in one minute and hour hand covers $\frac{1}{2}°$ in one minute

Let instant is x minutes past 12

$\Rightarrow 6x - \frac{x}{2} = 90$ or $270 \Rightarrow x = 16\frac{4}{11}$ or $49\frac{1}{11}$

Difference = $32\frac{8}{11} \Rightarrow a + b + c = 32 + 8 + 11 = 51$

मिनट की सुई एक मिनट में 6^o बनाती है तथा घंटे की सुई एक मिनट में $\frac{1}{2}°$ कोण बनाती है।

माना x उस क्षण 12 आगे है

$\Rightarrow 6x - \frac{x}{2} = 90$ or $270 \Rightarrow x = 16\frac{4}{11}$ or $49\frac{1}{11}$

अन्तर = $32\frac{8}{11} \Rightarrow a + b + c = 32 + 8 + 11 = 51$

8. How many positive integers n are there such that $3 \le n \le 100$ and $x^{2^n} + x + 1$ is divisible by $x^2 + x + 1$?

ऐसे कितने धनात्मक पूर्णांक n हैं जिनके लिए $3 \le n \le 100$ व $x^{2^n} + x + 1$ संख्या $x^2 + x + 1$ से भाज्य है?

Sol. **(49)**

Since ω and ω^2 are factors of $x^2 + x + 1$; so ω and ω^2 will be factors of $x^{2^n} + x + 1$

$\Rightarrow 2^n$ must be of the form $3k + 2$; $k \in I$. For this to happen, n must be odd.

So n = {3,5,7, …… 99}

Number of such numbers = 49

चूंकि ω तथा ω^2 , $x^2 + x + 1$ के गुणनखण्ड है ; इसलिए ω और ω^2 , $x^{2^n} + x + 1$ के भी गुणनखण्ड होगे

$\Rightarrow 2^n$ का रूप $3k + 2$; $k \in I$ होगा. इसके लिए n विषम होगा

इसलिए n = {3,5,7, …… 99}

इस प्रकार संख्याओ की सख्याऐं = 49

9. Let the rational number p/q be closest to but not equal to 22/7 among all rational numbers with denominator < 100. What is the value of p – 3q?

मान लो कि p/q ऐसा भिन्न है जो ऐसे भिन्न जिनका हर (denominator) < 100 हो और जो 22/7 से अलग हों, उनमें 22/7 के सबसे करीब पड़ता है। तो p – 3q का मान क्या होगा?

Sol. **(14)**

We have to have $\left|\frac{22}{7} - \frac{p}{q}\right|$ to be as small as possible for q < 100.

$\left|\frac{22}{7} - \frac{p}{q}\right| = \left|\frac{22q - 7p}{7q}\right|$ should be smallest.

$\Rightarrow |22q - 7p| = 1$

and q = 99

$\Rightarrow$ p = 311

p - 3q = 311 – 3 × 99 = 311 – 297 = 14

यहाँ $\left|\frac{22}{7}-\frac{p}{q}\right|$ छोटे से छोटा सम्भव होगा q < 100 के लिए

$\left|\frac{22}{7}-\frac{p}{q}\right| = \left|\frac{22q-7p}{7q}\right|$ न्यूनतम होगा

$\Rightarrow$ |22q – 7p| = 1

और q = 99

$\Rightarrow$ p = 311

p - 3q = 311 – 3 × 99 = 311 – 297 = 14

10. Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intersect Ω at A_1, B_1 and C_1 respectively and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2 and C_2, respectively. If the smallest angle of triangle ABC is 40°, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees?

मान लो कि ABC एक त्रिभुज है और Ω उसका बहिवृत्त है। कोण A, B व C के अन्तःसमद्विभाजक Ω से क्रमशः A_1, B_1 व C_1 में मिलते हैं और त्रिभुज $A_1B_1C_1$ के कोण A_1, B_1 व C_1 के अन्तःसमद्विभाजक Ω से क्रमशः A_2, B_2 व C_2, से मिलते है। अगर त्रिभुज ABC का सबसे छोटा कोण 40° है तो त्रिभुज $A_2B_2C_2$ के सबसे छोटे कोण का मान (degree) में क्या होगा?

Sol. **(55)**

$\angle A_1B_1C_1 = \frac{\pi}{2}-\frac{\angle ABC}{2}$

$\angle A_1C_1B_1 = \frac{\pi}{2}-\frac{\angle ACB}{2}$

$\angle B_1A_1C_1 = \frac{\pi}{2}-\frac{\angle BAC}{2}$

$\angle A_2B_2C_2 = \frac{\pi}{2}-\frac{\left(\frac{\pi}{2}-\frac{\angle ABC}{2}\right)}{2}$

$= \frac{\pi}{4}+\frac{\angle ABC}{4}$

similarly इसी प्रकार $\angle A_2C_2B_2 = \frac{\pi}{4}+\frac{\angle ACB}{4}$

and और $\angle B_2A_2C_2 = \frac{\pi}{4}+\frac{\angle BAC}{4}$

$\Rightarrow$ smaller angle of $\Delta A_2B_2C_2$ is 45º + $\left(\frac{40^o}{4}\right) = 55^o$

$\Rightarrow$ $\Delta A_2B_2C_2$ का सबसे छोटा कोण 45º + $\left(\frac{40^o}{4}\right) = 55^o$

11. How many distinct triangles ABC are there, up to similarity, such that the magnitudes of angles A, B and C in degrees are positive integers and satisfy

$$\cos A \cos B + \sin A \sin B \sin kC = 1$$

for some positive integer k, where kC does not exceed 360°?

ऐसे कितने अलग–अलग (मतलब असमरूप) त्रिभुज ABC हैं जिनमें कोण A, B व C का डिग्री में मान धनात्मक पूर्णांक है, और जो

$$\cos A \cos B + \sin A \sin B \sin kC = 1$$

समीकरण को किसी धनात्मक पूर्णांक k के लिए संतुष्ट करते हैं kC का मान 360° से ज्यादा नहीं है ?

Sol. **(6)**

$\cos A \cos B + \sin A \sin B \sin kC = 1$

$\Rightarrow \cos A \cos B + \sin A \sin B + \sin A \sin B \sin kC - \sin A \sin B = 1$

$\Rightarrow \sin A \sin B (\sin kC - 1) = 1 - \cos (A - B)$

So $(\sin kC - 1) = 0$ and $\cos (A - B) = 1$

$\Rightarrow kC = 90^{o}$ and $A = B$

Number of factors of 90^{o}

$90 = 2 \times 3^{2} \times 5$

Number of factors = $2 \times 3 \times 2 = 12$

for 6 factor A,B are integers

$\cos A \cos B + \sin A \sin B \sin kC = 1$

$\Rightarrow \cos A \cos B + \sin A \sin B + \sin A \sin B \sin kC - \sin A \sin B = 1$

$\Rightarrow \sin A \sin B (\sin kC - 1) = 1 - \cos (A - B)$

इसलिए $(\sin kC - 1) = 0$ और $\cos (A - B) = 1$

$\Rightarrow kC = 90^{o}$ और $A = B$

90^{o} के गुणनखण्डों की संख्या

$90 = 2 \times 3^{2} \times 5$

गुणनखड़ों की संख्या = $2 \times 3 \times 2 = 12$

12. A natural number k > 1 is called good if there exist natural numbers

$$a_1 < a_2 < \dots\dots\dots < a_k$$

such that $$\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \dots\dots + \frac{1}{\sqrt{a_k}} = 1$$

Let f(n) be the sum of the first n good numbers, $n \geq 1$. Find the sum of all values of n for which f(n + 5)/f(n) is an integer.

किसी प्राकृतिक संख्या k > 1 को हम अच्छा कहेंगे अगर ऐसी प्राकृतिक संख्याएँ

$$a_1 < a_2 < \dots\dots\dots < a_k$$

मौजूद हों जिससे कि

$$\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \dots\dots + \frac{1}{\sqrt{a_k}} = 1$$

हो। मान लो कि $n \geq 1$ के लिए f(n) पहली n अच्छी संख्याओं का योग है। ऐसी सभी संख्याओं n का योग ज्ञात करो जिनके लिए f(n + 5)/f(n) एक पूर्णांक है।

Ans **(18)**

Sol.

Let $a_1 = A_1^2, a_2 = A_2^2 ....a_k = A_k^2$

we have to check if it is possible for distinct natural number $A_1, A_2............A_k$ to satisfy,

$$\frac{1}{A_1}+\frac{1}{A_2}+........\frac{1}{A_k}=1$$

For k = 2; it is obvious that there do not exist distinct A_1, A_2, such that $\frac{1}{A_1}+\frac{1}{A_2}=1 \Rightarrow$ 2 is not a good number.

For k = 3; we have $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1 \Rightarrow$ 3 is a good number.

$\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)=1 \Rightarrow$ 4 is a good number.

k will be a good numbers for all $k \geq 3$

f(n) = 3 + 4 +… n terms = $\frac{n(n+5)}{2}$; f(n + 5) = $\frac{(n+5)(n+10)}{2}$

$\frac{f(n+5)}{f(n)} = \frac{n+10}{n} = 1 + \frac{10}{n}$

Which will be integer for n = 1, 2, 5 and 10

sum = 1 + 2 + 5 + 10 = 18.

माना $a_1 = A_1^2, a_2 = A_2^2 ....a_k = A_k^2$

यह जाँच करना है कि यह भिन्न प्राकृत संख्याओं $A_1, A_2............A_k$ के लिए संतुष्ठ करेगा

$$\frac{1}{A_1}+\frac{1}{A_2}+........\frac{1}{A_k}=1$$

k = 2 के लिए यह स्पष्ठ है कि यह A_1, A_2 भिन्न संख्याओ के लिए विद्यमान नहीं है $\frac{1}{A_1}+\frac{1}{A_2}=1 \Rightarrow$ 2 अच्छी संख्या नहीं है।

k = 3 के लिए $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1 \Rightarrow$ 3 अच्छी संख्या है

$\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)=1 \Rightarrow$ 4 अच्छी संख्या है।

f(n) = $\frac{n(n+5)}{2}$; f(n + 5) = $\frac{(n+5)(n+10)}{2}$

$\frac{f(n+5)}{f(n)} = \frac{n+10}{n} = 1 + \frac{10}{n}$

जो कि n = 1, 2, 5 और 10 के लिए पूर्णाक है

योगफल = 1 + 2 + 5 + 10 = 18.

13. Each of the numbers $x_1, x_2, \ldots\ldots\ldots x_{101}$ is ±1. What is the smallest positive value of $\sum_{1 \le i \le j \le 101} x_i x_j$?

$x_1, x_2, \ldots\ldots\ldots x_{101}$ में हर एक संख्या ±1 है। $\sum_{1 \le i \le j \le 101} x_i x_j$ का न्यूनतम धनात्मक मान क्या है ?

Sol. **(10)**

Let $S = \sum_{1 \le i \le j \le 10} x_i x_j$

We have

$$(x_1 + x_2 + x_3 + \ldots\ldots. x_{101})^2 = x_1^2 + x_2^2 + x_3^2 + \ldots\ldots.. + x_{101}^2 + 2S$$

$$\Rightarrow \quad 2S = \left(\sum_{i=1}^{101} x_i\right)^2 - \sum_{i=1}^{101} x_i^2$$

Since we have $x_i = \pm 1$, So $x_i^2 = 1$

So $2S = \left(\sum_{i=1}^{101} x_i\right)^2 - 101$

Since $\sum_{i=1}^{101} x_i$ will be an integer

So $\left(\sum_{i=1}^{101} x_i\right)^2$ will be a perfect square.

For smallest positive S ; $\left(\sum_{i=1}^{101} x_i\right)^2$ must be the smallest perfect square greater than 101.

So $\left(\sum_{i=1}^{101} x_i\right)^2 = 121$

$\Rightarrow \sum_{i=1}^{101} x_i = 11$ or -11

We can verify that the desired sum can be achieved by putting 45 x_i's to be –1 and 56 x_i's to be 1

So, 2S = 121 – 101 = 20

$\Rightarrow$ S = 10

माना $S = \sum_{1 \le i \le j \le 10} x_i x_j$

यहाँ

$$(x_1 + x_2 + x_3 + \ldots\ldots. x_{101})^2 = x_1^2 + x_2^2 + x_3^2 + \ldots\ldots.. + x_{101}^2 + 2S$$

$$\Rightarrow \quad 2S = \left(\sum_{i=1}^{101} x_i\right)^2 - \sum_{i=1}^{101} x_i^2$$

चूंकि $x_i = \pm 1$, इसलिए $x_i^2 = 1$

इसलिए $2S = \left(\sum_{i=1}^{101} x_i\right)^2 - 101$

चूंकि $\sum_{i=1}^{101} x_i$ पूर्णांक होगा

इसलिए $\left(\sum_{i=1}^{101} x_i\right)^2$ पूर्ण वर्ग होगा

न्यूनतम धनात्मक S के लिए ; $\left(\sum_{i=1}^{101} x_i\right)^2$, 101 से बड़ी सबसे छोटी पूर्ण वर्ग संख्या होगी

इसलिए $\left(\sum_{i=1}^{101} x_i\right)^2 = 121$

$\Rightarrow \sum_{i=1}^{101} x_i = 11$ or -11

यह असानी से ज्ञात किया जा सकता है अभीष्ठ योगफल को 45 x_i's, –1 हो और 56 x_i's , 1 हो

इसलिए, 2S = 121 – 101 = 20

⇒ S = 10

14. Find the smallest positive integer $n \geq 10$ such that n + 6 is a prime and 9n + 7 is a perfect square.
ऐसा सबसे छोटा धनात्मक पूर्णांक $n \geq 10$ ज्ञात करो जहाँ n + 6 अभाज्य हो और 9n + 7 एक पूर्ण वर्ग हो।

Sol. **(53)**

Since 9n + 7 is a perfect square
Let us assume that $9n + 7 = m^2$
Also n + 6 is a prime number ⇒ n + 6 must be odd number
⇒ n also must be an odd number
So Let us assume n = 2 k + 1 ….(ii) where k is an Integer
$9(2k + 1) + 7 = m^2$
$\Rightarrow 18k = m^2 - 16$
$\Rightarrow 18k = (m + 4)(m - 4)$ …..(iii)

Since 18k is even and m+4 and m–4 both are of some parity ⇒ m must also be even.
say m = 2p, where p is an Integer
Substituting in (iii)
$18k = (2p + 4)(2p - 4) = 4(p + 2)(p - 2)$
$\Rightarrow 9k = 2(p + 2)(p - 2)$
⇒ k must be even, say $k = 2\ell$ where ℓ is an Integer
$18\ell = 2(p + 2)(p - 2)$
$\Rightarrow 9\ell = (p + 2)(p - 2)$
⇒ p must be of the form 9q + 2 or 9q – 2, where q is an Integer
First take p = 9q – 2 ⇒ m = 2(9q – 2)
Take q = 1, m = 2 × 7 = 14 ⇒ $m^2 = 196$
9n + 7 = 196 ⇒ 9n = 189 ⇒ n = 21
⇒ n + 6 = 27 which is not prime
Next take p = 9q + 2 ⇒ m = 2(9q + 2)
Take q = 1 ⇒ m = 2 × 11 = 22 ⇒ $m^2 = 484$
9n + 7 = 484 ⇒ 9n = 477 ⇒ n = 53

$\Rightarrow$ n + 6 = 59 which is a prime

So n = 53 is the smallest such number

चूंकि 9n + 7 एक पूर्ण वर्ग है

माना कि $9n + 7 = m^2$ ……(i) जहां m एक पूर्णांक है तथा

n + 6 एक अभाज्य सख्या है धनात्मक पूर्णांक $\geq$ 10 के लिए $\Rightarrow$ n + 6 विषम संख्या होगी

$\Rightarrow$ n भी विषम संख्या होगी

इसलिए माना कि n = 2k + 1 ….(ii) जहां k एक पूर्णांक है

$9(2k + 1) + 7 = m^2$

$\Rightarrow 18k = m^2 – 16$

$\Rightarrow 18k = (m + 4)(m – 4)$ …..(iii)

चूंकि 18k सम है तथा m+4 और m–4 दोनों एक ही प्रकार के है $\Rightarrow$ m सम होगा

माना कि m = 2p, जहां p एक पूर्णांक है

(iii) में रखने पर

$18k = (2p + 4)(2p – 4) = 4(p + 2)(p – 2)$

$\Rightarrow 9k = 2(p + 2)(p – 2)$

$\Rightarrow$ k सम होगा माना कि $k = 2\ell$ जहां ℓ पूर्णांक है

$18\ell = 2(p + 2)(p – 2)$

$\Rightarrow 9\ell = (p + 2)(p – 2)$

$\Rightarrow$ p का रूप 9q + 2 या 9q – 2 के रूप में होगा , जहां q एक पूर्णांक है

पहले p = 9q – 2 लेने पर $\Rightarrow m = 2p = 2(9q – 2)$

q = 1 लेने पर, $m = 2 \times 7 = 14 \Rightarrow m^2 = 196$

$9n + 7 = 196 \Rightarrow 9n = 189 \Rightarrow n = 21$

$\Rightarrow$ n + 6 = 27 जो कि अभाज्य नहीं है

अब p = 9q + 2 लेने पर $\Rightarrow m = 2p = 2(9q + 2)$

q = 1 लेने पर $\Rightarrow m = 2 \times 11 = 22 \Rightarrow m^2 = 484$

$9n + 7 = 484 \Rightarrow 9n = 477 \Rightarrow n = 53$

$\Rightarrow$ n + 6 = 59 जो कि अभाज्य है

इसलिए n = 53 सबसे छोटी इस प्रकार की संख्या है

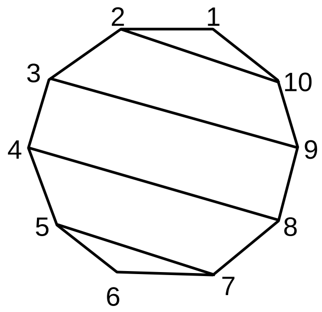
15. In how many ways can a pair of parallel diagonals of a regular polygon of 10 sides be selected

नियमित दसभुज के विकर्णों के ऐसे कितने जोड़े चुने जा सकते हैं जिसमें कि दोनों विकर्ण समानांतर हों ?

Sol. **(45)**

If we take gap of 2 sides then from figure shown we have ${}^4C_2$ ways

We can start with 1, 2, 3, 4, 5 so $5 \times {}^4C_2 = 5 \times 6$ = 30 ways

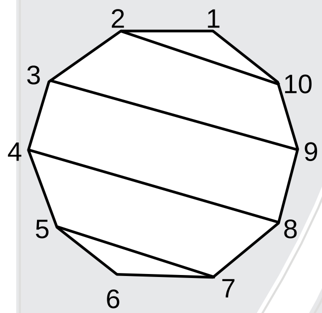


Now take 1 with 4 (gap of 3 sides) then we have 3 diagonals which are 14, 510, 69 so we have ${}^{3}C_{2}$ ways

We can start with 1, 2, 3, 4, 5 so $5 \times {}^{3}C_{2} = 5 \times 3 = 15$ ways

यदि चित्र से दो भुजाओं का अंतर लिया जाता है तब ${}^{4}C_{2}$ तरीके

यदि हम 1, 2, 3, 4, 5 से शुरूआत करते है तब $5 \times {}^{4}C_{2} = 5 \times 6 = 30$ तरीके



अब 1 को 4 (3 भुजाओं का अंतर लेते है।) तब 3 विकर्ण जो कि 14, 510, 69 में है के तरीके ${}^{3}C_{2}$

हम 1, 2, 3, 4, 5 के साथ शुरूआत कर सकते है। इसलिए $5 \times {}^{3}C_{2} = 5 \times 3 = 15$ तरीके

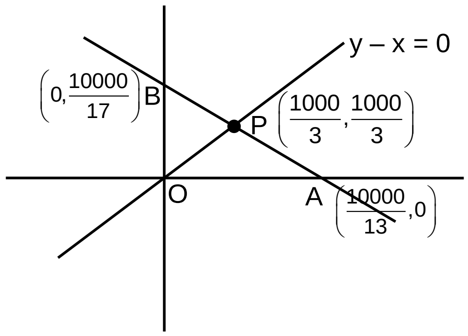
16. A pen costs Rs. 13 and a note book costs Rs. 17. A school spends exactly Rs. 10000 in the year 2017-18 to buy x pens and y note books such that x and y are as close as possible (i.e. |x – y| is minimum). Next year, in 2018-19, the school spends a little more than Rs. 10000 and buys y pens and x note books. How much more did the school pay ?

एक पेन का मूल्य रू. **13** है व एक कापी का मूल्य रू. **17** है। एक स्कूल **2017-18** में **x** पेन और **y** कापियॉं खरीदने में ठीक रू. **10000** खर्च करता है, ऐसे कि **x** व **y** जितना हो सकें उतना करीब हों (माने कि **|x – y|** न्यूनतम हो) अगले साल, **2018-19** में, स्कूल रू. **10000** से थोडा ज्यादा खर्च करता है, और **y** पेन व **x** कापियॉं खरीदता है। स्कूल ने कितना ज्यादा खर्च किया ?

Ans **(40)**

Sol. $13x + 17y = 10000$

$17x + 13y = 10000 + a$



$x = \frac{10000 - 17y}{13}$

y = 329

x = 339

$\Rightarrow$ 10040

17. Find the number of ordered triples (a,b,c) of positive integers such that 30a + 50b + 70c $\le$ 343.

धनात्मक पूर्णांकों के ऐसे कितने क्रमवार समुच्चय **(a,b,c)** हैं जिनके लिए **30a + 50b + 70c $\le$ 343** हैं।

Ans **(30)**

Sol. $30a + 50b + 70c \le 343$

$\Rightarrow 3a + 5b + 7c \le 34.3$

$\Rightarrow 3a + 5b + 7c \le 34$

$a, b, c \in N$, $a = 1 + p$, $b = 1 + q$, $c = 1 + r$

$3p + 5q + 7r \le 19$

If r = 0,

q = 0 $\Rightarrow$ p can take 7 values

q = 1 $\Rightarrow$ p can take 5 values

q = 2 $\Rightarrow$ p can take 4 values

q = 3 $\Rightarrow$ p can take 2 values

$\Rightarrow$ 18 values

If r = 1 then $3p + 5q \le 12$

q = 0 $\Rightarrow$ p can take 5 values

q = 1 $\Rightarrow$ p can take 3 values

q = 2 $\Rightarrow$ p can take 1 values

$\Rightarrow$ 9 values

If r = 2 then $3p + 5q \le 5$

q = 0 $\Rightarrow$ p can take 2 values

q = 1 $\Rightarrow$ p can take 1 value

$\Rightarrow$ 3 values

Total 18 + 9 + 3 = 30 values

$30a + 50b + 70c \le 343$

$\Rightarrow 3a + 5b + 7c \le 34.3$

$\Rightarrow 3a + 5b + 7c \le 34$

$a, b, c \in N$, $a = 1 + p$, $b = 1 + q$, $c = 1 + r$

$3p + 5q + 7r \le 19$

यदि $r = 0$,

$q = 0 \Rightarrow p$, 7 मान ले सकता है।

$q = 1 \Rightarrow p$, 5 मान ले सकता है।

$q = 2 \Rightarrow p$, 4 मान ले सकता है।

$q = 3 \Rightarrow p$,2 मान ले सकता है।

$\Rightarrow$ 18 मान

यदि $r = 1$ तब $3p + 5q \le 12$

$q = 0 \Rightarrow p$, 5 मान ले सकता है।

$q = 1 \Rightarrow p$, 3 मान ले सकता है।

$q = 2 \Rightarrow p$, 1 मान ले सकता है।

$\Rightarrow$ 9 मान

यदि $r = 2$ तब $3p + 5q \le 5$

$q = 0 \Rightarrow p$, 2 मान ले सकता है।

$q = 1 \Rightarrow p$, 1 मान ले सकता है।

$\Rightarrow$ 3 मान

कुल $18 + 9 + 3 = 30$ मान

18. How many ordered pairs (a, b) of positive integers with $a < b$ and $100 \le a,b \le 1000$ satisfy gcd (a,b) : lcm (a,b) = 1 : 495 ? .

धनात्मक पूर्णाकों के ऐसे कितने क्रमवार जोड़े **(a, b)** हैं जहाँ **a < b** व **100** $\le$ a,b $\le$1000 और जिनके लिए **gcd (a,b) : lcm (a,b) = 1 : 495 ?** (यहाँ **gcd =** महत्तम समापवर्तक **: lcm =** ल.स. लघुत्तम समापवर्तय**) .**

Sol. **(20)**

$ab = (gcd)^2.3^2.5.11$

$$\left(\frac{a}{gcd}\right)\left(\frac{b}{gcd}\right) = 3^2.5.11$$

$\left(\frac{a}{gcd}\right)$	$\left(\frac{b}{gcd}\right)$	gcd
3^2	5.11	12,13,….18 ($100 \le a,b \le 1000$)
5	3^2.11	no value कोई मान नहीं। ($100 \le a,b \le 1000$)
11	3^2.5	10, 11….22 ($100 \le a,b \le 1000$)
1	3^2.5.11	no value कोई मान नहीं। ($100 \le a,b \le 1000$)

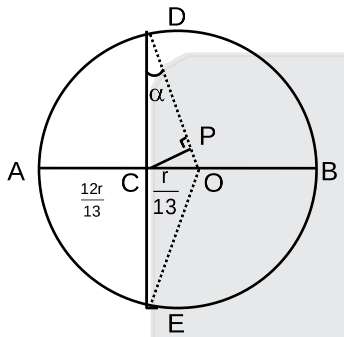
इसलिए so $7 + 13 = 20$ मान values

19. Let AB be a diameter of a circle and let C be a point on the segment AB such that AC : CB = 6 : 7. Let D be a point on the circle such that DC is perpendicular to AB. Let DE be the diameter through D. If [XYZ] denotes the area of the triangle XYZ. Find [ABD]/[CDE] to the nearest integer.

मन लो कि AB एक वृत का व्यास है और मान लो कि C रेखाखंड AB पर ऐसा बिंदु है जिससे कि AC : CB = 6 : 7 है। मान लो कि D वृत पर एक ऐसा बिंदु है कि DC खंड AB पर लंब है। मान लो कि DE बिंदु D से गुजरता हुआ वृत का व्यास है। अगर [XYZ] से हमारा मतलब त्रिभुज XYZ का क्षेत्रफल है तो [ABD]/[CDE] का मान सबसे करीबी पूर्णांक तक ज्ञात करो।

Ans **(13)**

Sol.



$$CD = \sqrt{r^2 - \frac{r^2}{13^2}} = \frac{r}{13}\sqrt{12\times14}$$

$$\tan\alpha = \frac{1}{\sqrt{12\times14}} \Rightarrow \sin\alpha = \frac{1}{\sqrt{12\times14+1}}$$

$$\frac{CP}{CD} = \sin\alpha = \frac{CP}{\frac{r}{13}\sqrt{12\times14}} = \frac{1}{\sqrt{12\times14+1}}$$

$$CP = \frac{r\sqrt{12\times14}}{13\sqrt{12\times14+1}}$$

$$\frac{|ABD|}{|CDE|} = \frac{\frac{1}{2}(2r)\times\frac{r}{13}\sqrt{12\times14}}{\frac{1}{2}(2r)\times\frac{r}{13}\frac{\sqrt{12\times14}}{\sqrt{12\times14+1}}} = \sqrt{12\times14+1} = 13$$

20. Consider the set E of all natural numbers n such that when divided by 11,12,13 respectively, the remainders, in that order, are distinct prime numbers in an arithmetic progression. If N is the largest number in E, find the sum of digits of N.

मन लो कि E ऐसी प्राकृतिक संख्याओं n का समुच्चय है जिन्हें 11,12,13 से भाग देने पर मिलने वाले शेष ऐसी अलग–अलग अभाज्य संख्याएँ हैं जो इसी क्रम में समान्तर श्रेणी में हैं। अगर N समुच्चय E की सबसे बड़ी संख्या है तो N के अंको का योग ज्ञात करो।

Ans **Bonus**

Sol. N can be of the type $(11\times12\times13\times\lambda) + 29$ (where λ belongs to integer) so there no largest value.

N का प्रकार $(11\times12\times13\times\lambda) + 29$ (जहाँ λ पूर्णाक है) इसलिए अधिकतम पूर्णाक मान नहीं।

21. Consider the set E = {5,6,7,8,9}. For any partition {A,B} of E, with both A and B non empty. Consider the number obtained by adding the product of elements of A to the product of elements of B. Let N be the largest prime number among these numbers. Find the sum of the digits of N.

समुच्चय E = {5,6,7,8,9} को लो। E के किसी भी विभाजन {A,B} के लिए, जहाँ A और B दोनों अरिक्त हैं, A के सदस्यों के गुणनफल का B के सदस्यों के गुणनफल से योग लेने पर मिलने वाली संख्या को लो। N इन सभी संख्याओं में सबसे बड़ा अभाज्य है। N के अंकों (digits) का योग ज्ञात करो।

Sol. **(17)**

one of the set A or set B contain only odd number

समुच्चय A या समुच्चय B में से केवल एक समुच्चय की विषम संख्या हो सकती है।

	set A	set B	Comment
Case I	5	6,7,8,9	5 + 6 × 7 × 8 × 9 = 3029 Which is divisible by 13
II	7	5,6,8,9	7 + 5 × 6 × 8 × 9 = 2167 Which is divisible by 11
III	9	5,6,7,8	9 + 5 × 6 × 7 × 8 is multiple of 3
IV	5,7	6,8,9	5 × 7 + 6 × 8 × 9 = 467 which is prime
V	5,9	6,7,8	5 × 9 + 6 × 7 × 8 is divisible by 3
VI	7,9	5,6,8	7 × 9 + 5 × 6 × 8 is divisible by 3
VII	5,7,9	6,8	5 × 7 × 9 + 6 × 8 is divisible by 3

Hence N is 467

Sum of the digit of N is 17

	समुच्चय A	समुच्चय B	तर्क
स्थिति I	5	6,7,8,9	5 + 6 × 7 × 8 × 9 = 3029 जो कि 13 से विभाजित है।
II	7	5,6,8,9	7 + 5 × 6 × 8 × 9 = 2167 जो कि 11 से विभाजित है।
III	9	5,6,7,8	9 + 5 × 6 × 7 × 8, 3 का गुणज है।
IV	5,7	6,8,9	5 × 7 + 6 × 8 × 9 = 467 जो कि अभाज्य है।
V	5,9	6,7,8	5 × 9 + 6 × 7 × 8 , 3 से विभाजित है।
VI	7,9	5,6,8	7 × 9 + 5 × 6 × 8, 3 से विभाजित है।
VII	5,7,9	6,8	5 × 7 × 9 + 6 × 8, 3 से विभाजित है।

अतः N का मान 467 है।

N के अंकों का योगफल 17 है।

22. What is the greatest integer not exceeding the sum $\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}$

ऐसा सबसे बड़ा पूर्णांक ज्ञात करो जो कि योग $\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}$ से बड़ा ना हो।

Sol. **(78)**

$$\int_1^{1600} \frac{dx}{\sqrt{x}} < \sum_{n=1}^{1599} \frac{1}{\sqrt{n}} < 1 + \int_1^{1599} \frac{dx}{\sqrt{x}}$$

$$\left|2\sqrt{x}\right|_1^{1600} < \sum_{n=1}^{1599} \frac{1}{\sqrt{n}} < 1 + \left|2\sqrt{x}\right|_1^{1599}$$

$$78 < \sum_{n=1}^{1599} \frac{1}{\sqrt{n}} < 2\sqrt{1599} - 1$$

$$78 < \sum_{n=1}^{1599} \frac{1}{\sqrt{n}} < 78.97.....$$

$$\Rightarrow \left[\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}\right] = 78$$

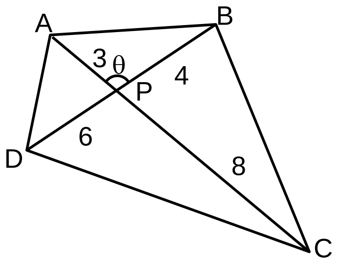
23. Let ABCD be a convex cyclic quadrilateral. Suppose P is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of ABCD is the least. If {PA, PB, PC, PD} = {3, 4, 6, 8}

What is the maximum possible area of ABCD ?

मान लो कि ABCD एक उत्तल (convex) चक्रिय (cyclic) चतुर्भज है। मान लो कि P चतुर्भज के तल में ऐसा बिंदु है जिसकी चतुर्भज ABCD के शीर्षो से दूरियों का योग न्यूनतम है। अगर {PA, PB, PC, PD} = {3, 4, 6, 8} तो ABCD का अधिकतम संभव क्षेत्रफल कितना है ?

Ans **(55)**

Sol. P must be point of intersection of diagonals AC and BD



Let $\angle APB = \theta$, then

area of $\Delta PAB = \frac{1}{2} \times 3 \times 4 \times \sin\theta$

area of $\Delta PAD = \frac{1}{2} \times 3 \times 6 \times \sin(\pi - \theta)$

area of $\Delta PDC = \frac{1}{2} \times 8 \times 6 \times \sin\theta$

area of $\Delta PCB = \frac{1}{2} \times 8 \times 4 \times \sin(\pi - \theta)$

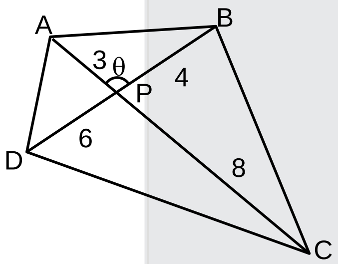
$\Rightarrow$ area of quadrilateral ABCD is

$\frac{1}{2}(12 + 18 + 48 + 32) \sin\theta$

$= (6 + 9 + 24 + 16) \sin\theta$

Maximum area of quadrilateral ABCD is $6 + 9 + 24 + 16 = 55$.

P, विकर्ण AC और BD का प्रतिच्छेद बिन्दु होगा।



माना $\angle APB = \theta$, तब

ΔPAB का क्षेत्रफल $= \frac{1}{2} \times 3 \times 4 \times \sin\theta$

ΔPAD का क्षेत्रफल $= \frac{1}{2} \times 3 \times 6 \times \sin(\pi - \theta)$

ΔPDC का क्षेत्रफल $= \frac{1}{2} \times 8 \times 6 \times \sin\theta$

ΔPCB का क्षेत्रफल $= \frac{1}{2} \times 8 \times 4 \times \sin(\pi - \theta)$

$\Rightarrow$ चतुर्भुज ABCD का क्षेत्रफल

$\frac{1}{2}(12 + 18 + 48 + 32) \sin\theta = (6 + 9 + 24 + 16) \sin\theta$

चतुर्भुज ABCD का अधिकतम क्षेत्रफल $6 + 9 + 24 + 16 = 55$.

24. A $1 \times n$ rectangle ($n \geq 1$) is divided into n unit (1×1) squares. Each square of this rectangle is coloured red, blue or green. Let f(n) be the number of colourings of the rectangle in which there are an even number of red squares. What is the largest prime factor of f(9) / f(3) ? (The number of red squares can be zero)

एक $1 \times n$ आयत ($n \geq 1$) को n इकाई (1×1) वर्गों में बॉट देते हैं। इस आयत के हर वर्ग को लाल,नीला या हरा रंग देते हैं। मान लो कि f(n) वह संख्या है जितने तरीकों से ऐसे रंग सकते है कि लाल रंग के आयतों की संख्या सम हो। f(9) / f(3) का सबसे बड़ा अभाज्य गुणनखंड क्या है ? (लाल वर्गों की संख्या शून्य हो सकती है।)

Sol. **(37)**

$f(n) = {}^nC_0\, 2^n + {}^nC_2 2^{n-2} + {}^nC_4\, 2^{n-4} + \ldots\ldots\ldots\ldots$

Now अब $(2 + 1)^n = {}^nC_0 2^n + {}^nC_1 2^{n-1} + {}^nC_2 2^{n-2} + {}^nC_3\, 2^{n-3} + \ldots\ldots\ldots\ldots$ (1)

$(2 - 1)^n = {}^nC_0 2^n - {}^nC_1 2^{n-1} + {}^nC_2 2^{n-2} - {}^nC_3\, 2^{n-3} + \ldots\ldots\ldots\ldots$ (2)

add (1) and (2)

(1) व (2) को जोड़ने पर

$3^n + 1 = 2[{}^nC_0\, 2^n + {}^nC_2 2^{n-2} + {}^nC_4\, 2^{n-4} + \ldots\ldots\ldots\ldots]$

$f(n) = \frac{3^n + 1}{2}$

$f(9) = \frac{3^9 + 1}{2}$, $f(3) = \frac{3^3 + 1}{2}$

$\frac{f(9)}{f(3)} = \frac{3^9 + 1}{3^3 + 1} = 3^6 - 3^3 + 1 = 703 = 19 \times 37$

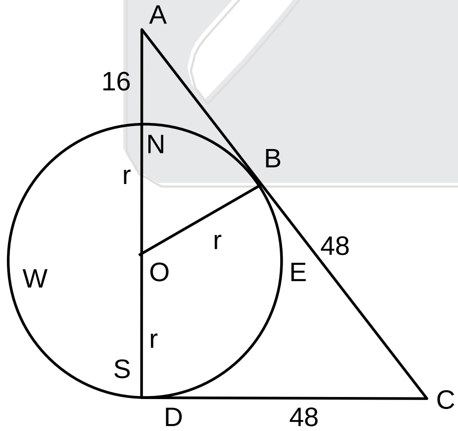
$\Rightarrow$ Largest prime factor is 37.

अधिकतम अभाज्य गुणनखण्ड 37 है।

25. A village has a circular wall around it, and the wall has four gates pointing north, south east and west. A tree stands outside the village, 16m north of the north gate, and it can be just seen appearing on the horizon from a point 48 m east of the south gate. What is the diameter in meters, of the wall that surrounds the village ?

एक गॉव के चारों तरफ एक गोल दीवार है जिसमें पूरब, पश्चिम, उत्तर व दक्षिण की तरफ चार द्वार हैं। गॉव के बाहर लगा एक पेड़ उत्तर द्वार के 16 मी. उत्तर में है और वह दक्षिण द्वार के 48 मी. पूरब स्थित बिन्दु से बस क्षितिज पर उभरता हुआ थोडा सा दिखाई देता है। गॉव को घेरती हुई दीवार का मीटर में व्यास क्या है ?`

Sol. **(48)**



Let radius equals to r

माना त्रिज्या r है।

$AB = \sqrt{AO^2 - OB^2} = \sqrt{(16 + r)^2 - r^2} = \sqrt{256 + 32r}$

$AD^2 + CD^2 = AC^2$

$\Rightarrow (48)^2 + (2r + 16)^2 = \left(48 + \sqrt{256 + 32r}\right)^2$

$\Rightarrow r^2 + 8r = 24\sqrt{256 + 32r}$

$\Rightarrow r(r + 8)\_ = 24 \times 4\sqrt{2}\ \ \sqrt{r + 8}$

$\Rightarrow r\sqrt{r + 8} = 24\sqrt{32}$

$\Rightarrow r = 24$

Diameter equal to 48

व्यास 48 के बराबर है।

26. Positive integers x, y, z satisfy xy + z = 160. Compute the smallest possible value of x + yz.

धनात्मक पूर्णांक x, y, z समीकरण xy + z = 160 को संतुष्ट करते हैं। x + yz का न्यूनतम संभव मान ज्ञात करो।`

Sol. **(50)**

$x + yz = \frac{160 - z}{y} + yz$

$\frac{160}{y} + \frac{z(y^2 - 1)}{y} = \frac{(160 - z)}{y} + \frac{zy^2}{y} = \frac{160 - z}{y} + zy$

At particle value of z it is greater

than equal to $2\sqrt{z(160 - z)}$

$\Rightarrow$ least value is $2\sqrt{z(160 - z)}$

but integer also

Now least for least value z is also

Case I

$z = 1, x + y\,z = \frac{159}{y} + y$

$\Rightarrow$ minimum value is at y = 3 which is 56

Case II

$z = 2, x + y\,z = \frac{158}{y} + 2y$

$\Rightarrow$ minimum value at y = 2 which is 83 (rejected)

Case III

$z = 3, x + y\,z = \frac{157}{y} + 3y$

$\Rightarrow$ minimum at y = 1 which is 160 (rejected)

Case IV

$z = 4, x + y\,z = \frac{156}{y} + 4y$

$\Rightarrow$ minimum at y = 6 which is 50 (accept)

Case V

$z = 5, x + yz = \frac{155}{y} + 5y$

minimum value at y = 5 which is 56 (reject)

Case VI

$z = 6, x + yz = \frac{154}{y} + 6y \geq 2\sqrt{924} > 50$

similarly in all the cases minimum value is greater than 50. then answer is 50

Hindi $x + yz = \frac{160 - z}{y} + yz$

$$\frac{160}{y} + \frac{z\left(y^2 - 1\right)}{y} = \frac{\left(160 - z\right)}{y} + \frac{zy^2}{y} = \frac{160 - z}{y} + zy$$

z का एक मान

$2\sqrt{z\left(160 - z\right)}$ से बड़ा होगा

$\Rightarrow$ न्यूनतम मान $2\sqrt{z\left(160 - z\right)}$

परन्तु पूर्णांक भी

अब z का न्यूनतम से न्यूनतम मान के लिए

स्थिति I

$z = 1, x + yz = \frac{159}{y} + y$

$\Rightarrow$ y = 3 पर न्यूनतम मान 56 है

स्थिति II

$z = 2, x + yz = \frac{158}{y} + 2y$

$\Rightarrow$ y = 2 पर न्यूनतम मान 83 है (अस्वीकार्य)

स्थिति III

$z = 3, x + yz = \frac{157}{y} + 3y$

$\Rightarrow$ y = 1 पर नयूनतम मान 160 है (अस्वीकार्य)

स्थिति IV

$z = 4, x + yz = \frac{156}{y} + 4y$

$\Rightarrow$ y = 6 न्यूनतम मान 50 है (स्वीकार्य)

स्थिति V

$z = 5, x + yz = \frac{155}{y} + 5y$

y = 5 न्यूनतम मान 56 है (अस्वीकार्य)

स्थिति VI

$\not z = 6, x + y \not z = \frac{154}{y} + 6y \ge 2\sqrt{924} > 50$

इसी प्रकार सभी स्थितियों में न्यूनतम मान 50 से बड़ा है तब उत्तर 50 है

27. We will say that a rearrangement of the letters of a word has no fixed letters if. When the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, H B R A T A is a rearrangement with no fixed letters of B H A R A T. How many distinguishable rearrangement with no fixed letters does B H A R A T have ? (The two A's are considered identical)

हम कहेंगें कि किसी शब्द के अक्षरों के क्रमचय में कोई स्थिर अक्षर नहीं है अगर, जब हम क्रमचय को शब्द के ठीक नीचे लिखें, तो किसी भी स्तंभ में एक ही अक्षर ऊपर नीचे दोनों जगह नहीं होगा। जैसे कि शब्द B H A R A T के क्रमचय H B R A T A में कोई स्थिर अक्षर नहीं है। B H A R A T के अलग–अलग ऐसे कितने क्रमचय हैं जिनमें कोई स्थिर अक्षर नही हैं ? (यहाँ मानेंगे कि दोनों A में कोई भेद नहीं है।)

Sol. **(84)**

Let use assume the 2A's to A_1 and A_2

$BHA_1\ RA_2T$

Number of rearrangements of these 6.

$$= \lfloor 6 \left(\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} + \frac{1}{\lfloor 6} \right) = 265$$

Let P be the set when A_2 occupies place of A_1 and θ be the set when A_1 occupies place of A_2

$n(P) = 53, \quad n(Q) = 53$

$n(n \cap Q) = 9$

Hence required arrangements

$= \frac{1}{2}(265 - n(P \cup Q))$

$= \frac{1}{2}(265 - 106 + 9)$

$= \frac{168}{2} = 84$

माना कि 2A's इस प्रकार A_1 और A_2

$BHA_1\ RA_2T$

इन 6 के पुनः व्यवस्थीकरण की संख्या

$$= \lfloor 6 \left(\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} + \frac{1}{\lfloor 6} \right) = 265$$

माना P एक समुच्चय है जब A_2 , A_1 के स्थान पर आता है और θ समुच्चय है जब A_1 , A_2 के स्थान पर आता है

$n(P) = 53, \quad n(Q) = 53$

$n(n \cap Q) = 9$

अतः अभीष्ठ क्रमचय

$= \frac{1}{2}(265 - n(P \cup Q))$

$= \frac{1}{2}(265 - 106 + 9)$

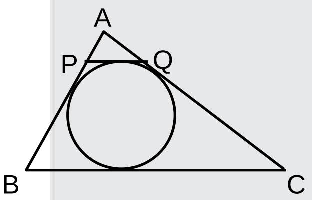
$= \frac{168}{2} = 84$

28. Let ABC be a triangle with sides 51, 52, 53. Let Ω denote the incircle of ΔABC. Draw tangents to Ω which are parallel to the sides of ABC. Let r_1, r_2, r_3 be the inradii of the three corner triangles so formed. Find the largest integer that does not exceed $r_1 + r_2 + r_3$.

मान लो कि ABC एक त्रिभुज है जिसकी भुजाओं का मान 51, 52, 53 है। मान लो कि Ω त्रिभुज ΔABC का अंतः वृत्त है। Ω के वह स्पर्शक बनाओ जो कि ABC की भुजाओं के समानांतर हैं। मान लो कि कोनों पर बने तीन त्रिभुजों का अंतःव्यास r_1, r_2, r_3 है। ऐसा अधिकतम पूर्णांक कौनसा होगा जो $r_1 + r_2 + r_3$ के मान से ज्यादा न हो ?

Ans **(15)**

Sol.



Let PQ be one of the required tangents is parallel to BC and meets sides AB and AC at P and Q respectively.

Further Let PQ = x

and BC = 51

Now ΔABC is similar to ΔAPQ so $\frac{x}{a} = \frac{r_1}{r}$

Similarly $\frac{y}{6} = \frac{r_2}{r}$

Hence $\frac{r_1 + r_2 + r_3}{r} = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$

Now $\frac{x}{a} = \frac{h - 2r}{h}$ where h is altitude

$\frac{x}{a} = 1 - \frac{2\Delta}{s.2\Delta}$, $x = 1 - \frac{a}{s}$

$\sum \frac{x}{a} = 3 - \frac{a + b + c}{s} = 1$

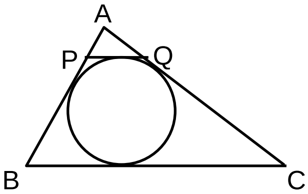
so $r_1 + r_2 + r_3 = r$

Now $r = \sqrt{\frac{78(27)(26)(25)}{78}}$

$= \sqrt{\frac{3^3.5^2.2.13}{2.3.13}}$

$= \sqrt{3^2.5^2} = 15$

Sol.



माना PQ स्पर्श रेखा में से एक स्पर्श रेखा है जो BC के समान्तर है तथा AB और AC को क्रमशः P और Q पर मिलती है

माना Let PQ = x

और BC = 51

अब ΔABC, ΔAPQ के समरूप है $\frac{x}{a} = \frac{r_1}{r}$

इसी प्रकार $\frac{y}{6} = \frac{r_2}{r}$

अतः $\frac{r_1 + r_2 + r_3}{r} = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$

अब $\frac{x}{a} = \frac{h - 2r}{h}$ जहां h शीर्ष लम्ब है

$$\frac{x}{a} = 1 - \frac{2\Delta}{s.2\Delta}, \; x = 1 - \frac{a}{s}$$

$$\sum \frac{x}{a} = 3 - \frac{a+b+c}{s} = 1$$

इसलिए $r_1 + r_2 + r_3 = r$

अब $r = \sqrt{\frac{78(27)(26)(25)}{78}}$

$= \sqrt{\frac{3^3.5^2.2.13}{2.3.13}}$

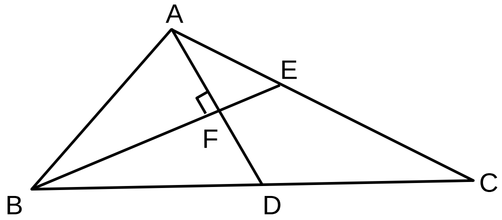
$= \sqrt{3^2.5^2} = 15$

29. In a triangle ABC, the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If AD = 7 and BE = 9, find the integer nearest to the area of triangle ABC.

एक त्रिभुज ABC में मध्यिका AD (जहाँ D भुजा BC पर है) और कोण–समद्विभाजक BE (जहाँ E भुजा AC पर है) लम्ब हैं। अगर AD = 7 व BE = 9 तो ABC के क्षेत्रफल के सबसे करीबी पूर्णांक का मान ज्ञात करो।

Ans **(47)**

Sol.



Let AD and BE meet at F

Now $\angle ABF = \angle FBD = \frac{B}{2}$ and $\angle AFB = \angle BFD = 30^o$ and BF is common to triangles ABF and BFD

hence the two triangles are

Cengruent so AF = FD= $\frac{7}{2}$

Also, AB = BD, $\Rightarrow$ AB : BC = 1 :2

$\Rightarrow$ AE : EC = 1 : 2

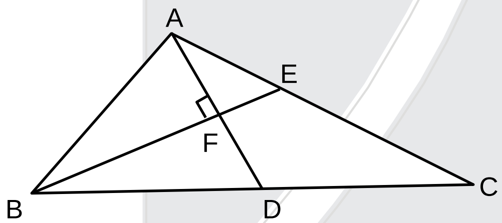
So area of triangle ABC = 3(Area of ΔABE)

$= 3\left(\frac{1}{2} \times AF \times BE\right)$

$= \frac{3}{2} \times \frac{7}{2} \times 9 = \frac{189}{4} = 47.25$

Hence nearest integer = 47

Sol.



माना AD और BE बिन्दु F पर मिलते है

अब $\angle ABF = \angle FBD = \frac{B}{2}$ और $\angle AFB = \angle BFD = 30^o$ और BF त्रिभुज ABF और BFD में उभयनिष्ठ है

अतः दोनों त्रिभुज सर्वागसम है

इसलिए AF = FD= $\frac{7}{2}$

तथा, AB = BD, $\Rightarrow$ AB : BC = 1 :2

$\Rightarrow$ AE : EC = 1 : 2

त्रिभुज ABC का क्षेत्रफल = 3(त्रिभुज ΔABE का क्षेत्रफल)

$= 3\left(\frac{1}{2} \times AF \times BE\right)$

$= \frac{3}{2} \times \frac{7}{2} \times 9 = \frac{189}{4} = 47.25$

अतः निकटतम पूर्णाक = 47

30. Let E denote the set of all natural numbers n such that 3 < n < 100 and the set {1, 2, 3,….,n } can be partitioned in to 3 subsets with equal sums. Find the number of elements of E.

मान लो कि E ऐसी प्राकृतिक संख्यों n का समुच्चय है जिनके लिए 3 < n < 100 और जिनके लिए समुच्चय {1, 2, 3,….,n } को ऐसे तीन उपसमुच्चयों में विभाजित किया जा सकता है जिनका योग बराबर है। E के सदस्यों की संख्या ज्ञात करो।

Ans **(64)**

Sol. {1, 2………..n}

This set can be partitioned into 3 subsets with equal sums so total sum is divisible by 3 ,

$\frac{n(n+1)}{2}$ is divisible by 3.

So n will be of the form 3 λ, 3λ + 2

or for convenience we can take n = 6λ, 6λ +2, 6λ +3, 6λ +5

If n = 6λ then we can group numbers in bundles of 6.

In each bundle we can select numbers like 1, 2, 3, 4, 5, 6 (16, 25, 34)

If n = 6λ + 2 then we can club last bundle of 8 numbers rest can be partitioned and those 8 numbers can be done 1, 2, 3, 4, 5, 6, 7, 8 (1236, 48)

If n = 6λ + 3, we can club last nine numbers and rest can be partitioned 1, 2, 3, 4, 5, 6, 7, 8, 9 (12345, 69, 78)

If n = 6λ + 5 we can take last five numbers. Rest can be partitioned 1, 2, 3, 4, 5 (14, 23, 5)

Hence we can select any number of form 6λ(16), 6λ + 2(16), 6λ + 3(16), 6λ + 5(16)

so total 64 numbers.

Sol. {1, 2………..n}

इस समुच्चय को 3 उपसमुच्चय में विभाजित किया जाता है जो योगफल में बराबर है इसलिए कुल योगफल 3 से विभाजित है

$\frac{n(n+1)}{2}$, 3 से विभाजित है.

इसलिए n का रूप 3 λ, 3λ + 2 का होगा

या सुविधा के लिए हम n = 6λ, 6λ +2, 6λ +3, 6λ +5 ले सकते है

यदि n = 6λ तब इन संख्याओ को 6 समुह में लिख सकतें है

प्रत्येक बड़ल में चुनी गई संख्याऐं लिख सकते है 1, 2, 3, 4, 5, 6 (16, 25, 34)

यदि n = 6λ + 2 तब इन 8 संख्याओं के अन्य भागों में विभाजित किया जा सकता है। तथा वे 8 संख्याऐ निम्न तरह कि हो सकती है 1, 2, 3, 4, 5, 6, 7, 8 (1236, 48)

यदि n = 6λ + 3, हम अन्तिम 9 संख्याओं को एक साथ रख सकते है तथा शेष संख्याओं को विभाजित कर सकते है1, 2, 3, 4, 5, 6, 7, 8, 9

(12345, 69, 78)

यदि n = 6λ + 5 हम अन्तिम 5 संख्याऐ ले सकते है और शेष संख्याओ को विभाजित कर सकते है 1, 2, 3, 4, 5 (14, 23, 5)

अतः हम 6λ(16), 6λ + 2(16), 6λ + 3(16), 6λ + 5(16) के रूप की किसी भी संख्या को ले सकते है इसलिए कुल 64 संख्याऐ है।

INSTRUCTION

Number of Questions: 30 **Max. Marks: 102**

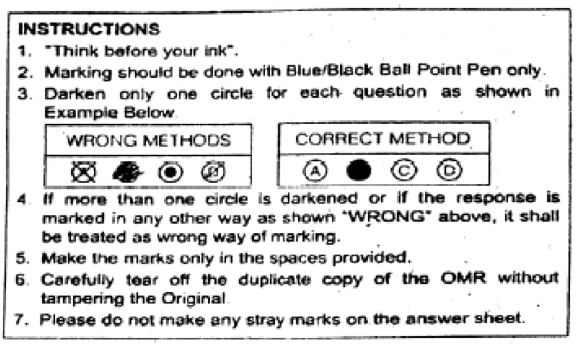
1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

2. The correction is done by machine through scanning. On OMR sheet, darken bubbles completely with a black pencil or a black blue pen. Darken the bubbles completely only after you are sure of your answer: else, erasing lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name, email address and date of birth entered on the OMR sheet will be your login credentials for accessing your PROM score.

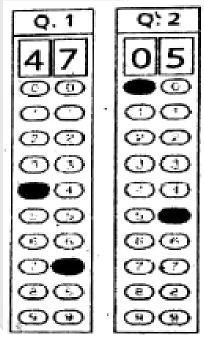
4. Incompletely, incorrectly and carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.


INSTRUCTIONS
1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.

WRONG METHODS	CORRECT METHOD

4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.



6. The answer you write on OMR sheet is irrelevant. The darken bubble will be considered as your final answer.

7. Questions 1 to 6 carry 2 marks each: Questions 7 to 21 carry 3 marks each: Questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current Olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

1. Consider the sequence of number $\left[n+\sqrt{2n}+\frac{1}{2}\right]$ for n ≥ 1, where [x] denotes the greatest integer not exceeding x. If the missing integers in the sequence are $n_1 < n_2 < n_3 < \ldots$ then find the n_{12}.

Sol. **(78)**

$S_n = n + \left[\sqrt{2n} + 0.5\right],\ n \geq 1$

[·] = G.I.F.

n	$\sqrt{2n}$	$\left[\sqrt{2n}+0.5\right]$	$n+\left[\sqrt{2n}+0.5\right]$
1	1.4	1	2
2	2	2	4
3	2.4	2	5
4	2.8	3	7
5	3.1	3	8
6	3.4	3	9
7	3.7	4	11
8	4	4	12
9	4.2	4	13
10	4.4	4	14
11	4.6	5	16
12	4.8	5	17
13	5.0	5	18
14	5.2	5	19
15	5.4	5	20
16	5.6	6	22
17	5.8	6	23
18	6	6	24
19	6.1	6	25
20	6.3	6	26
21	6.4	6	27
22	6.6	7	29

By observing pattern,

Missing numbers = 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78,....

+2 +3 +4 +5 +6

$\therefore$ 12th number in series = 78

2. If $x = \sqrt{2}+\sqrt{3}+\sqrt{6}$ is a root of $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c, d are integers, what is the value of $|a+b+c+d|$?

Sol. **(93)**

$x = \sqrt{2}+\sqrt{3}+\sqrt{6}$ root of equation :

$x^4 + ax^3 + bx^2 + cx + d = 0$; a, b, c, d $\in$ Z

Now,

$$\left(x-\sqrt{2}\right)^2 = \left(\sqrt{3}+\sqrt{6}\right)^2 \Rightarrow \quad x^2 - 2\sqrt{2}x + 2 = 9 + 6\sqrt{2}$$

$$\Rightarrow \quad x^2 - 7 = 2\sqrt{2}\left(3+x\right)$$

(On squaring both sides)

$\Rightarrow \quad x^4 - 14x^2 + 49 = 8\left(x^2 + 6x + 9\right)$

$\Rightarrow \quad x^4 - 22x^2 - 48x - 23 = 0$

$\therefore \quad$ a = 0, b = – 22, c = – 48 and d = – 23

$\therefore \quad$ |a + b + c + d| = 93

3. Find the number of positive integers less than 101 that can not be written as the difference of two squares of integers.

Sol. **(25)**

Let, c $\rightarrow$ be positive integer ≤ 100 such that $c^2 = a^2 - b^2$; a, b $\in$ Z.

C – I : Difference of a and b is 1.

$\therefore \quad 1^2 - 0^2 = 1$

$2^2 - 1^2 = 3$

$3^2 - 2^2 = 5$

$4^2 - 3^2 = 7$

$5^2 - 4^2 = 9$

$\vdots$

$50^2 - 49^2 = 99$

All odd number upto 100 can be expressed as difference of two squares.

C – II : Difference of a, b is 2

$2^2 - 0^2 = 4$

$3^2 - 1^2 = 8$

$4^2 - 2^2 = 12$

$\vdots$

$26^2 - 24^2 = 100$

All multiples of ‘4’ upto 100 can be expressed as difference of two squares.

C – III : Difference of a, b is 3

This case will give odd numbers which are already counted

C – IV : Difference of a, b is 4

This case will give multiples of 4 which are already counted

Similarly, the remaining cases will all give number already counted in

C – I and C – II

$\therefore \quad$ No. of numbers which can be expressed as difference of squares

= (odd numbers) + (multiples of 4) = 50 + 25 = 75

$\therefore \quad$ Required numbers which cannot be expressed as a difference of squares of two integers are = 100 – 75 = 25

4. Let $a_1 = 24$ and form the sequence a_n, $n \ge 2$ by $a_n = 100a_{n-1} + 134$. The first few terms are

24, 2534, 253534, 25353534, ……..

What is the least value of n for which a_n is divisible by 99?

Sol. **(88)**

$a_1 = 24$ & $a_n = 100 \,.\, a_{n-1} + 134$

First few terms :

$a_1 = 24$

$a_2 = 2534$

$a_3 = 253534$

$a_4 = 25353534$

$\therefore a_n = 2\underbrace{535353........53}_{(n-1)\text{Times}'53'}4$

Now, an $\rightarrow$ divisible by 99 $\Rightarrow$ by 9 & 11 both

Sum of digits = 6 + 8 (n – 1)

To be divisible by 9,

n = 7, 16, 25, 34, 43, 52, 61, 70, 79, 88, ….

$a_7 = 2\underbrace{535353535353}_{6\text{Times}'53'}4$

But $a_7 \rightarrow$ Not divisible by 11.

$a_{16} = 2\underbrace{53535353553......53}_{15\text{Times}'53'}4$

Similarly, $a_{16} \rightarrow$ Not divisible by 11.

Now, n = 88

$a_{88} = 2\underbrace{5353......53}_{87\text{Times}'53'}4$

Divisibility by $11 \rightarrow \left|(2+3+3.....)-(5+5+....4)\right|$

$= \left|263-439\right|$

$= 176$

$\therefore$ Least n = 88

5. Let N be the smallest positive integer such that $N+2N+3N+....+9N$ is a number all whose digits are equal. What is the sum of the digits of N?

Sol. **(37)**

$N \in Z^+$

P = N + 2N + 3N + ……… + 9N = 45N

$45 = \underbrace{555..........5}_{a\text{ times}}$

Only 55 ………. 5 can be repeated as 45N will have units place = 0 or 5

Also as 45N $\rightarrow$ multiple of '9' also.

$\therefore$ If a = 9 $\Rightarrow \dfrac{555555555}{45} = N$

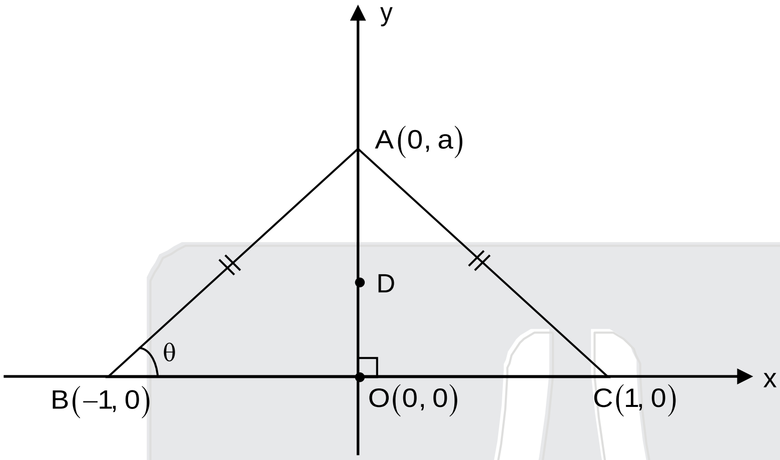
$\therefore N = 12345679$

Sum of digits of N = 37.

6. Let ABC be a triangle such that AB = AC. Suppose the tangent to the circumcircle of ABC at B is perpendicular of AC. Find $\angle ABC$ measured in degrees.

Sol. **(30)**

Consider the Isosceles triangle with vertex A, B, C such that AB = AC



By symmetry $\rightarrow$ circumcentre will lie on y – axis

$\therefore$ D $\equiv$ (0 , k) $\rightarrow$ circumcenter

Radius of circumcircle = DA = DB = DC

$\therefore \sqrt{(a-k)^2} = \sqrt{k^2+1}$

$\Rightarrow a^2 + k^2 - 2ak = k^2 + 1$

$\Rightarrow a^2 - 2ak = 1$ ...(1)

Now

Circumcircle : $x^2 + (y-k)^2 = k^2 + 1$

$\therefore$ Slope of tangent at B,

$2x + 2(y-k)y' = 0$

$\Rightarrow y'|_B = \left.\frac{-x}{y-k}\right|_B = \frac{1}{-k}$

Also, $m_{AC} = -a$

By given condition, $(-a) \times \left(-\frac{1}{k}\right) = -1$

$\Rightarrow a = -k$ ...(2)

From (1) & (2), $a^2 + 2a^2 = 1$

$\Rightarrow a = \pm\frac{1}{\sqrt{3}}$

If $a = \frac{1}{\sqrt{3}}$ $\Rightarrow k = -\frac{1}{\sqrt{3}}$

$\therefore \tan\theta = m_{AB} = \frac{1}{\sqrt{3}}$ $\Rightarrow \theta = 30°$

$\Rightarrow \angle ABC = 30°$

7. Let s(n) denote the sum of the digits of a positive integer n in base 10. If s(m) = 20 and s(33m) = 120, what is the value of s(3m)?

Sol. **(60)**

S(m) = 20 and S (33m) = 120 is possible only for/m/ having digits ‘0’ or ‘1’

$\therefore$ S(3m) = 60

8. Let $F_k(a,b) = (a+b)^k - a^k - b^k$ and let S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. For how many ordered pairs (a, b) with a, b $\in$ S and $a \le b$ is $\frac{F_5(a,b)}{F_3(a,b)}$ an integer?

Sol. **(22)**

$$\frac{F_5(a,b)}{F_3(a,b)} = \frac{(a+b)^5 - a^5 - b^5}{(a+b)^3 - a^3 - b^3}$$

$$= \frac{5}{3}(a^2 + b^2 + ab) \qquad \in I$$

a = $3k_1$ or $3k_1+1$ or $3k_1 + 2$

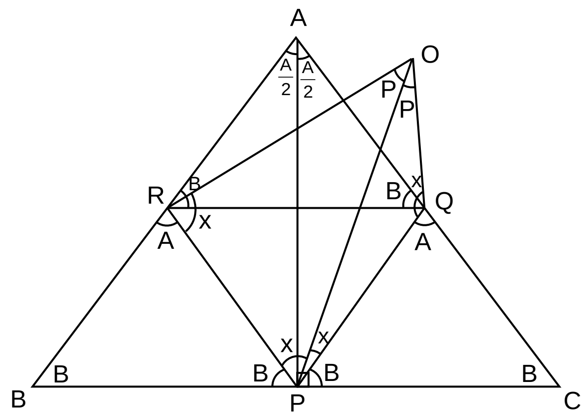
and b = $3k_2$ or $3k_2+1$ or $3k_2 + 2$

only when a and b give same remainder $a^2 + b^2 + ab$ is divisible by 3.

$\therefore$	a = 1,	b = 1, 4, 7, 10	$\rightarrow$ 4
	a = 2,	b = 2, 5, 8	$\rightarrow$ 3
	a = 3,	b = 3, 6, 9	$\rightarrow$ 3
	a = 4,	b = 4, 7, 10	$\rightarrow$ 3
	a = 5,	b = 5, 8	$\rightarrow$ 2
	a = 6,	b = 6, 9	$\rightarrow$ 2
	a = 7,	b = 7, 10	$\rightarrow$ 2
	a = 8,	b = 8	$\rightarrow$ 1
	a = 9,	b = 9	$\rightarrow$ 1
	a = 10,	b = 10	$\rightarrow$ 1
		Total	22 ordered pair

9. The centre of the circle passing through the midpoints of the sides of an isosceles triangle ABC lies on the circumcircle of triangle ABC. If the larger angle of triangle ABC is α^o and the smaller one β^o then what is the value of $\alpha - \beta$?

Sol. **(90)**



$A + 2B = 180$

$\frac{A}{2} = 90 - B$

$2x + 2B = 180$

$x = 90 – B$

$x = \frac{A}{2}$

$\Rightarrow OP \perp BC$

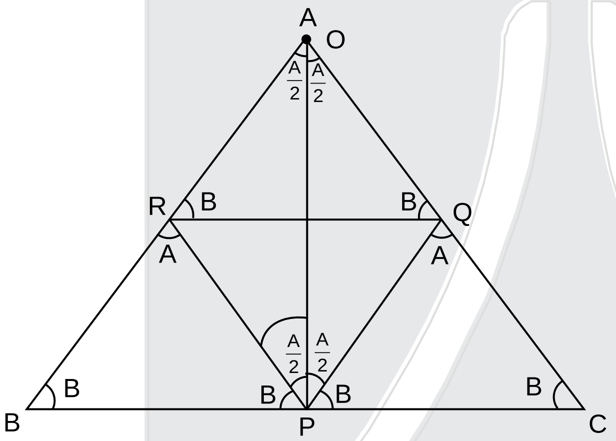
$\therefore$ O and A coincide

$\therefore \angle ORP = \angle OPR = \angle RAP = 60°$

$\therefore \alpha = 120° \; \beta = 30°$

$\alpha - \beta = 90°$

By drawing properly we get



10. One day I went for a walk in the morning at x minutes past 5'O clock, where x is a two digit number. When I returned, it was y minutes past 6'O clock, and I noticed that (i) I walked exactly for x minutes and (ii) y was a 2 digit number obtained by reversing the digits of x. How many minutes did I walk?

Sol. **(42)**

At x = ab minutes past 5 hr = 5 x 60 + 10 a + b minutes

At y = bc minutes past 6 hr = 6 x 60 + 10 b + a minutes

Total minutes of walk = 60 + 9b – 9a = 10 a+b

= 60 + 8b = 19a

a = 4, b = 2

I walked for 42 minutes

11. Find the largest value of a^b such that the positive integers a, b > 1 satisfy $a^b b^a + a^b + b^a = 5329$.

Sol. **(81)**

$(a^b + 1)(b^a + 1) = 5330 = 2 \times 5 \times 13 \times 41$

$\Rightarrow \quad a^b = 81 = 3^4$ and $b^a = 64 = 4^3$

Or $a^b = 64 = 4^3$ and $b^a = 81 = 3^4$

$\therefore a^b = 81$

12. Let N be the number of ways of choosing a subset of 5 distinct numbers from the set

$$\{10a + b : 1 \le a \le 5, 1 \le b \le 5\}$$

where a, b are integers, such that no two of the selected numbers have the same units digit and no two have the same tens digit. What is the remainder when N is divided by 73?

Sol. **(47)**

$10a + b$; $1 \le a \le 5, 1 \le b \le 5$

Let us divide numbers into different sets, such as

Set 1 = {11, 12, 13, 14, 15}

Set 2 = {21, 22, 23, 24, 25}

Set 3 = {31, 32, 33, 34, 35}

Set 4 = {41, 42, 43, 44, 45}

Set 5 = {51, 52, 53, 54, 55}

Now to make a number having no two digits and no two ten's digit same, we can select any 1 number from each of set 1, set 2, set 3, set 4, set 5.

$\therefore$ No. of ways = ${}^5C_1 \times {}^4C_1 \times {}^3C_1 \times {}^2C_1 \times {}^1C_1 = 120$

$\therefore$ $120 \div 73$

$\Rightarrow$ Remainder = 47

13. Consider the sequence

1, 7, 8, 49, 50, 56, 57, 343, …

which consists of sums of distinct powers of 7, that is, 7^0, 7^1, $7^0 + 7^1$, 7^2, …, in increasing order. At what position will 16856 occur in this sequence?

Sol. **(36)**

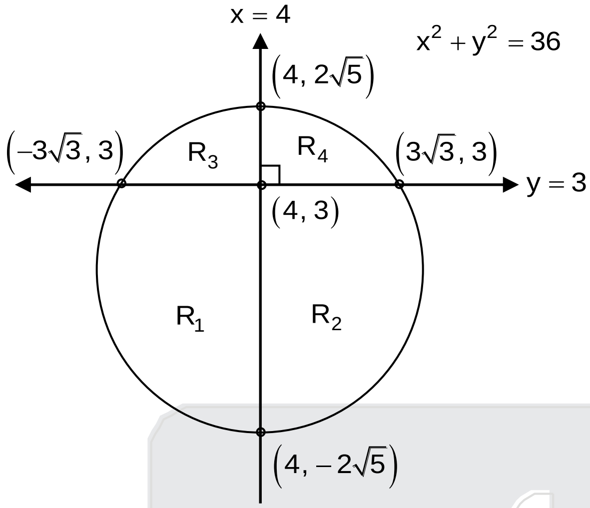
7^0	7^0+7^3	7^2+7^4	7^5
7^1	7^1+7^3	7^3+7^4	7^0+7^5
7^0+7^1	7^2+7^3	$7^0+7^1+7^4$	7^1+7^5
7^2	$7^0+7^1+7^3$	$7^0+7^2+7^4$	7^2+7^5
7^0+7^2	$7^0+7^2+7^3$	$7^0+7^3+7^4$	$=16856$
7^1+7^2	$7^1+7^2+7^3$	$7^1+7^2+7^4$	
$7^0+7^1+7^2$	$7^0+7^1+7^2+7^3$	$7^1+7^3+7^4$	
7^3	7^4	$7^0+7^1+7^2+7^4$	
	7^0+7^4	$7^0+7^1+7^3+7^4$	
	7^1+7^4	$7^0+7^2+7^3+7^4$	
		$7^1+7^2+7^3+7^4$	
		$7^2+7^3+7^4$	

i.e. we get 16856 at 36^{th} position

14. Let R denote the circular region in the xy-plane bounded by the circle $x^2 + y^2 = 36$. The lines x = 4 and y = 3 divide R into four regions R_i, i = 1, 2, 3, 4. If $|R_i|$ denotes the area of the region R_i and if $|R_1| > |R_2| > |R_3| > |R_4|$, determine $|R_1| - |R_2| - |R_3| + |R_4|$.

[Here $|\Omega|$ denotes the area of the region Ω in the plane.]

Sol. **(48)**



$$R_1 = \int_{-2\sqrt{5}}^{3} \sqrt{36-y^2}\, dy = \frac{1}{2}y\sqrt{36-y^2} + \frac{1}{2}(36)\sin^{-1}\frac{y}{6}$$

Similarly for using integration. we can find

$$R_1 + R_4 = \frac{1}{2}\pi(6)^2 + 2(3)(4)$$

$\therefore$ Remaining circle area = $R_2 + R_3 = \frac{1}{2}\pi(6)^2 - 2(3)(4)$

Hence,

$$R_1 + R_4 - (R_2 + R_3) = 24 + 24 = 48$$

15. In base – 2 notation, digits are 0 and 1 only and the places go up in powers of – 2. For example, 11011 stands for $(-2)^4 + (-2)^3 + (-2)^1 + (-2)^0$ and equals number 7 in base 10. If the decimal number 2019 is expressed in base – 2 how many non zero digits does it contain?

Sol. **(06)**

Since 2019 is closest to $2^{11} = 2048$

But since base is -2

Hence we will require $(-2)^{12}$ as maximum no. as well as $(-2)^{11}$ to compensate

The series can then be worked out as the following permutation.

1100000100111

Hence no. of non zero digits = 6

16. Let N denote the number of all natural numbers n such that n is divisible by a prime $p > \sqrt{n}$ and $p < 20$. What is the value of N?

Sol. **(69)**

Since $P > \sqrt{n}$ and $P > 20$

Hence for the times from 2 to 20 we calculate possibilities that that $n < P^2$

Prime No.	Natural Number n
2	2
3	3, 6
5	5, 10, 15, 20
7	7, 14, 21, 28, 35, 42
11	11, 22, 33, 44, 55, 66, 77, 88, 99, 110
13	13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156
17	17, 34, 51, 68, 85, 102, 119, 136, 153, 170, 187, 204, 221, 238, 255, 272
19	19, 38, 57, 76, 95, 114, 133, 152, 171, 190, 209, 228, 247, 266, 285, 304, 323, 342

Total numbers = 69

17. Let a, b, c be distinct positive integers such that b + c – a, c + a – b and a + b – c are all perfect squares. What is the largest possible value of a + b + c smaller than 100?

Sol. **(91)**

b + c – a, c + a – b, a + b – c …... integer

Add these three

b + c – a + c+ a – b + a + b – c = a + b+ c

$\because$ a + b + c < 100

$\because$ Possible value of b + c – a , c + a – b, a + b – c are 1, 4, 9, 16, 25, 36, 49, 64, 81.

Either b + c – a, a + b – c, c + a – b all three will odd or all will even.

$\therefore$ Largest possible value of a + b + c is possible.

If b + c – a = 81

c + a – b = 9

a + b – c = 1

$\therefore$ a + b + c = 91

18. What is the smallest prime number p such that $p^3 + 4p^2 + 4p$ has exactly 30 positive divisors?

Sol. **(43)**

$p^3 + 4p^2 + 4p$

$= p(p^2 + 4p + 4)$

$= p'(p+2)^2$

Number of divisors for $a^m.b^n.c^p$…. is (m + 1) (n + 1) (p+1) …

For 30 divisors we will need 15 divisors from $(p+2)^2$

If p = 43 [for less than 43 we will have maximum 5 divisors] number of divisors

$(p+2)^2 = (43+2)^2 = 45^2$

$= 3^{2\times 2} \times 5^2$

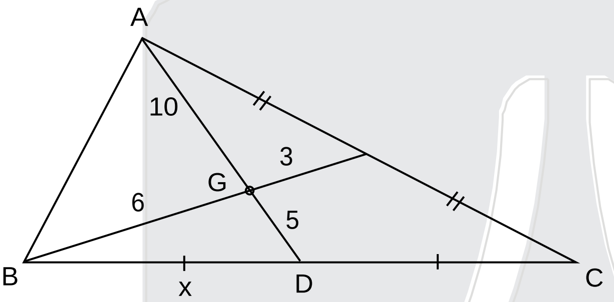
$\therefore$ Number of divisors = (4 + 1) (2 +1) = 15

$\therefore$ Ans. is 43

19. If 15 and 9 are lengths of two medians of a triangle, what is the maximum possible area of the triangle to the nearest integer?

Sol. **(90)**

Area of ΔBDG



$S = \dfrac{11+x}{2}$

Area = $\sqrt{\dfrac{11+x}{2}\left(\dfrac{11+x}{2}-x\right)\left(\dfrac{11+x}{2}-5\right)\left(\dfrac{11+x}{2}-6\right)}$

$= \sqrt{\dfrac{11+x}{2}\left(\dfrac{11-x}{2}\right)\left(\dfrac{x+1}{2}\right)\left(\dfrac{x-1}{2}\right)}$

$= \sqrt{\dfrac{(121-x^2)(x^2-1)}{16}}$

Let y = $\dfrac{(121-x^2)(x^2-1)}{16}$

For max area $\dfrac{dy}{dx} = 0$

$-2x(x^2-1)+(121-x^2)2x = 0$

$-x^2+1+121-x^2 = 0$

$2x^2 = 122$

$x^2 = 61$

Area of $\Delta BDG = \sqrt{\dfrac{(121-61)(61-1)}{16}}$

$= \sqrt{\dfrac{60\times 60}{16}} = \dfrac{60}{4} = 15$

Area of ΔABC = 6 x 15 = 90

20. How many 4-digit numbers $\overline{abcd}$ are there such that a < b < c < d and b – a < c – b < d – c?

Sol. **(07)**

a < b < c < d
b – a < c – b < d – c
possibilities are
a = 1, b = 2, c = 4, d = 7
a = 1, b = 2, c = 4, d = 8
a = 1, b = 2, c = 4, d = 9
a = 1, b = 2, c = 5, d = 9
a = 1, b = 3, c = 6, d = no possible
a = 2, b = 3, c = 5 d = 8
a = 2, b = 3, c = 5 d = 9
a = 2, b = 3, c = 6, d = not possible
a = 3, b = 4, c = 6, d = 9
a = 4, b = 5, c = 7, d = not possible
Ans. 7

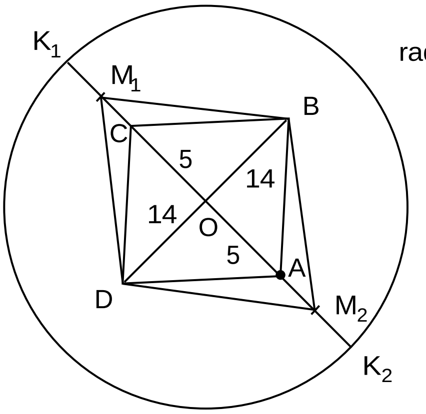
21. Consider the set E of all positive integers n such that when divided by 9, 10, 11 respectively, the remainders (in that order) are all > 1 and form a non-constant geometric progression. If N is the largest element of E, find the sum of digits of E.

Sol. **(Bonus)**

Remainder will be less than 11, only possible set of remainders is 1,2,4 or 2,4,8 or 1,3,9. But r greater than 1 so 2, 4, 8. But Since E is set of numbers, how can we find the sum of digits of E.
They meant "sum of digits of N.
But question should have been
"Smallest element of E is N" and sum of N.
Then there is answer. 74 : for ratio 2,4,8 as ratio greater than 1
So Bonus

22. In parallelogram ABCD, AC = 10 and BD = 28. The points K and L in the plane of ABCD move in such a way that AK = BD and BL = AC. Let M and N be the midpoints of CK and DL, respectively. What is the maximum value of $\cot^2(\angle BMD/2) + \tan^2(\angle ANC/2)$?

Sol. **(02)**



radius = 28

$CK_1 = 18$

$OM_1 = 5 + 9 = 14$

$\therefore \quad \angle BM_1D = 90°$ ($\angle$ in semicircle)

Similarly, $CK_2 = 10 + 28 = 38$

$\therefore \quad CM_2 = 19$

$AM_2 = 9$

$OM_2 = 5 + 9 = 14$

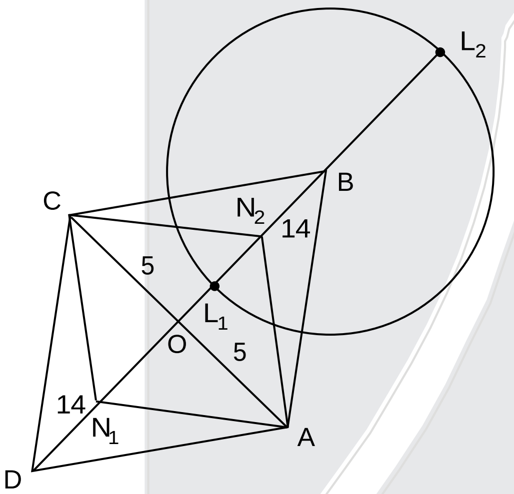
$\therefore \quad \angle BM_2D = 90°$ ($\angle$ in semicircle)

If K_1 or K_2 moves $\angle BMD$ will increase

By observation : $\frac{\angle BMD}{2} \uparrow \qquad \therefore \cot\left(\frac{\angle BMD}{2}\right) \downarrow$

$\therefore \quad$ For maximum $\angle BMD = 90°$

$$\cot^2\left(\frac{\angle BMD}{2}\right) = 1$$



$BL_1 = 10$, $OL_1 = 4$, $DL_1 = 18$, $DN_1 = 9$, $ON_1 = 5$

$\therefore \angle AN_1C = \frac{\pi}{2}$ ($\angle$ in semicircle)

For L_2 point,

$DL_2 = 14 + 14 + 10$

$DL_2 = 38$

$DN_2 = 19$

$ON_2 = 5$

$\angle AN_2C = \frac{\pi}{2}$ ($\angle$ in semicircle)

Now if L_1 or L_2 moves,

$\angle ANC \downarrow \qquad \frac{\angle ANC}{2} \downarrow \qquad \tan\frac{\angle ANC}{2} \downarrow$ (Observation)

Hence for maximum,

$$\angle ANC = \frac{\pi}{2}$$

$$\therefore \quad \tan^2\left(\frac{\pi}{4}\right) = 1$$

$\therefore \quad$ Maximum = 1 + 1 = 2

23. Let t be the area of a regular pentagon with each side equal to 1. Let P(x) = 0 be the polynomial equation with least degree, having integer coefficients, satisfied by x= t and the gcd of all the coefficients equal to 1. If M is the sum of the absolute values of the coefficients of P(x). What is the integer closest to $\sqrt{M}$? ($\sin 18^\circ = \left(\sqrt{5}-1\right)/2$).

Sol. **(16)**

Area of regular pentagon $= \dfrac{a^2 n}{4\tan\left(\dfrac{180}{n}\right)}$

n $\Rightarrow$ No. of sides

a $\Rightarrow$ Length of side

$\therefore$ For regular pentagon by side length 1,

$$\text{Area (t)} = \frac{5}{4\tan 36^\circ}$$

$$= \frac{5}{4(0.73)}$$

$$= 1.71$$

Now, P(1.71) = 0 to be found with least degree and integer coefficient soon that gcd of all coefficient is 1.

Let x = 1.71

100x = 171

$\therefore$ P(x) = 100x – 171 = 0 is the polynomial which satisfied all the conditions.

$\therefore$ m = 100 + 171 = 271

$\therefore$ $\sqrt{m}$ = 16.46

$\therefore$ Nearest integer = 16

But this question can have multiple solutions as student can take tan 36 as 0.72, 0.726 or even 0.7. Every time we will get different answers. So this question should be Bonus.

24. For n $\geq$ 1, let a_n be the number beginning with n 9's followed by 744 ; e.g., a_4 = 9999744. Define f(n) = max {m $\in$ N | 2^m divides a_n}, for n $\geq$ 1. Find f(1) + f(2) + f(3) +…. + f(10).

Sol. **(75)**

$\because$ $n \geq 1$

$a_1 = 9744$

$a_2 = 99744$

$a_3 = 999744$

$\vdots$

$a_{10} = \underbrace{999...9}_{10\text{ times}}744$

$\because$ $f(n) = \max\left\{m \in \dfrac{N}{2^m} \text{ divides } a_n\right\}$

$f(1) = 2^m$ divides 9744

$9744 = 2 \times 2 \times 2 \times 2 \times 609$

$= 2^4 \times 609$

9744 is divided by 2^4

$\therefore$ m = 4

$f(2) = 2^m$ should divide 99744

$$99744 = 2 \times 2 \times 2 \times 2 \times 2 \times 3117$$

$$99744 = 2^5 \times 3117$$

$a_2 =$ 99744 divided by 2^5

$\therefore$ m = 5

f (3) = 2^m should divide 999744

$$999744 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 15621$$

$$999744 = 2^6 \times 15621$$

$a_3 =$ 999744 divided by 2^6

$\therefore$ m = 6

Similarly a_4 divided by 2^7

$\therefore$ m = 7

a_5 divides by 2^{13}

a_6 divides by 2^8

a_7 divides by 2^8

a_8 divides by 2^8

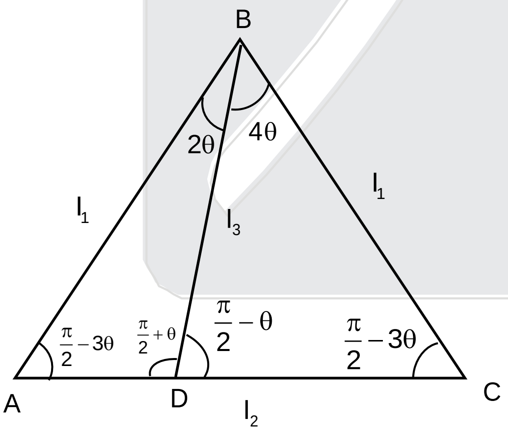
a_9 divides by 2^8

a_{10} divides by 2^8

$\therefore$ f(1) + f(2) + f(3) ………+ f(10)
= 4 + 5 + 6 + 7 + 13 + 8 + 8 + 8 + 8 + 8
= 75

25. Let ABC be an isosceles triangle with AB = BC. A trisector of $\angle$B meets AC at D. If AB, AC and BD are integers and AB – BD = 3, find AC.

Sol. **(26)**



$$\frac{l_1}{\sin\left(\frac{\pi}{2}+\theta\right)} = \frac{l_3}{\sin\left(\frac{\pi}{2}-3\theta\right)}$$

$\Rightarrow$ put $l_3 = l_1 - 3$

$\Rightarrow$ $l_1 = \frac{3}{4}\text{cosec}^2\theta$

If $\text{cosec}^2\theta = 36$

i.e. $\text{cosec}^2\theta = 6$

i.e. $\sin\theta = \frac{1}{6}$

$l_1 = \frac{3}{4}\times 36 = 27$

$l_3 = 24$

$$\frac{l_1}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{DC}{\sin 4\theta} \quad \& \quad \frac{l_1}{\sin\left(\frac{\pi}{2}+\theta\right)} = \frac{AD}{\sin 2\theta}$$

$$\Rightarrow \; l_2 = \frac{l_1 \sin 4\theta}{\cos\theta} + \frac{l_1 \sin 2\theta}{\cos\theta}$$

$$\Rightarrow \; l_2 = l_1 \times 2 \times \sin 3\theta$$

$$\Rightarrow \; \frac{3}{4}\operatorname{cosec}^2\theta \times 2 \times \sin 3\theta$$

$$\Rightarrow \; \frac{3}{2}\left(\frac{3}{\sin\theta} - 4\sin\theta\right)$$

$$l_2 = \frac{3}{2}\left(3\times 6 - \frac{4}{6}\right)$$

$$= \frac{3}{2}\left(18 - \frac{2}{3}\right)$$

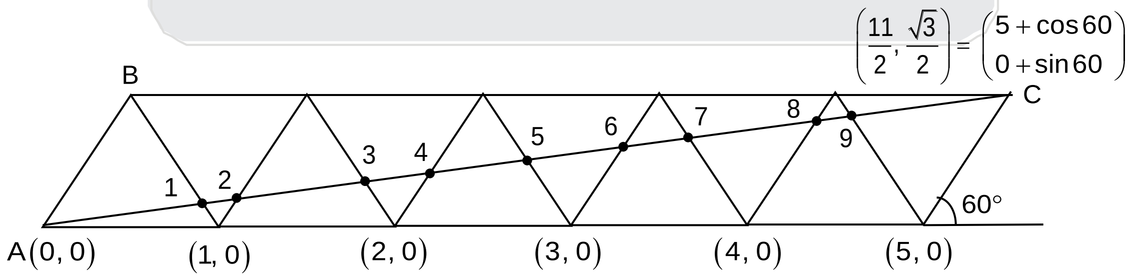
$$= \frac{3}{2}\left(\frac{54-2}{3}\right)$$

$$= \frac{3}{2}\times\frac{52}{3}$$

$l_2 = 26$

26. A friction-less board has the shape of an equilateral triangle of side length 1 meter with bouncing walls along the sides. A tiny super bouncy ball is fired from vertex A towards the side BC. The ball bounces off the walls of the board nine times before it hits a vertex for the first time. The bounces are such that the angle of incidence equal the angle of reflection. The distance traveled by the ball in meters is of the form $\sqrt{N}$, where N is an integer. What is the value of N ?

Sol. **(31)**

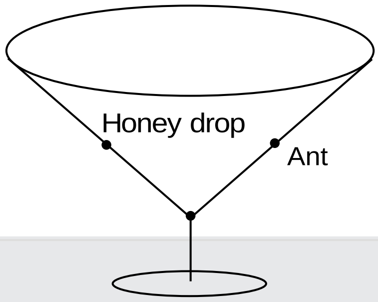


$AX = \sqrt{N}$

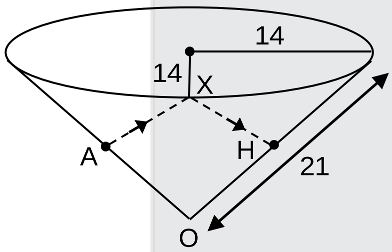
$$AX^2 = \sqrt{\frac{121}{4} + \frac{3}{4}} = \sqrt{31}$$

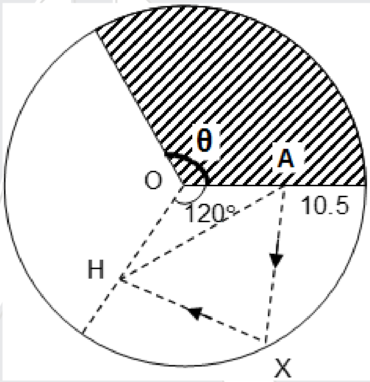
N = 31

27. A conical glass is in the form of a right circular cone. The slant height is 21 and the radius of the top rim of the glass is 14. An ant at the mid point of a slant line on the outside wall of the glass sees a honey drop diametrically opposite to it on the inside wall of the glass. (see the figure). If d the shortest distance it should crawl to reach the honey drop, what is the integer part of d ? (Ignore the thickness of the glass)



Sol. **(36)**



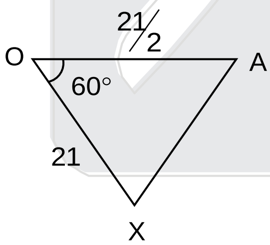


Assume we open the cone, it will become a section of a bigger circle with radius 21.

Section of the circle used to make cone = $\frac{2\pi \cdot 14}{2\pi \cdot 21} = \frac{2}{3}$ $\quad (\therefore \theta = 120^o)$

$\therefore$ minimum distance 'd' = AX + XH = 2AX

In ΔDAX ,



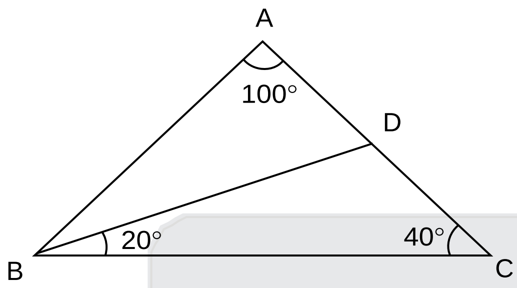
$$\text{Cos}60^o = \frac{(21)^2 + \left(\frac{21}{2}\right)^2 - (AX)^2}{2 \cdot 21 \cdot \frac{21}{2}} \quad \Rightarrow AX = 21\left(\frac{\sqrt{3}}{2}\right)$$

$\therefore$ d = 36.37, [d] = 36

28. In a triangle ABC, it is known that $\angle A = 100°$ and AB = AC. The internal angle bisector BD has length 20 units. Find the length of BC to the nearest integer, given that $\sin 10° \approx 0.174$.

Sol. **(27)**

ΔBCD, by sine rule



$$\frac{BC}{120°} = \frac{BD}{\sin 40°} = \frac{20}{\sin 40°}$$

$$\therefore\ BC = \frac{10\sqrt{3}}{\sin 40°}$$

Given, $\cos 80° (\sin 10°) \approx 0.174$

$$\sin 40° \approx \sqrt{\frac{1-\cos 80°}{2}} \approx 0.643$$

$\therefore$ Nearest integer to BC= 27

29. Let ABC be an acute angled triangle with AB =15 and BC = 8. Let D be a point on AB such that BD = BC. Consider points E on AC such that $\angle DEB = \angle BEC$. If α denotes the product of all possible value of AE, find $[\alpha]$ the integer part of α.

Sol. **(Bonus)**

The problem, as stated, has infinite solutions. If you take ANY triangle ABC with AB = 15 and BC= 8 construct point

D on AB such that BD = 8 and draw the bisector BE, then you have a perfectly valid triangle.

As the triangle is acute-angled, then AE can take any value between $\frac{15}{23}\sqrt{161} \approx \sqrt{8.275}$

(case when $\angle BCA$=90° and $\left(\frac{15}{23}\right)17 \approx 11.087$

(case when $\angle ABC$=90°)

Question will have sense if only integer values for AE were allowed. If that is the case, then the possible integer values for AE are : 9, 10, 11

30. For any real number x, let [x] denotes the integer part of x; {x} be the fractional part of x ({x} = x − |x|). Let A denote the set of all real numbers x satisfying

$$\{x\} = \frac{x + [x] + [x + (1/2)]}{20}$$

If S is the sum of all numbers in A, find [S]

Sol. **(21)**

Let, x = I + f

$\therefore$ given equation reduces to,

$$f = \frac{I + f + I + I + \left[f + \frac{1}{2}\right]}{20}$$

(I) $f \in \left[0, \frac{1}{2}\right)$

$\left[f + \frac{1}{2}\right] = 0$

$\therefore$ $19f = 3I \Rightarrow f = \frac{3I}{19}$

$0, < \frac{3I}{19} < \frac{1}{2}$

$\Rightarrow$ $I \in \left[0, \frac{19}{6}\right) \Rightarrow$ I = 0, 1, 2, 3

$\therefore$ $f = 0, \frac{3}{19}, \frac{6}{19}, \frac{9}{19}$

$\therefore$ $x = 0, \frac{22}{19}, \frac{44}{19}, \frac{66}{19}$

(II) $f \in \left[\frac{1}{2}, 1\right)$

$\left[f + \frac{1}{2}\right] = 1$

$20f = 3I + 1 + f$

$f = \frac{3I + 1}{19}$

$\frac{1}{2} \le \frac{3I + 1}{19} < 1 \Rightarrow \frac{17}{6} \le I < 6 \Rightarrow$ I = 3, 4, 5

$\therefore$ $f = \frac{10}{19}, \frac{13}{19}, \frac{16}{19}$

$\therefore$ $x = \frac{67}{19}, \frac{89}{19}, \frac{111}{19}$

$\therefore$ $S = \frac{399}{19}$

[S] = 21