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Mathematics

कक्षा / Class XII

2025-26

विद्यार्थी सहायक सामग्री
Student Support Material

केन्द्रीय विद्यालय संगठन~Kendriya Vidyalaya Sangathan

संदेश

विद्यालयी शिक्षा में शैक्षिक उत्कृष्टता प्राप्त करना एवं नवाचार द्वारा उच्च - नवीन मानक स्थापित करना केन्द्रीय विद्यालय संगठन की नियमित कार्यप्रणाली का अविभाज्य अंग है। राष्ट्रीय शिक्षा नीति 2020 एवं पी. एम. श्री विद्यालयों के निर्देशों का पालन करते हुए गतिविधि आधारित पठन-पाठन, अनुभवजन्य शिक्षण एवं कौशल विकास को समाहित कर, अपने विद्यालयों को हमने ज्ञान एवं खोज की अद्‌भुत प्रयोगशाला बना दिया है। माध्यमिक स्तर तक पहुँच कर हमारे विद्यार्थी सैद्धांतिक समझ के साथ-साथ, रचनात्मक, विश्लेषणात्मक एवं आलोचनात्मक चिंतन भी विकसित कर लेते हैं। यही कारण है कि वह बोर्ड कक्षाओं के दौरान विभिन्न प्रकार के मूल्यांकनों के लिए सहजता से तैयार रहते हैं। उनकी इस यात्रा में हमारा सतत योगदान एवं सहयोग आवश्यक है - केन्द्रीय विद्यालय संगठन के पांचों आंचलिक शिक्षा एवं प्रशिक्षण संस्थान द्वारा संकलित यह विद्यार्थी सहायक-सामग्री इसी दिशा में एक आवश्यक कदम है । यह सहायक सामाग्री कक्षा 9 से 12 के विद्यार्थियों के लिए सभी महत्वपूर्ण विषयों पर तैयार की गयी है। केन्द्रीय विद्यालय संगठन की विद्यार्थी सहायक- सामग्री अपनी गुणवत्ता एवं परीक्षा संबंधी सामग्री संकलन की विशेषज्ञता के लिए जानी जाती है और शिक्षा से जुड़े विभिन्न मंचों पर इसकी सराहना होती रही है। मुझे विश्वास है कि यह सहायक सामग्री विद्यार्थियों की सहयोगी बनकर निरंतर मार्गदर्शन करते हुए उन्हें सफलता के लक्ष्य तक पहुँचाएगी।

शुभाकांक्षा सहित ।

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CHAPTER-1 - RELATIONS AND FUNCTIONS

Gist / Summary of the lesson:

- **Types Of Relations:**
  Empty relation, Universal relation, Reflexive, Symmetric, Transitive and Equivalence relations.
- Types Of Functions: One – one (or injective) functions, onto (or surjective) functions, One-one and onto (or bijective)

Definitions:

- A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., R = ø ⊂ A × A
- A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., R = A × A
- A relation R in a set A is called
  (i) reflexive, if (a, a) ∈ R, for every a ∈ A,
  (ii) symmetric, if $(a_1, a_2) \in$ R ⇒ $(a_2, a_1) \in$ R, for all $a_1, a_2 \in$ A.
  (iii) transitive, if $(a_1, a_2) \in$ R and $(a_2, a_3) \in$ R ⇒ $(a_1, a_3) \in$ R, for all $a_1, a_2, a_3 \in$ A.
- A relation R in a set A is said to be an equivalence relation iff R is reflexive, symmetric and transitive
- A function f : X → Y is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in$ X, $x_1 \neq x_2$. implies $f(x_1) \neq f(x_2)$. Otherwise, f is called many-one.
- A function f : X → Y is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every y ∈ Y, there exists an element x in X such that f(x) = y. In other words f:X→ Y is onto if and only if Range of f = Y.
- A function f : X → Y is said to be one-one and onto (or bijective), if f is both one-one and onto.

Formulae:

- If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and number of relations from set A to set B=2^{pq}.
- If A is a non-empty finite set containing n elements, then number of reflexive relations on set A=$2^{n(n-1)}$.
- If A is a non-empty finite set containing n elements, then number of symmetric relations on set A =$2^{\frac{n(n+1)}{2}}$
- If A and B are non-empty finite sets containing m and n elements respectively, then
  (i) Number of functions from A to B = n^m
  (ii) Number of one-one functions from A to B =$\begin{cases} nP_m, if\ m \leq n \\ 0, \ \ if\ m > n \end{cases}$
  (iii)Number of onto functions from A to B
  =$\begin{cases} nC_0 n^m - nC_1 (n-1)^m + nC_2 (n-2)^m - \ldots\ldots\ldots, if\ n \leq m \\ 0, \qquad if\ n > m \end{cases}$

  (iv) Number of one-one and onto functions

  i.e., bijective functions from A to B= $\begin{cases} m!, if\ m = n \\ 0, if\ m \neq n \end{cases}$

MULTIPLE CHOICE QUESTIONS

1) Let A={3,5}. Then number of reflexive relations on A is

(A) 2 (B) 4 (C) 0 (D) 8

Solution: If A is a non-empty finite set containing n elements, then number of reflexive relations on set A=$2^{n(n-1)}$.

Here n=2

No. of reflexive relations on A= $2^{2(2-1)}$=2^2=4 **Answer: B**

2) The number of possible symmetric relations on a set consisting of 4 elements is

(A) 512 (B)1024 (C)256 (D)32

Solution: If A is a non-empty finite set containing n elements, then number of symmetric relations on set A =$2^{\frac{n(n+1)}{2}}$

Here $n = 4$

No. of symmetric relations on A=$2^{\frac{4(4+1)}{2}}$ =$2^{\frac{4x5}{2}}$=2^{10}=1024 **Answer: B**

3) A relation R on set G = $\{All\ the\ students\ in\ a\ certain\ mathematics\ class\}$ is defined as, $R = \{(x, y): x\ and\ y\ have\ the\ same\ mathematics\ teacher\}$.Which of the following is true about R ?

(A) R is reflexive and transitive but not symmetric.

(B)R is transitive and symmetric but not reflexive.

(C) R is reflexive and symmetric but not transitive.

(D) R is an equivalence relation.

Answer: D

4) Let the relation R in the set $A = \{x \in Z: 0 \le x \le 12\}$, given by $R = \{(a, b): |a - b|\ is\ a\ multiple\ of\ 4\}$. Then the equivalence class [1] is

(A){1,5,9} (B){0,1,2,5} (C){1} (D)A

Solution: Given $A = \{0,1,2,3,4, \dots ,12\}$

Now $[1] = \{x \in A: |x - 1|\ is\ a\ multiple\ of\ 4\} = \{1,5,9\}$ **Answer: A**

5) A and B are two sets with m elements and n elements respectively(m < n). How many onto functions can be defined from set A to set B?

(A) 0 (B)m! (C) n! (D)n^m

Solution: Given $n(A) = m, n(B) = n$

We know that onto function requires every element of set B to be mapped by at least one element from set A. Since it is given that m < n, there is no such onto function.

Therefore number of onto functions from set A to set B where m < n is zero. **Answer: A**

6) For real x, let $f(x) = x^3. Then$

(A) f is one-one but not onto on R (B) f is onto on R but not one-one.

(c) f is one-one and onto on R (D) f is neither one-one nor onto on R

Solution: Let $f(x_1) = f(x_2)\ \forall\ x_1, x_2 \in R$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

$So\ f(x)\ is\ one - one.$

$Let f(x) = x^3 = y$

$x = y^{\frac{1}{3}}, \forall y \in R$

Every image $y \in R$ has a unique pre image in R.

f is onto.

f is one-one and onto.

Answer: C

7) The function $f : R \to [-1,1]$ $defined\ by\ f(x) = cos\ x$ is :

(A) Both one-one and onto (B) not one-one but onto

(C) One-one but not onto (D) neither one-one nor onto

Answer: B

8) A function f: R→ A defined as f(x)=$x^2 + 1$ is onto, if A is

(A) $(-\infty, \infty)$ (B) $(1, \infty)$ (C) [1,∞) (D)[-1,∞)

Solution: x∈ $R \Rightarrow x^2 \geq 0 \Rightarrow x^2 + 1 \geq 0 + 1 \Rightarrow f(x) \geq 1$

Range of f = [1,∞)

Thus for f to be onto , A=[1,∞) **Answer: C**

9) Let L denotes the set of all straight lines in a plane. Let a relation R be defined by lRm if and only if l is perpendicular to m ∀ l, m ∊ L. Then R is:

(A) reflexive (B) symmetric (C) transitive (D) Equivalence relation

Answer: B

10) A relation R on set A=$\{x: x \in Z\ and\ 0 \leq x \leq 10\}$ as R={(x,y):x=y} is given to be an equivalence relation. The number of equivalence classes is

(A)1 (B)2 (C)10 (D)11

Solution: A={0,1,2,3,4,5,6,7,8,9,10}

R={(0,0),(1,1), (2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10)}

We observe that each element in set A is only related to itself in relation R.

[0] ={0}, [1] ={1}, [2] ={2}, [3] ={3}, [4] ={4}, [5] ={5}, [6] ={6}, [7] ={7}, [8] ={8}, [9] ={9}, [10] ={10}

Number of equivalence classes is 11 **Answer: D**

ASSERTION AND REASON BASED QUESTIONS

Questions numbers 1 to 10 are Assertion and Reason based questions. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R).

Select the correct answer from the codes (A), (B), (C) and (D) as given below.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

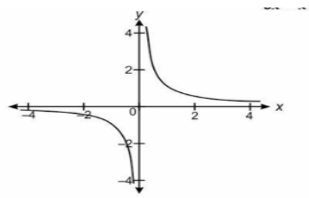
1) X = {0, 2, 4, 6, 8}. P is a relation on X defined by P = {(0, 2), (4, 2), (4, 6), (8, 6), (2, 4), (0, 4)}.

Assertion (A): The relation P on set X is a transitive relation.

Reason (R): The relation P has a subset of the form {(a, b), (b, c), (a, c)}, where a, b, c ∈ X is transitive.

Solution: P is not transitive as (4,2) (2,4)∊P but (4,4) $\notin \boldsymbol{P}$. **Answer: D**

Shown below is the graph of the function $f(x) = \frac{9-x^2}{9x-x^3}$



Assertion (A): The function f is not onto.

Reason (R): 3 ∈ R (co-domain of f) has no pre-image in the domain of f.

Solution: A is true but R is false. 0∈ $R\ does\ not$ have a pre-image. **Answer: C**

2) Assertion (A): Let $f(x) = e^x$ and $g(x) = logx$. Then $(f+g)x = e^x + log\ x$ where domain of $(f+g)\ is\ R.$

Reason (R) : $Dom(f + g) = Dom(f) \cap Dom(g)$

Solution: Domain of $f = R$.

Domain of $g = (0, \infty)$

Domain of $(f + g) = R \cap (0, \infty) = (0, \infty)$ *not* R

Here Assertion is false, Reason is true. **Answer: D**

3) Assertion (A): Let A=$\{x \in R : -1 \leq x \leq 1\}$. *If* $f: A \rightarrow A$ *be defined as* $f(x) = x^2$, *then f is not an onto function.*

Reason (R): If y=-1$\in A$, *then* $x = \pm\sqrt{-1}$ *not an element of A.*

Solution: Here assertion and reason are true and R is the correct explanation of A.

Answer: A

4) Assertion (A): Let Z be the set of integers. A function f: Z$\rightarrow Z$ *defined as* $f(x) = 3x - 5, \forall x \in Z$ *is a bijective.*

Reason (R): A function is bijective if it is both surjective and injective.

Solution: Here $f(x) = 3x - 5$ is not a bijective function. **Answer :D**

5) $f: X \rightarrow X$ is a function on the finite set X.

Assertion (A): If f is onto, then f is one-one and if f is one-one, then f is onto..

Reason (R): Every one-one function is always onto and every onto function is always one-one.

Solution: For function on finite sets, a one-one function implies onto and vice-versa. This is not necessarily true for infinite sets.

So reason is not true in this context. **Answer: C**

6) Assertion (A): If n(A)=m, then the number of reflexive relation on A is m.

Reason(R):A relation R on set A is said to be reflexive if $(a, a) \in R \;\forall\; a \in A$.

Solution: In a reflexive relation, every element of a set is connected to itself only. So number of reflexive relation is 2^{m^2-m}. **Answer: D**

7) Let A and B be two finite sets such that n(A)=5 and n(B)=2.Then

Assertion (A):Number of one-one functions from A to B is $5P_2$

Reason (R): Number of onto functions from A to B is 30.

Solution: Since n(A)> $n(B)$, *Number of one − one functions from A to B is zero.*

Assertion(A) is false. Reason (R) is true. **Answer: D**

9) Assertion (A) : If A and B are two sets having 3 and 5 elements respectively, then the total number of functions that can be defined from A to B is 5^3.

Reason (R): A function from set A to set B relates elements of set A to elements of set B.

Solution : A function from set A to set B relates every element of set A to a unique element in set B.

Consequently R is not true.

Since each element of set A can be associated to any one of five elements in B and there are 3 elements in set A

Total number of functions from A to B $= 5 \times 5 \times 5 = 5^3$. **Answer : C**

10) Assertion (A):The relation $f: \{1,2,3,4\} \rightarrow \{x, y, z, p\}$ defined by $f = \{(1, x), (2, y), (3, z)\}$ is a bijective function

Reason (R): The relation $f: \{1,2,3,4\} \rightarrow \{x, y, z, p\}$ defined by $f = \{(1, x), (2, y), (3, z), (4, p)\}$ is one-one.

Solution: Assertion is false since f is not a function .4 has no image under f. **Answer: D**

VERY SHORT ANSWER TYPE QUESTIONS

1) Let $A = \{a, b, c\}$ and the relation R be defined on A as follows: R = {(a, a), (b, c), (a, b)}. Then, write minimum number of ordered pairs to be added in R to make R reflexive and transitive.

Solution : We have relation $R = \{(a, a), (b, c), (a, b)\}$

To make R reflexive, we must add (b, b) *and* (c, c).
To make R transitive, we must add (a, c) *to* R.
Minimum number of ordered pairs to be added in R are (b, b) , (c, c) *and* (a, c).

2) Let C be the set of complex numbers. Prove that the mapping f : C→R given by $f(z) = |z|, \forall z \in C$, is neither one-one nor onto.
Solution: We have f : C→ R given by f (z) = |z|, ∀ z ∈ C
$f(3 + 4i)=|3 + 4i| = \sqrt{3^2 + 4^2} = 5$
$f(3 - 4i) = |3 - 4i| = \sqrt{3^2 + 4^2} = 5$
Thus f(z) is many-one.
Also $|z| \geq 0, \forall z \in C$
But co-domain given is R.
Hence $f(z)$ is not onto.

3) Let A={1,2,3},B ={4,5,6,7} and let f = {(1,4), (2,5), (3,6)} be a function from A to B. State whether f is one-one or not. Justify your answer.
Solution: Given f = {(1,4), (2,5), (3,6)}
Since every element of A has one and only one image in B under f, f is a one-one function.

4) A function f : A→ B *defined as* $f(x) = 2x$ is both one-one and onto. If A={1,2,3,4}then find the set B.
Solution: $When\ x = 1, f(1) = 2.1 = 2$
$When\ x = 2, f(2) = 2.2 = 4$
$When\ x = 3, f(3) = 2.3 = 6$
$When\ x = 4, f(4) = 2.4 = 8$
Since f is both one-one and onto Range of f = Codomain of f
B={2,4,6,8}

5) State whether the following statement is true or false. Justify your answer.
"The sine function is bijective in nature when the domain is $[0, 4\pi]$ ".
Solution : Given function f(x)=sin x, x∈ $[0, 4\pi]$
Let $x_1, x_2 \in [0, 4\pi]$ *such that* $f(x_1) = f(x_2)$
⇨ $\sin x_1 = \sin x_2$
⇨ $x_1 = n\pi + (-1)^n x_2 , n \in \{0,1,2,3\}$
⇨ $x_1 = x_2 \quad or \quad x_1 = \pi - x_2 \quad or \quad x_1 = 2\pi + x_2 \quad or \quad x_1 = 3\pi - x_2$
⇨ sin x is not one-one in $[0, 4\pi]$.
⇨ sine function is not bijective in $[0, 4\pi]$.
⇨ The given statement is false.

6) Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {(a, b) : b = a + 1} is reflexive, symmetric or transitive.
Solution: Let A ={1, 2, 3, 4, 5, 6}
Relation on R is defined as R = {(a, b) : b = a + 1}
In roster form
R={(1,2), (2,3), (3,4), (4,5), (5,6), (6,7)}
R is not reflexive since (1,1),(2,2),(3,3),(4,4),(5,5),(6,6) ∉R.
(1,2)∈ R *but* $(2,1) \notin R$.
R is not symmetric.
$(1,2), (2,3) \in R$ *but* $(1,3) \notin R$.
R is not transitive.
Hence R is neither reflexive, symmetric nor transitive.

7) Let A= {1,2,3,4} . Let R be the equivalence relation on A x A defined by

$(a,b)R(c,d)$ iff $a+d=b+c$. Find the equivalence class [(1,3)] .

Solution: Given on set A = {1,2,3,4} an equivalence relation R on A x A is defined by $(a,b)R(c,d)$ iff $a+d=b+c$

Let (1,3) R (x, y) for all (x, y) ∈ A x A

$$\Rightarrow\ 1+y\ =\ 3+x$$

$$\Rightarrow\ \ y-x\ =\ 2$$

Therefore (x, y) will be (1,3) and (2,4)

Hence [(1,3)] = {(1,3) , (2,4)}

8) A={1,3,5,7,….} and B={2,4,6,8,…}.Define a function from A to B that is neither one-one nor onto.

Solution: Given sets A={1,3,5,7,….} and B={2,4,6,8,…}.

Let f :A→ B $be\ defined\ by\ f(x)=2,\forall\ x\in A$.

Since all the elements of A have the same image 2 in B, f is not one-one.

Only the element 2 in B have a pre-image in A. Hence f is not onto.

f is neither one-one nor onto.

9) Let n be a fixed positive integer. Define a relation R in Z as follows: ∀ a,b ∈ Z, a R b if and only if a – b is divisible by n . Show that R is an equivalence relation.

Solution: Given that ∀ a,b ∈ Z, a R b if and only if a – b is divisible by n .

aRa => (a-a) is divisible by n, which is true for any integer a, as zero is divisible by n.

Hence R is reflexive.

aRb => (a-b) is divisible by n

- ⇨ - (b-a) is divisible by n
- ⇨ (b-a) is divisible by n
- ⇨ bRa
- ⇨ R is symmetric.

Let aRb and bRc.

=> (a-b) is divisible by n and (b-c) is divisible by n

=> (a-b) + (b-c) is divisible by n

=>a-c is divisible by n.

=> aRc

Hence R is transitive.

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

10) $f(x)=\frac{2\,tanx}{1-tan^2x}$. Find the range of f(x) for x ∈ R.

Solution: Given $f(x)=\frac{2\,tanx}{1-tan^2x}$

⇨ $f(x)=tan2x$

We know that $tan\ x\in(-\infty,\infty)$

$tan2x\in(-\infty,\infty)$

Range of f(x) is $(-\infty,\infty)$. i.e., R.

SHORT ANSWER TYPE QUESTIONS

1) Let f: A→ B be defined by $f(x)=\frac{x-2}{x-3}$, where A=R- {3} and $B=R-\{1\}$.

Discuss the bijectivity of the function.

Solution: To check f is one-one: $f(x_1)=f(x_2)$

$$\frac{x_1-2}{x_1-3}=\frac{x_2-2}{x_2-3}$$

=> $(x_1-2)(x_2-3)=(x_2-2)(x_1-3)$

=> $x_1x_2-3x_1-2x_2+6=x_1x_2-3x_2-2x_1+6$

=> $-3x_1+2x_1=-3x_2+2x_2$

=> $-x_1 = -x_2$

=> $x_1 = x_2$

Hence if $f(x_1) = f(x_2)\ then\ x_1 = x_2$

f is one-one

To check f is onto.

$$f(x) = \frac{x-2}{x-3}$$

Let f(x)=y such that y ∈ $B = R - \{1\}$

$$y = \frac{x-2}{x-3}$$

=> $y(x-3) = x-2$

=> $xy - 3y = x - 2$

=> $xy - x = 3y - 2$

=> $x(y-1) = 3y - 2$

=> $x = \frac{3y-2}{y-1}$

$For\ y = 1, x\ is\ not\ defined.$

$But\ it\ is\ given\ that\ y \in R - \{1\}$

Hence x = $\frac{3y-2}{y-1} \in R - \{1\}$

Checking value for y= f(x)

Putting value of x in f(x)

$$f(x) = f\left(\frac{3y-2}{y-1}\right)$$

=> f(x) = $\frac{\frac{3y-2}{y-1} - 2}{\frac{3y-2}{y-1} - 3}$

=>$f(x) = \frac{3y-2-2(y-1)}{3y-2-3(y-1)}$

=> f(x) = y

Thus for every $y \in B, there\ exists\ x \in A\ such\ that\ f(x) = y.$

Hence f is onto.

Since f is one-one and onto f is a bijective function.

2) Let N be the set of natural numbers and R be the relation on N$XN\ defined\ by\ (a,b)\ R\ (c,d)\ iff\ ad = bc\ for\ all\ a,b,c,d \in N. Show\ that\ R\ is\ an\ equivalence\ relation.$

Solution: Given a relation R on N× $N\ defined\ by\ (a,b)\ R\ (c,d)$
$iff\ ad = bc\ for\ all\ a,b,c,d \in N.$

Reflexive: Let (a,b) ∈ $N \times N\ such\ that\ (a,b)\ R\ (a,b)$

⇨ ab=ba (Product of two natural numbers is commutative)

⇨ R is reflexive

Symmetric: Let (a,b), (c,d) ∈ $N \times N\ such\ that\ (a,b)\ R\ (c,d)$

⇨ ad=bc => bc = ad => cb = da =>(c,d) R (a,b)

⇨ R is symmetric

Transitive: Let (a,b), (c,d),(e,f) ∈ $N \times N\ such\ that$

$(a,b)\ R\ (c,d)$=> ad=bc -----(i) and

(c,d) R (e,f) => cf = de ------(ii)

Multiplying (i) and (ii) we get

acdf = bcde => af = be => (a,b) R (e,f)

R is transitive.

Since R is reflexive, symmetric and transitive, the given relation R on N x N, is an equivalence relation.

3) Check whether the relation S in the set of real numbers R defined by S $=\{(a,b); a-b+\sqrt{2}\ is\ an\ irrational\ number\}$ is reflexive, symmetric or transitive.

Solution: Reflexive : For a $\in S$, $a-a+\sqrt{2}\ =\sqrt{2}\ is\ an\ irrational\ number.$

⇨ (a,a)$\in S$

Thus S is a reflexive relation.

Symmetric : Let (a,b)$\in S$

⇨ a – b + $\sqrt{2}\ is\ an\ irrational\ number$

But b – a + $\sqrt{2}\ may\ not\ be\ an\ irrational\ number.$

For example: $(\sqrt{2},1)\in S\ \Rightarrow \sqrt{2}-1+\sqrt{2}\ =2\sqrt{2}-1\ is\ an\ irrational\ number.$

$(1,\sqrt{2})\notin\ S\ as\ 1-\sqrt{2}\ +\sqrt{2}\ =1\ is\ not\ an\ irrational\ number.$

Here (a, b)$\in S$ but (b,a) $\notin$ ***S***. So S is not a symmetric relation.

Transitive : Let (a, b) $\in S$ => a – b + $\sqrt{2}\ is\ an\ irrational\ number$ and

(b, c) $\in S$ => b – c + $\sqrt{2}$ is an irrational number.

But a – c + $\sqrt{2}$ may not be an irrational number.

For Example :

$(1,\sqrt{3})\in\ S\ as\ 1-\sqrt{3}\ +\sqrt{2}\ is\ an\ irrational\ number.$

$(\sqrt{3},\sqrt{2})\in\ S\ as\ \sqrt{3}-\sqrt{2}\ +\sqrt{2}\ =\sqrt{3}\ is\ an\ irrational\ number$

But $(1,\sqrt{2})\notin\ S\ as\ 1-\sqrt{2}\ +\sqrt{2}\ =1\ is\ not\ an\ irrational\ number.$

=> (a, c) $\notin\ S. So\ S\ is\ not\ a\ transitive\ relation.$

Thus S is reflexive but neither symmetric nor transitive relation.

4) A student wants to pair up natural numbers in such a way that they satisfy the equation $2x\ +\ y=41, x,y\in\ N$.Find the domain and range of the relation. Check if the relation thus formed is reflexive, symmetric and transitive. Hence, state whether it is an equivalence relation or not.

Solution: We have R={(x, y):x$\in N, y\in N, 2x+y=41\}$

Since y ∊ N, Domain $=\{1,2,3,\ldots,20\}$

$$R=\{(1,39),(2,37),(3,35)\ldots\ldots,(19,3),(20,1)\}$$

Range = $\{1,3,5,\ldots\ldots,39\}$

R is not reflexive as (2,2) $\notin$ R as $2\times 2+2\neq 41$.

Also R is not symmetric as (1,39)$\in R\ but\ (39,1)\ \notin R$.

Further R is not transitive as (11,19)$\in R, (19,3)\in R but (11,3)\notin R$.

Hence R is neither reflexive, symmetric nor transitive.

Therefore R is not an equivalence relation.

5) A function f : R → R is defined by $f(x)=ax+b$,such that f(1)=1 and f(2)=3. Find function f(x). Hence check whether function f(x) is one-one and onto or not.

Solution: Given f(1)=1

⇨ $a+b=1$------(1)

$f(2)=3$

⇨ $2a+b=3$-----(2)

Solving equations (1) and (2) we get a=2 and b = -1.

$f(x)\ =\ 2x-1$ is the required function.

One-one : Let $x_1,x_2\in R\ such\ that\ x_1\neq x_2$

=> $2x_1\neq 2x_2$

=> $2x_1-1\neq 2x_2-1$

=> $f(x_1)\neq f(x_2)$

Hence f(x) is a one-one function.

Onto: Let y=f(x) $\in R$

$\Rightarrow$ y=2x-1

$\Rightarrow$ x=$\frac{y+1}{2}$

Thus for all y $\in R, \exists\, x = \frac{y+1}{2} \in R$ such that $f(x) = f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1 = y$

f(x) is an onto function.

Hence f is one-one and onto.

6) Show that the function f: R $\rightarrow R$ $defined\ by\ f(x) = \frac{x}{x^2+1}, \forall x \in R\ is\ neither\ one - one\ nor\ onto.$

Solution :

Let $x_1, x_2 \in R$ $such\ that\ f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1(x_2^2+1) = x_2(x_1^2+1)$$

$$\Rightarrow x_1x_2^2 + x_1 = x_2x_1^2 + x_2$$

$$\Rightarrow x_1x_2^2 - x_2x_1^2 = x_2 - x_1$$

$$\Rightarrow x_1x_2(x_2 - x_1) = x_2 - x_1$$

$$\Rightarrow x_1x_2(x_2 - x_1) - (x_2 - x_1) = 0$$

$$\Rightarrow (x_2 - x_1)(x_1x_2 - 1) = 0$$

$$\Rightarrow x_2 - x_1 = 0 \quad or \quad (x_1x_2 - 1) = 0$$

$\Rightarrow x_2 = x_1$ or $x_1x_2 = 1$

Here $f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2 \quad or \quad x_1 = \frac{1}{x_2}$

f is not one-one.

Let k ∊ R be any arbitrary element and let f(x)=k

$$\frac{x}{x^2+1} = k$$

$\Rightarrow \quad x = k(x^2+1)$

$\Rightarrow \quad kx^2 + k = x$

$$kx^2 - x + k = 0$$

- ⇨ $x = \frac{1 \pm \sqrt{1-4k^2}}{2k} \notin R, if\ 1 - 4k^2 < 0$
- ⇨ $or\ if\ (1-2k)(1+2k) < 0$
- ⇨ i.e., k> $\frac{1}{2}$ $or\ k < \frac{-1}{2}$
- ⇨ f is not onto.

Hence f is neither one-one nor onto.

7) Let f : R → R be the function defined by $f(x) = \frac{1}{2-cosx}$ ∀x ∈ R. Then, find the range of f.

Solution:

f : R → R, $f(x) = \frac{1}{2-cosx}$ ∀x ∈ R

Let $y = \frac{1}{2-cosx}$

$$2y - ycosx = 1$$

Cos x =$\frac{2y-1}{y}$

Cos x =2 - $\frac{1}{y}$

We know that -1≤ $cos\ x\ \leq 1$

$$-1 \le 2 - \frac{1}{y} \le 1$$

$$-1-2 \le 2 - \frac{1}{y} - 2 \le 1-2$$

$$-3 \le -\frac{1}{y} \le -1$$

$$1 \le \frac{1}{y} \le 3$$

$$\frac{1}{3} \le y \le 1$$

Range is $\left[\frac{1}{3}, 1\right]$

8) Show that the function f : (-∞, 0) → $(-1,0)$ *defined by* $f(x) = \frac{x}{1+|x|}$,

$x \in (-\infty, 0)$ *is one − one and onto.*

Solution: Given $f(x) = \frac{x}{1+|x|}$, $x \in (-\infty, 0)$

Since $x \in (-\infty, 0), |x| = -x$

Therefore $f(x) = \frac{x}{1-x}$

One-one: Let $x_1, x_2 \in (-\infty, 0)$

Now $f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$

$\Rightarrow x_1(1 - x_2) = x_2(1 - x_1)$

$\Rightarrow x_1 - x_1x_2 = x_2 - x_2x_1$

$\Rightarrow x_1 = x_2$

=>Hence f is one-one.

Onto :

Let y = $\frac{x}{1+|x|}$

$y = \frac{x}{1-x}$ *Since* $x \in (-\infty, 0)$

$\Rightarrow y(1 - x) = x$

- ⇨ $y - yx = x$
- ⇨ $y = x + yx$
- ⇨ $Y = x(1+y)$
- ⇨ $x = \frac{y}{1+y}$

For each $y \in (-1, 0)$ *there exists* $x \in (-\infty, 0)$ *such that*

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\frac{y}{1+y}}{1 - \frac{y}{1+y}} = y$$

Hence f is onto.

Thus f is both one-one and onto.

LONG ANSWER TYPE QUESTIONS

1) Prove that the relation R in the set of integers Z defined as R=$\{(a, b): 2$ *divides* $(a + b)\}$ *is an equivalence relation. Also determine* [3]

Solution: $a + a = 2a$, which is divisible by 2, ∀ a ∈ Z

$\Rightarrow (a, a) \epsilon R, \forall a \in Z$

⇒R is reflexive.

Symmetric: Let (a, b) ∈ R ⇒ 2 divides (a+b)

⇒ 2 divides (b+a)

⇒ (b, a) ∈ R
R is symmetric
Transitive: Let (a, b), (b, c)∈R
⇒ 2 divides (a+b) and (b+c) both
⇒ (a + b) =2m and (b +c) = 2n
⇒a+2b+c=2m+2n
⇒a+c=2(m+n-b)
⇒2 divides (a+c) ⇒ R is transitive.
Since R is reflexive, symmetric and transitive,
R is an equivalence relation.

$$[3] = \{x : x \; is \; an \; odd \; integer\}$$

Or $[3] = \{.....-1,1,3,5,7,.....\}$

2) If N denotes the set of all natural numbers and R is the relation on N × N defined by (a, b) R (c, d) if $ad(b + c) = b\,c(a + d)$. Show that R is an equivalence relation.

Solution:
Reflexive: By commutative law under addition and multiplication of natural numbers b+ $a = a + b \; and \; ab = ba \; \forall a, b \in N.$
$ab(b + a) = ba(a + b) \quad \forall a, b \in N$
$=> (a, b)R(a, b)$
Hence, R is reflexive.
Symmetric: Let $(a, b), (c, d) \in N \times N \; such \; that \; (a, b)R\,(c, d)$
$=> ad(b + c) = bc(a + d).$
$=> bc(a + d) = ad(b + c)$
$=> cb(d + a) = da(c + b)$
$=> (c, d)\;R\;(a, b)$
Hence, R is symmetric.
Transitive: Let (a, b), (c, d), (e, f) ∈ N X N such that
(a, b) R (c, d) and (c, d) R (e, f)
⇒ ad(b+c) = bc(a + d) and cf(d+e) = de(c +f)
=>$\frac{b+c}{bc} = \frac{a+d}{ad}$ and$\frac{d+e}{de} = \frac{c+f}{cf}$
=>$\frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a}$ $.......(1)$ and
$\frac{1}{e} + \frac{1}{d} = \frac{1}{f} + \frac{1}{c}$ ……………..(2)

$$Adding \; (1) \; and \; (2)$$

=>$\frac{1}{c} + \frac{1}{b} + \frac{1}{e} + \frac{1}{d} = \frac{1}{d} + \frac{1}{a} + \frac{1}{f} + \frac{1}{c}$

=>$\frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f}$

=>$\frac{e+b}{be} = \frac{f+a}{af}$
=>af(b+e)=be(a+f)
=>(a,b) R (e, f)
Hence, R is transitive.
Thus, R is reflexive, symmetric and transitive.
Therefore, R is an equivalence relation.

3) In the set of natural numbers N, define a relation R as follows:

∀n, m ∈ N, n R m if on division by 5 each of the integers n and m leaves the remainder less than 5, i.e. one of the numbers 0, 1, 2, 3 and 4. Show that R is an equivalence relation. Also, obtain the pairwise disjoint subsets determined by R.

Solution:

R is reflexive since for each a ∈ N, aRa.

R is symmetric since if aRb, then bRa for a, b ∈ N.

Also, R is transitive since for a, b, c ∈ N, if aRb and bRc, then aRc.

Hence R is an equivalence relation in N which will partition the set N into the pairwise disjoint subsets.

The equivalent classes are as mentioned below:

A_0 = {5, 10, 15, 20 ...}

A_1 = {1, 6, 11, 16, 21 ...}

A_2= {2, 7, 12, 17, 22, ...}

A_3 = {3, 8, 13, 18, 23, ...}

A_4 = {4, 9, 14, 19, 24, ...}

It is evident that the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^{4} A_i = N$$

4) Show that the function f: R→ R $defined\ as\ f(x) = x^2 + x + 1\ is\ neither\ one - one\ nor\ onto. Also\ find\ all\ the\ values\ of\ x\ for\ which\ f(x) = 3.$

Solution:

Given a function f: R→ $R\ defined\ as\ f(x) = x^2 + x + 1$

Let $x_1, x_2 \in R\ such\ that\ f(x_1) = f(x_2)$

=> $x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$

⇨ $(x_1^2 - x_2^2) + (x_1 - x_2) = 0$

⇨ $(x_1 - x_2)(x_1 + x_2) + (x_1 - x_2) = 0$

⇨ $(x_1 - x_2)(x_1 + x_2 + 1) = 0$

⇨ $(x_1 - x_2) = 0\ or\ (x_1 + x_2 + 1) = 0$

⇨ $x_1 = x_2\ or\ x_1 = -(x_2 + 1)$

⇨ f is not one-one.

Onto:

Let y=f(x)

⇨ Y = $x^2 + x + 1$

⇨ $x^2 + x + 1 - y = 0$

For x∈ $R, discriminant\ D \geq 0$

⇨ $1^2 - 4 \times 1 \times (1 - y) \geq 0$

⇨ 1-4+4y≥ 0

⇨ 4y-3≥ 0

⇨ 4y≥ 3

⇨ y≥ $\frac{3}{4}$

⇨ y∈ $[\frac{3}{4}, \infty)$

⇨ Range of f(x) is $[\frac{3}{4}, \infty) \neq Co - domain\ of\ f\ i.e., R$

⇨ f is not onto.

Hence f is neither one-one nor onto.

Also we have f(x)=3

$$\Rightarrow x^2 + x + 1 = 3$$
$$\Rightarrow x^2 + x - 2 = 0$$
$$\Rightarrow x^2 + 2x - x - 2 = 0$$
$$\Rightarrow x(x+2) - (x+2) = 0$$
$$\Rightarrow (x+2)(x-1) = 0$$
$$\Rightarrow x = 1, -2$$

CASE BASED QUESTIONS

1) Students of a school are taken to a railway museum to learn about railways heritage and its history.

An exhibit in the museum depicted many rail lines on the track near the railway station. Let L be the set of all rail lines on the railway track and R be the relation on L defined by R = $\{(l_1, l_2): l_1 \textit{is parallel to } l_2\}$.
On the basis of the above information answer the following questions:
(i) Find whether the relation R is symmetric or not.
(ii) Find whether the relation R is transitive or not.
(iii) (a) If one of the rail lines on the railway track is represented by the equation y=3x+2, then find the set of rail lines in R related to it. OR
(b)Let S be the relation defined by S = $\{(l_1, l_2): l_1 \textit{is perpendicular to } l_2\}$.
Check whether the relation S is symmetric and transitive.

Solution:

(i)Let $(l_1, l_2) \in R \Rightarrow l_1 \| l_2 \Rightarrow l_2 \| l_1 \Rightarrow (l_2, l_1) \in R \Rightarrow$ R is a symmetric relation.
(ii) Let $(l_1, l_2) \in R \Rightarrow l_1 \| l_2$ and Let $(l_2, l_3) \in R \Rightarrow l_2 \| l_3$
Since $l_1 \| l_2$ *and* $l_2 \| l_3, l_1 \| l_3 \Rightarrow (l_1, l_3) \in R$
Hence R is a transitive relation**.**
(iii)(a) The set is $\{l : l \textit{ is a line of type } y = 3x + c, c \in R\}$
(b) Let $(l_1, l_2) \in S \Rightarrow l_1 \perp l_2 \Rightarrow l_2 \perp l_1 \Rightarrow (l_2, l_1) \in S$
$\Rightarrow$ S is symmetric.
Let $(l_1, l_2) \in S \Rightarrow l_1 \perp l_2$ and
Let $(l_2, l_3) \in S \Rightarrow l_2 \perp l_3$

$$l_1 \perp l_2 \textit{ and } l_2 \perp l_3 \Rightarrow l_1 \| l_3$$

$\Rightarrow (l_1, l_3)$ *is not an element of S.*
S is not a transitive relation.

2) A class room teacher is keen to assess the learning of her students the concept of "relations" taught to them. She writes the following five relations each defined on the set A={1,2,3}.

$$R_1 = \{(2,3), (3,2)\}$$
$$R_2 = \{(1,2), (1,3), (3,2)\}$$
$$R_3 = \{(1,2), (2,1), (1,1)\}$$
$$R_4 = \{(1,1), (1,2), (3,3), (2,2)\}$$
$$R_5 = \{(1,1), (1,2), (3,3), (2,2), (2,1), (2,3), (3,2)\}$$

The students are asked to answer the following questions about the above relations:
(i) Identify the relation which is reflexive, transitive but not symmetric.

(ii) Identify the relation which is reflexive and symmetric but not transitive.

(iii) Identify the relations which are symmetric but neither reflexive nor transitive.

OR

(iv) What pairs should be added to the relation R_2 to make it an equivalence relation ?

Solution:

(i) R_4 (ii) R_5 (iii) R_1 (iv)$\{(1,1),(2,2),(3,3),(2,1),(3,1),(2,3)\}$

3) Let A be the set of 50 students of class XII in a school. Let f : A→ N, N is the set of natural numbers such that the function f(x) = Roll Number of student x. On the basis of the given information, answer the following:

(i) Is f a bijective function ?

(ii) Give reasons to support your answer to (i)

(iii) Let R be a relation defined by the teacher to plan the seating arrangement of students in pairs, where $R = \{(x,y): x,y\ are\ roll\ numbers\ of\ students\ such\ that\ y = 3x\}.$ *List the elements of R*. Is the relation R reflexive, symmetric and transitive? Justify your answer.

Solution :

(i) No

(ii) No two different students of the class can have same roll number. Therefore, f must be one-one.

We can assume without any loss of generality that roll numbers of students are from 1 to 50. This implies that 51 in N is not roll number of any student of the class, so that 51 cannot be image of any element of X under f.

Hence, f is not onto.

$(iii)\ R = \{(1,3),(2,6),(3,9),(4,12),(5,15),(6,18),(7,21),(8,24),(9,27),(10,30),(11,33),(12,36),(13,39),(14,42),(15,45),(16,48)\}$

R is not reflexive since (1,1), (2,2)…. ∉R

R is not symmetric

Example: (1,3)∈R but (3,1) ∉R

R is not transitive.

Example: (1,3)∈R,(3,9)∈R but (1,9) ∉R

Since R is not reflexive, symmetric and transitive, R is not an equivalence relation.

CHAPTER-2: INVERSE TRIGONOMETRIC FUNCTIONS

Gist/Summary of the lesson:

In mathematics, the trigonometric functions are also called as circular functions, angle functions.

Inverse trigonometric functions are the inverse functions of the basic trigonometric functions which are sine, cosine, tangent, cotangent, secant, and cosecant functions.

Definition: Inverse trigonometric functions are the inverse functions of the basic trigonometric functions which are sine, cosine, tangent, cotangent, secant, and cosecant functions

Principal Value Branches:

FUNCTION	DOMAIN	RANGE (Principal Value Branch)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\text{cosec}^{-1} x$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	R	$(0, \pi)$

Formulae:

- $\sin^{-1}(-x) = -\sin^{-1} x, x \in [-1,1]$
- $\tan^{-1}(-x) = -\tan^{-1} x, x \in R$
- $cosec^{-1}(-x), = -cosec^{-1}(x),\ x \in R - (-1,1)$
- $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1,1]$
- $\sec^{-1}(-x) = \pi - \sec^{-1} x, x \in R - (-1,1)$
- $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$
- $\sin^{-1} x + \cos^{-1} x = \pi/2, x \in [-1,1]$
- $\tan^{-1} x + \cot^{-1} x = \pi/2, x \in R$
- $\sec^{-1} x + cosec^{-1} x = \pi/2, x \in R - [-1,1]$

Additional Formulae

- $\sin^{-1} x + \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$
- $\sin^{-1} x - \sin^{-1} y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$
- $\cos^{-1} x + \cos^{-1} y = \cos^{-1}(xy - \sqrt{1-y^2}\sqrt{1-x^2})$
- $\cos^{-1} x - \cos^{-1} y = \cos^{-1}(xy + \sqrt{1-y^2}\sqrt{1-x^2})$
- $\tan^{-1} x + \tan^{-1} y = tan^{-1}\left(\frac{x+y}{1-xy}\right), if\ xy < 1$
- $\tan^{-1} x - \tan^{-1} y = tan^{-1}\left(\frac{x-y}{1+xy}\right), if\ xy > -1$
- $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$
- $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$
- $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$
- $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$
- $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$
- $3\tan^{-1} x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

MULTIPLE CHOICE QUESTIONS

1.The principal value of $\cos^{-1}\left(\frac{-1}{2}\right)$`

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{-\pi}{3}$ (d) $\frac{-\pi}{6}$

Solution: We have $cos^{-1}(-x) = \pi - cos^{-1}(x)$

$$\cos^{-1}\left(\frac{-1}{2}\right) = \pi - cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Ans: (a)

2. The principal value of $\sin^{-1}\left(\sin\left(\frac{3\pi}{5}\right)\right)$`

(a) $\frac{3\pi}{5}$ (b) $\frac{2\pi}{5}$ (c) $\frac{-2\pi}{5}$ (d) $\frac{\pi}{5}$

Solution: We have $\sin^{-1}\sin\left(\frac{3\pi}{5}\right) = \sin^{-1}\sin\left(\pi - \frac{3\pi}{5}\right) = \sin^{-1}\sin\left(\frac{2\pi}{5}\right) = \frac{2\pi}{5}$

Ans. (b)

3.The value of: $tan^{-1}\sqrt{3} - sec^{-1}(-2)$ is

(a) $\frac{\pi}{6}$ (b) $\frac{-\pi}{6}$ (c) $\frac{-\pi}{3}$ d) 0

Solution: We have $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$.

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \frac{\pi}{3} - (\pi - \frac{\pi}{3}) = \frac{-\pi}{3}$$

Ans**: (c)**

4. The value of $sin\left[\frac{\pi}{3} - sin^{-1}\left(\frac{-1}{2}\right)\right]$ is

(a) 0 (b) 1 (c) -1 (d) 2

Solution: We have $\sin^{-1}(-x) = -\sin^{-1}(x)$

$$\therefore \sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right] = \sin\left[\frac{\pi}{3} - \left(\frac{-\pi}{6}\right)\right] = \sin\left(\frac{\pi}{2}\right) = 1$$

Ans: (b)

5. The principal value of $cos^{-1}\left(cos\left(\frac{-7\pi}{3}\right)\right)$ is

(a) $\frac{7\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{-\pi}{3}$ (d) $\frac{-7\pi}{3}$

Solution: We have $cos(-x) = cosx$

$$\therefore \cos^{-1}\cos\left(\frac{-7\pi}{3}\right) = \cos^{-1}\cos\left(\frac{7\pi}{3}\right) = \cos^{-1}\cos\left(2\pi + \frac{\pi}{3}\right) = \cos^{-1}\cos\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$$ **Ans: (b)**

6.The value of $tan^{-1}\sqrt{3} - cot^{-1}(-\sqrt{3})$ is

(a) $\frac{\pi}{2}$ (b) $\frac{-\pi}{2}$ (c) $\frac{-\pi}{3}$ (d) $\frac{\pi}{6}$

Solution: $\cot^{-1}(-\sqrt{3}) = \pi - \cot^{-1}(\sqrt{3}) = \pi - \frac{\pi}{6} = 5\frac{\pi}{6}$

$$\therefore \tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \frac{5\pi}{6} = -\frac{\pi}{2}$$

Ans: **(b)**

7. The value of x if $tan^{-1}\sqrt{3} + cot^{-1}x = \frac{\pi}{2}$

(a) $\sqrt{3}$ (b) $-\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{\pi}{6}$

Solution: $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}\sqrt{3} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \quad \therefore x = \cot\frac{\pi}{6} = \sqrt{3}$

Ans: (a)

8. The value of x if $sec^{-1}2 + cos\,e\,c^{-1}x = \frac{\pi}{2}$

(a) $\sqrt{3}$ (b) 2 (c) $\frac{\sqrt{3}}{2}$ (d) -2

Solution:

$\cos ec^{-1}x = \frac{\pi}{2} - \sec^{-1}2 = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \quad \therefore x = \cos ec\frac{\pi}{6} = 2$

Ans: (b)

9. If $sin^{-1}x = y$ then the principal value of y is:

(a) $0 \le y \le \pi$ (b) $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ (c) $\frac{-\pi}{2} < y < \frac{\pi}{2}$ (d) $0 < y < \pi$

Ans: (b)

10. If $tan^{-1}x = y$ then the principal value of y is:

(a) $0 \le y \le \pi$ (b) $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$ (c) $\frac{-\pi}{2} < y < \frac{\pi}{2}$ (d) $0 < y < \pi$

Ans: (c)

ASSERTION-REASON BASED QUESTIONS

In the following questions, a statement of assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choices.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true but R is not the correct explanation of A.

(c) A is true but R is false.

(d) A is false but R is true.

1. ASSERTION (A): Principal value of $cos^{-1}cos\left(\frac{7\pi}{6}\right)$ is $\frac{5\pi}{6}$

   REASON (R): Range of principal branch of $\cos^{-1}x$ is $[0, \pi]$ and $\cos^{-1}\cos x = x$ if $x \in [0, \pi]$

   Ans: (a)

2. ASSERTION (A): Principal value of $sin^{-1}sin\left(\frac{13\pi}{6}\right)$ is $\frac{\pi}{6}$

   REASON (R): $\sin^{-1}(-x) = -\sin^{-1}(x)$

   Ans:(b)

3. ASSERTION (A): Principal value of $\sin^{-1}(-1) = \frac{-\pi}{2}$

   REASON (R): $\sin^{-1}(-x) = -\sin^{-1}(x)$

   Ans: (a)

4. ASSERTION (A): Principal value of $sin^{-1}sin\left(\frac{3\pi}{5}\right) = \frac{3\pi}{5}$

   REASON (R): $\sin^{-1}\sin(x) = x, x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

   Ans: (d)

5. ASSERTION (A): The principal value of $\cos^{-1}\left(\cos\left(\frac{-\pi}{2}\right)\right)$ is $\frac{-\pi}{2}$ `

   REASON (R): Cosine function is an even function, therefore $cos(-x) = cosx$.

Ans: (d)

6. ASSERTION (A): The principal value of $\cos^{-1}\left(\frac{-1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right)$`

REASON (R): Range of $\cos^{-1}x$ is $[0, \pi]$

Ans: (b)

7. ASSERTION (A): The principal value of $\tan^{-1}\sin\left(\frac{-\pi}{2}\right) = \frac{-\pi}{2}$`

REASON (R): $\tan^{-1}(-x) = -\tan^{-1}(x)$

Ans: (d)

8. ASSERTION (A): The principal value of $\tan^{-1}\tan\left(\frac{-\pi}{4}\right) = \frac{-\pi}{4}$`

REASON (R): Range of $\tan^{-1}x$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\tan^{-1}(\tan x) = x$ if $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Ans: (a)

9. ASSERTION(A): One branch of $\cos^{-1}x$ other than the principal value branch is $[\pi, 2\pi]$

REASON (R): $cos\left(\frac{-\pi}{2}\right) = -1$

Ans: (c)

10. ASSERTION (A): A branch of $\sin^{-1}x$ other than principal branch is $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

REASON (R): $\sin\left(\frac{3\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Ans: (c)

VERY SHORT ANSWER TYPE QUESTIONS

1. Find the value of $sin^{-1}\left(sin\left(\frac{2\pi}{3}\right)\right) + cos^{-1}\left(cos\left(\frac{2\pi}{3}\right)\right)$.

Solution: We have $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

Value of $cos^{-1}\left(cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$

$\therefore$ $sin^{-1}\left(sin\left(\frac{2\pi}{3}\right)\right) + cos^{-1}\left(cos\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3} + \frac{2\pi}{3} = \pi$

2. Find the value of : $tan^{-1}\left[2\, cos\left(sin^{-1}\left(\frac{1}{2}\right)\right)\right]$

Solution: $tan^{-1}\left[2\, cos\left(sin^{-1}\left(\frac{1}{2}\right)\right)\right] = tan^{-1}\left[2\, cos\left(\frac{\pi}{6}\right)\right] = tan^{-1}\left[2 \times \frac{\sqrt{3}}{2}\right] = tan^{-1}\sqrt{3} = \frac{\pi}{3}$

3. Find the value of : $tan^{-1}\left[2\, sin\left(2\, cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]$

Solution: $\tan^{-1}\left[2\sin\left(2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right] = \tan^{-1}\left[2\sin\left(2\frac{\pi}{6}\right)\right] = \tan^{-1}\left[2 \times \frac{\sqrt{3}}{2}\right] = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

4. If $cot^{-1}\left(\frac{1}{5}\right)$ then find the value of $\sin x$

Solution: $\cot x = \frac{1}{5}$ $\therefore \sin x = \frac{5}{\sqrt{26}}$

5. Find the value of $sin^{-1}\left(\frac{-1}{2}\right) + 2\, cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

Solution: $\sin^{-1}\left(\dfrac{-1}{2}\right) = -\sin^{-1}\left(\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$,

$$\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\sin^{-1}\left(\frac{-1}{2}\right) + 2\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{-\pi}{6} + 2 \times \frac{5\pi}{6} = \frac{3\pi}{2}$$

6. Show that for $|x| < 1, \sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$

Solution: Let $\tan^{-1} x = y$, $\tan y = x$

$\therefore$ L H S = $\sin y = \dfrac{x}{\sqrt{1+x^2}} = R\ H\ S$

7. Prove that : $\tan\left(\dfrac{1}{2}\sin^{-1}\dfrac{3}{4}\right) = \dfrac{4-\sqrt{7}}{3}$

Solution: Let $\sin^{-1}\dfrac{3}{4} = x$ $\therefore$ $\sin x = \dfrac{3}{4}$ $\therefore$ $\cos x = \dfrac{\sqrt{7}}{4}$

L H S = $tan\frac{x}{2} = \sqrt{\frac{1-cos\,x}{1+cos\,x}} = \sqrt{\frac{1-\frac{\sqrt{7}}{4}}{1+\frac{\sqrt{7}}{4}}} = \frac{4-\sqrt{7}}{3} = RHS$

8. Find the value of $\tan\left(2\tan^{-1}\dfrac{1}{5} - \dfrac{\pi}{4}\right)$

Solution: Let $\tan^{-1}\dfrac{1}{5} = x$ $\therefore$ $\tan x = \dfrac{1}{5}$

$\therefore$ $tan\left(2\,tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = tan\left(2x - \frac{\pi}{4}\right) = \frac{tan\,2x - tan\frac{\pi}{4}}{1+tan\,2x\,tan\frac{\pi}{4}} = \frac{-7}{17}$

where, $tan\,2\,x = \frac{2\,tan\,x}{1-tan^2 x} = \frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2} = \frac{5}{12}$

9. Find the value of $\boldsymbol{sin^{-1}\left(\frac{-1}{2}\right) + cos^{-1}\left(\frac{-1}{2}\right) + tan^{-1}(1)}$

Solution: $sin^{-1}\left(\frac{-1}{2}\right) = -sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$

$cos^{-1}\left(\frac{-1}{2}\right) = \pi - cos^{-1}\left(\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ $\quad \tan^{-1}(1) == \dfrac{\pi}{4}$

$\therefore sin^{-1}\left(\frac{-1}{2}\right) + cos^{-1}\left(\frac{-1}{2}\right) + tan^{-1}(1) = \frac{-\pi}{6} + \frac{2\pi}{3} + \frac{\pi}{4} = \frac{3\pi}{4}$

10. Find the value of $\sin\left(2\sin^{-1}\dfrac{3}{5}\right)$

Solution: Let $\sin^{-1}\left(\dfrac{3}{5}\right) = \theta$ $\therefore$ $\sin\theta = \dfrac{3}{5}$ $\therefore$ $\sin\left(2\sin^{-1}\dfrac{3}{5}\right) = \sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times \dfrac{3}{5} \times \dfrac{4}{5} = \dfrac{24}{25}$

SHORT ANSWER TYPE QUESTIONS

1. Write the principal value of $\cos^{-1}[\cos(680)°]$

Solution: We know that, principal value branch of $\cos^{-1} x$ is [0, 180°].

Since, 680° ∉ [0, 180°], so write 680° as 2 × 360° – 40°

Now, $\cos^{-1} [\cos(680)°] = \cos^{-1} [\cos(2 \times 360° – 40°)]$

$= \cos^{-1} (\cos 4o°)$ [∵ cos(4π – θ) = cos θ]

Since, 40° ∈ [0, 180°]

∴ $\cos^{-1}[\cos(680°)] = 40°$ [∵ $\cos^{-1} (\cos \theta) = \theta$; ∀ θ ∈ [0, 1 80°]]

2. Write the principal value of the following. $\cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(\frac{-1}{2}\right)$

Solution: $\cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(\frac{-1}{2}\right) = \cos^{-1}\frac{\sqrt{3}}{2} + \pi - \cos^{-1}\left(\frac{1}{2}\right)$

$= \cos^{-1}\left(\cos\frac{\pi}{6}\right) + \pi - \cos^{-1}\left(\cos\frac{\pi}{3}\right) = \frac{\pi}{6} + \pi - \frac{\pi}{3} = \frac{5\pi}{6}$

3. Find the value of $\cos^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) + \operatorname{cosec}^{-1}(\frac{1+\sqrt{x}}{1-\sqrt{x}})$

Solution: $\cos^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) + \operatorname{cosec}^{-1}(\frac{1+\sqrt{x}}{1-\sqrt{x}}) = \cos^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) + \sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) = \frac{\pi}{2}$

4. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$

Solution: Let $\tan^{-1}\frac{1}{5} = \theta \Longrightarrow \tan\theta = \frac{1}{5}$

$$\tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{\frac{2\times1}{5}}{1-\left(\frac{1}{5}\right)^2} = \frac{5}{12}$$

5. $3\cos^{-1}x = \cos^{-1}(4x^3 – 3x)$, $x \in [\frac{1}{2},1]$

Solution: Consider, RHS = $\cos^{-1} (4x^3 – 3x)$

Let $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

RHS = $\cos^{-1} (4 \cos^3 \theta – 3 \cos \theta)$

$= \cos^{-1} (\cos 3\theta)$ [∵ $\cos 3A = 4 \cos^3 A – 3 \cos A$] $= 3 \cos^{-1} x$ = LHS

6. $3\sin^{-1}x = \sin^{-1}(3x – 4x^3)$, $x \in [-\frac{1}{2},\frac{1}{2}]$

Solution: Consider, RHS = $\sin^{-1} (3x – 4x^3)$ ……(i)

Let $x = \sin\theta$, then $\theta = \sin^{-1} x$

from Eq. (i), we get RHS = $\sin^{-1} (3 \sin\theta – 4 \sin^3\theta) = \sin^{-1} (\sin 3\theta)$

[∵ $\sin 3A = 3\sin A – 4\sin^3 A$] $= 3\theta = 3 \sin^{-1} x$ [∵ $\theta = \sin^{-1} x$] = LHS

7. Write in simplest form: $\tan^{-1}\left(\frac{\cos x-sinx}{cosx+sinx}\right)$

Solution: $\tan^{-1}\left(\frac{\cos x-sinx}{cosx+sinx}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}(tan\left(\frac{\pi}{4}-x\right)) = \frac{\pi}{4}-x$

8. Find the value of $\sec(\tan^{-1}(-\sqrt{3}))$=

Solution: $\sec(\tan^{-1}(-\sqrt{3})) = \sec(-\tan^{-1}(\sqrt{3})) = \sec(\tan^{-1}(\sqrt{3}))$

$= \sec(\tan^{-1}\left(\tan\frac{\pi}{3}\right)) = \sec\frac{\pi}{3} = 2$

9. If $\cos^{-1}\frac{1}{x} = \theta$, then write tanθ in terms of x

Solution: $\cos\theta = \frac{1}{x} \Rightarrow \sec\theta = x \Rightarrow \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{x^2-1}$

10. Prove that $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{1+x}\right) = \frac{1}{2}\cos^{-1}x$

Solution: $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{1+x}\right) = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$ Let $x = \cos\theta$ => $\theta = \cos^{-1}x$

$=\tan^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)=\tan^{-1}\left[\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right]=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\cos^{-1}x$

LONG ANSWER TYPE QUESTIONS

1. Prove that $\sin^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{8}{17}\right)=\cos^{-1}\left(\frac{84}{85}\right)$

Solution: Let $\sin^{-1}\left(\frac{3}{5}\right)=A$ and $\sin^{-1}\left(\frac{8}{17}\right)=B$

Thus, we can write $\sin A=\frac{3}{5}$ and $\sin B=\frac{8}{17}$

Now, find the value of cosA and cosB

$\cos A=\sqrt{[1-\sin^2 A]}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$

Thus, the value of $\cos A=\frac{4}{5}$

$\cos B=\sqrt{[1-\sin^2 B]}=\sqrt{1-\left(\frac{8}{17}\right)^2}=\sqrt{1-\frac{64}{289}}=\sqrt{\frac{225}{289}}=\frac{15}{17}$

Thus, the value of $\cos B=\frac{15}{17}$

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

substitute the values

$\cos(A-B)=\left(\frac{4}{5}\right)\times\left(\frac{15}{17}\right)+\left(\frac{3}{5}\right)\times\left(\frac{8}{17}\right)$

$\cos(A-B)=\frac{60+24}{17\times5}$

$\cos(A-B)=\frac{84}{85}$

$(A-B)=\cos^{-1}\left(\frac{84}{85}\right)$

Substituting the values of A and B $\sin^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{8}{7}\right)=\cos^{-1}\left(\frac{84}{85}\right)$

2. i) Find the value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$.

ii) If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$; $xy<1$, then write the value of $x+y+xy$

Solution: i). The value of $\cos^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{3}$.

The value of $\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{6}$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}+2\left(\frac{\pi}{6}\right),=\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$

ii) $\frac{x+y}{1-xy}=\tan\frac{\pi}{4}=1\Rightarrow x+y=1-xy\Rightarrow x+y+xy=1$

3. Write the principal value of $\tan^{-1}(1)+\cos^{-1}\left(\frac{-1}{2}\right)$.

Solution: $\tan^{-1}(1)+\cos^{-1}\left(\frac{-1}{2}\right)=\frac{\pi}{4}+\pi-\cos^{-1}\left(\frac{1}{2}\right)$

$=\frac{\pi}{4}+\pi-\cos^{-1}\left(\cos\frac{\pi}{4}\right)=\frac{\pi}{4}+\pi-\frac{\pi}{4}=\frac{11\pi}{12}$

4. Write the value of $\tan^{-1}\left(2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right)$

Solution: $\tan^{-1}\left(2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right)=\tan^{-1}\left(2\sin\left(2\cos^{-1}\left(\cos\frac{\pi}{6}\right)\right)\right)$

$=\tan^{-1}\left(2\sin\left(2\times\frac{\pi}{6}\right)\right)=\tan^{-1}\left(2\sin\left(\frac{\pi}{3}\right)\right)=\tan^{-1}\left(2\times\frac{\sqrt{3}}{2}\right)$

$=\tan^{-1}(\sqrt{3})=\tan^{-1}\left(\tan\frac{\pi}{3}\right)=\frac{\pi}{3}$

5. If $\tan^{-1}a+\tan^{-1}b+\tan^{-1}c=\pi$ prove that a + b + c = abc

Solution: $\tan^{-1}a=A,\tan^{-1}b=B,\tan^{-1}c=C$

- ⇨ a=tan A   b=tan B   c = tan C
- ⇨ A+B+C= π => A+B= $\pi-C$

⇨ $\tan(A+B) = \tan(\pi - C) = -\tan C$

⇨ $\frac{\tan A+\tan B}{1-\tan A\tan B} = -\tan C \Rightarrow \tan A + \tan B = \tan A \tan B \tan C - \tan C$

⇨ $\tan A + \tan B + \tan C = \tan A \tan B \tan C \Rightarrow a + b + c = abc.$

6. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, then show that $xy + yz + zx = 1$

Solution: $\tan^{-1} x = A$, $\tan^{-1} y = B$, $\tan^{-1} z = C \Rightarrow x = \tan A, y = \tan B, z = \tan C$

$$\Rightarrow A+B+C=\frac{\pi}{2} \Rightarrow \tan(A+B) = \tan\left(\frac{\pi}{2} - C\right) = \cot C = \frac{1}{\tan C}$$

$$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B} = \frac{1}{\tan C} \Rightarrow \tan A\tan C + \tan B\tan C = 1 - \tan A\tan B\tan C$$

$$\Rightarrow \tan A\tan C + \tan B\tan C + \tan A\tan C = 1$$

7. Prove that $\cot^{-1}\left(\frac{\sqrt{1+sinx}+\sqrt{1-\sin x}}{\sqrt{1+sinx}-\sqrt{1-sinx}}\right) = \frac{x}{2}$

Solution: $\cot^{-1}\left(\frac{\sqrt{1+sinx}+\sqrt{1-\sin x}}{\sqrt{1+\text{sinx}}-\sqrt{1-\text{sinx}}}\right)$

$$\Rightarrow \cot^{-1}\left(\frac{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\cos\frac{x}{2}sin\frac{x}{2}}+\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\cos\frac{x}{2}sin\frac{x}{2}}}{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\cos\frac{x}{2}sin\frac{x}{2}}-\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\cos\frac{x}{2}sin\frac{x}{2}}}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right)$$

$$\Rightarrow \cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}+\sin\frac{x}{2}}\right) = \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2}$$

8. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$, $\frac{-1}{\sqrt{2}} \le x \le 1$

Solution: Let $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$$\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2cos^2\frac{\theta}{2}}-\sqrt{2sin^2\frac{\theta}{2}}}{\sqrt{2cos^2\frac{\theta}{2}}+\sqrt{2sin^2\frac{\theta}{2}}}\right) = \tan^{-1}\left(\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)$$

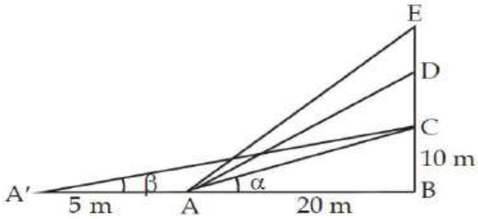
$$= \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$$

CASE BASED QUESTIONS

1) The Government of India is planning to fix a hoarding board at the face of a building on the road of a busy market for awareness on COVID-19 protocol. Ram, Robert and Rahim are the three engineers who are working on this project. 'A' is considered to be a person viewing the hoarding board 20 metres away from the building, standing at the edge of a pathway nearby, Ram Robert and Rahim suggested to the film to place the hoarding board at three different locations namely C, D and E. 'C' is at the height of 10 metres from the ground level. For the viewer 'A', the angle of elevation of 'D' is double the angle of

elevation of 'C'. The angle of elevation of 'E' is triple the angle of elevation of 'C' for the same viewer.

Look at the figure given and based on the above information answer the following:



(i) Measure of ∠CAB =

Ans. $\tan^{-1}(\frac{1}{2})$

(ii) Measure of ∠DAB =

Ans. $\tan^{-1}(\frac{4}{3})$

(iii) Measure of ∠EAB

Ans. $\tan^{-1}(\frac{11}{21})$

2). The following table gives inverse trigonometric functions along with domain and range

FUNCTION	DOMAIN	RANGE (Principal Value Branch)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1} x$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	R	$(0, \pi)$

i. Value of $\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right)$ is Ans: $\frac{11\pi}{12}$

ii. Value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) + \cot^{-1}\frac{1}{\sqrt{3}}$ is Ans. $\frac{\pi}{6}$

iii. Principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ Ans. $\frac{\pi}{3}$

OR

iv. Principal value of $\sec^{-1}(-2)$ Ans. $\frac{2\pi}{3}$

CHAPTER 3 : MATRICES

Gist of the Lesson:

- Matrix representation and order of matrix
- Types of Matrices
- Operations on Matrices
- Transpose of a Matrix
- Symmetric and Skew Symmetric Matrices
- Invertible Matrices

Definitions:

Order of Matrix:

If a matrix has m rows and n columns, then it is known as the matrix of order m x n.

Representation of matrix

A general matrix of order $m \times n$ can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

$$= [a_{ij}]_{m\times n}, where\ i = 1,2, \dots m\ and\ j = 1,2, \dots n$$

Types of Matrices:

Depending upon the order and elements, matrices are classified as:

- Column matrix
- Row matrix
- Square matrix
- Diagonal matrix
- Scalar matrix
- Identity matrix
- Zero matrix

Operation of Matrices:

- **ADDITION OF MATRICES:** Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices of the same order. Then $A + B$ is defined to be the matrix of order of $m \times n$ obtained by adding corresponding elements of $A\ and\ B$

  i.e $A + B = [a_{ij} + b_{ij}]_{m\times n}$

- **DIFFERENCE OF MATRICES:** Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices of the same order. Then $A - B$ is defined to be the matrix of order of $m \times n$ obtained by subtracting corresponding elements of $A\ and\ B$

i.e $A - B = [a_{ij} - b_{ij}]_{m\times n}$

- **MULTIPLICATION OF MATRICES:** The product of two matrices A and B is defined if the number of columns of A is equal to the number of *rows* of B.

  Let $= [a_{ij}]_{m\times n}$ *and* $B = [b_{jk}]_{n\times p}$. Then the product of the matrices A and B is the matrix C of order m × p

- **MULTIPLICATION OF A MATRIX BY A SCALAR:** Let $A = [a_{ij}]_{m\times n}$ and k is a scalar, then $kA = k[a_{ij}]_{m\times n} = [k.a_{ij}]_{m\times n}$

➢ **TRANSPOSE OF A MATRIX:** If $A = [a_{ij}]_{m\times n}$ be an m × n matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A'or A^T.

If $A = [a_{ij}]_{m\times n}$, then $A' = [a_{ji}]_{n\times m}$

➢ **SYMMETRIC MATRIX:** A square matrix If $A = [a_{ij}]$ is said to be symmetric if $A^T = A$

Example: $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{5} & \mathbf{12} \\ \mathbf{5} & \mathbf{7} & \mathbf{3} \\ \mathbf{12} & \mathbf{3} & \mathbf{6} \end{bmatrix}$, .

➢ **SKEW-SYMMETRIC MATRIX:** A square matrix $A = [a_{ij}]$ is said to be skew symmetric matrix if $A^T = -A$.

Example: $\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{5} & \mathbf{-12} \\ \mathbf{-5} & \mathbf{0} & \mathbf{-3} \\ \mathbf{12} & \mathbf{3} & \mathbf{0} \end{bmatrix}$

➢ **INVERTIBLE MATRICES:** If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the inverse matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

➢ **PROPERTIES OF MATRICES:**

- $A + B = B + A$ $A - B \neq B - A$ $AB \neq BA$ (AB)C=A(BC)
- (A')'=A AI=IA=A AB=BA=I then $A^{-1} = B$ and B^{-1}=A
- $AB = 0 \implies$ *it is not necessary that one the matrix is zero*
- $A(B + C) = AB + AC$
- Every square matrix can be expressed as the sum of symmetric and skew-symmetric matrices.
- $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A'}) + \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A'})$ where $\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A'})$ is symmetric matrix and $\frac{1}{2}(\boldsymbol{A} - \boldsymbol{A'})$ is skew-symmetric matrices.

MULTIPLE CHOICE QUESTIONS

1. A is 2×2 matrix and A=$[a_{ij}]$ where $a_{ij} = (i + j)^2$, then A is

(a) $\begin{bmatrix} 4 & 9 \\ 9 & 16 \end{bmatrix}$ (b) $\begin{bmatrix} 9 & 4 \\ 16 & 9 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ (d) *none of the above*

Solution: $a_{11} = 4$, $a_{12} = 9$, $a_{21} = 9$, $a_{22} = 16$ Option (a) $\begin{bmatrix} 4 & 9 \\ 9 & 16 \end{bmatrix}$

2. A and B are two matrices such that $AB = A$ *and* $BA = B$ then B^2 is

(a) A (b) B (c) 0 (d) I

Solution: B^2=BB=(BA)B=B(AB)=BA=B Option: (b) B

3. If A = $\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is unit matrix, then what is value of x?

(a) 1 (b) 2 (c) 0 (d) -1

Solution: $A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$x^2 + 1 = 1$, x=0 Option (c) 0

4. A = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then A^{10} is

(a) 10A (b)9A (c) 2^9A (d)2^{10}A

Solution: $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ =2A, A^3 =4I=2^2A so $A^{10} = 2^9$I

Option: (c) 2^9A

5. A is a 3 × 4 matrix. A matrix B is such that A'B and BA' are defined. Then the order of B is

(a) 4 × 3 (b) 3 × 3 (c) 4 × 4 (d) 3 × 4

Solution: Let $O(B) = m \times n$ A'B is defined. So $m = 3$. BA' is defined.
So $n = 4$. Option: (d) 3 × 4

6. A and B are symmetric matrices of same order , then $AB^T - BA^T$ is always

(a) symmetric matrix (b) skew symmetric matrix

(c) zero matrix (d) unit matrix

Solution: $(AB^T - BA^T)^T = (B^T)^T A^T - (A^T)^T B^T = BA^T - AB^T$

Option: (b)skew symmetric matrix

7. The number of all possible matrices of order 3 × 3 with each entry 0 or 1 is

(a) 32 (b) 64 (c) 512 (d) none of these

Solution: There are nine places. Each can be filled in two ways. 2^9ways

Option: (c) 512

8. A is a matrix of order 2×3 and B is a matrix of order 3×2. $C = AB$ *and* $D = BA$, then order of CD is

(a) 3×3 (b) 2×2 (c) 3×2 (d) CD not defined

Solution: O(C)=2×2 and O(D)=3×3. The number of columns of A not equal to number of rows of B. Therefore CD not defined

Option: (d) CD not defined

9. If A= $\begin{bmatrix} 1 & 12 & 4y \\ 6x & 5 & 2x \\ 8x & 4 & 6 \end{bmatrix}$ is a symmetric matrix, then (2x+y) is

(a) -8 (b) 0 (c) 6 (d) 8

Solution: Since A is symmetric, 6x=12 x=2; 8x=4y,y=4 then 2x+y = 8

Option: (d) 8

10.If $\begin{bmatrix} 2x-1 & 3x \\ 0 & y^2-1 \end{bmatrix} = \begin{bmatrix} x+3 & 12 \\ 0 & 35 \end{bmatrix}$, then the value of x – y is

(a) 2 or 10 (b) -2 or 10 (c) 2 or -10 (d) -2 or -10

Solution: 2x-1=x+3, x=4; $y^2 - 1$=35, y=$\pm$6, then x-y = -2 or 10

Option: (b) -2 or 10

ASSERTION REASON BASED QUESTIONS

In the following questions from 1 to 10 , a statement of Assertion(A) is followed by a statement of Reason (R).Choose the correct answer out of the following choices.

(a) Both(A) and (R) are true and (R) is the correct explanation of (A).
(b) Both(A) and (R) are true and (R) is not the correct explanation of (A)
(c) (A) is true and (R) is false
(d) (A) is false and (R) is true

1) **Assertion (A):** If a matrix is skew symmetric, then its diagonal elements must be zero.
   Reason (R): A matrix A is skew symmetric if $A^T = -A$
   Solution: By definition of skew symmetric matrix $a_{ij} = -a_{ji}$. So $a_{jj} = -a_{jj}.\ hence\ \ a_{jj} = 0$
   Both the statements are true and second is not the reason for first.
   Option (b)
2) **Assertion (A):** A= $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and B= $\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}$, Then $(A + B)^2 = A^2 + 2AB + B^2$
   Reason (R): A and B for two matrices$(A + B)^2 = A^2 + 2AB + B^2$ if AB=BA
   Solution: $(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$ only if $AB + BA$ both the statements are true and second statement is reason for first.
   Option: (a)
3) **Assertion (A):**If A and B are symmetric matrices of same order, AB-BA is skew symmetric matrix
   Reason (R): If A and B are symmetric matrices of same order, AB+BA is symmetric matrix
   Solution: Both the statements are correct. . But second statement is not the reason for first statement.
   $(AB + BA)^T$=$(AB)^T + (BA)^T = B^TA^T + A^TB^T = BA + AB$ and
   $(AB - BA)^T$=$(AB)^T - (BA)^T = B^TA^T - A^TB^T = BA - AB = -(AB - BA)$ Option: (b)
4) **Assertion (A):** The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 5 & -7 & 8 \\ 0 & -1 & 9 \end{bmatrix}$ can be expressed as sum of a symmetric and a skew symmetric matrices
   Reason (R): If A and B, are skew symmetric matrices of same order, then AB is symmetric if AB=BA
   Solution: Assertion is correct, since any matrix A can be written as $A = P + Q\ where \frac{A+A^T}{2}\ and\ Q = \frac{A-A^T}{2}$ where P symmetric and Q skew symmetric.
   Reason is also correct, since $(AB)^T = B^TA^T = -B \times -A = BA$. AB is symmetric if AB=BA But second statement is not correct reason for first. Option: (b)
5) **Assertion (A):** If a matrix is skew symmetric, then its diagonal elements must be zero.
   Reason (R): A matrix A is skew symmetric if $A^T = -A$
   Solution: By definition of skew symmetric matrix $a_{ij} = -a_{ji}$. So $a_{jj} = -a_{jj}.\ hence\ \ a_{jj} = 0$
   Both the statements are true and second is not the reason for first Option (a)
6) Assertion (A):The points $A(a, b + c), B(b, c + a)\ and\ C(c, a + b)$ are collinear
   Reason (R): Area of a triangle with three collinear points is zero.
   Solution: Assertion is true. Reason is correct explanation of assertion Option (a)
7) **Assertion (A):** If the product of two matrices is the identity matrix, both matrices must be invertible.
   Reason (R): The identity matrix is only obtained when a matrix is multiplied by its inverse.
   Solution: By definition inverse of a matrix A exists if and only if $AA^{-1} = I = A^{-1}A$. So both are correct and second is correct explanation for first Option: (a)
8) **Assertion (A):**If A is a square matrix s and $A^2 = A$, then$(A + I)^3 - 7A = I$

Reason (R): AI=I=AI where I is unit matrix.
Solution: $(I+A)^3 - 7A = I^3 + 3A^2 + 3A + A^3 - 7A = I + 3A + 3A + A - 7I = I$ since $A^2 = A$ and $A^3 = A$ So both the statements are correct. Second statement is reason for first. Option (a)

9) **Assertion(A):** For a sqaure of order 2 x 2 , $A^{-1} = \left[\frac{1}{3}\right](adjA)$, so |5A| = 75.

Reason: (R): For a square matrix A of order n x n, , $A^{-1} = \frac{adjA}{|A|}$ and $|adjA| = |A|^{n-1}$

Solution: A (adjA) = |A| I = (adjA)A Option : (a)

10) **Assertion (A):** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. then $A^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
Reason (R): A is unit matrix..
Solution: Given matrix is unit matrix. By property of unit matrix, $I \times I = I$.Both are correct. and second is correct explanation for first Option (a)

VERY SHORT ANSWER TYPE QUSTIONS

1) If $A = \begin{bmatrix} \cos\propto & -\sin\propto \\ \sin\propto & \cos\propto \end{bmatrix}$ and $A+A^T=I$. then find $\propto$
Solution: finding A^T and sub in $A+A^T=I$ $\propto = \frac{\pi}{3}$

2) If $[2x \quad 3] \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ 8 \end{bmatrix}$ =O find the value of x

Solution: getting $2x^2+23x = 0$, x=0 and x=-23/2

3) Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$ Prove that $A^2 - 9A + 2I = O$ and express A^{-1} in terms of A.

Solution: proving $A^2 - 9A + 2I = O$ and finding $A^{-1} = \frac{9I-A}{2}$

4) If A and B is symmetric matrices of same order, show that AB is symmetric iff AB=BA
Solution: $(AB)^T = B^TA^T = BA$ and $(AB)^T = AB$ iff $AB = BA$

5) Construct a matrix of order 3x3 whose elements are given by $a_{ij} = 1$ if $i \neq j$. $a_{ij} = 0$ if $i = j$.

Solution: $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

6) If $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$, then the value of $I-A+A^2-A^3+\ldots\ldots$.
Solution: finding A^2 as null matrix, Then A^3, A^4….are all zero.

Then the value of $I-A+A^2-A^3+\ldots\ldots$ Is $\begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$

7) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $A^2 + 7I = kA$, then find k.

Solution: finding $A^2 + 7I$ as $\begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$ equating to kA and finding k = 5

8) Simplify: $\cos\emptyset \begin{bmatrix} \cos\emptyset & \sin\emptyset \\ -\sin\emptyset & \cos\emptyset \end{bmatrix} + \sin\emptyset \begin{bmatrix} \sin\emptyset & -\cos\emptyset \\ \cos\emptyset & \sin\emptyset \end{bmatrix}$

Solution: on multiplication we get
$\begin{bmatrix} \cos^2\emptyset + \sin^2\emptyset & \sin\emptyset\cos\emptyset - \sin\emptyset\cos\emptyset \\ -\sin\emptyset\cos\emptyset + \sin\emptyset\cos\emptyset & \cos^2\emptyset + \sin^2\emptyset \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

SHORT ANSWER TYPE QUESTIONS

1. Express $\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ as the sum of symmetric and skew symmetric matrices.

Solution: writing $\frac{A+A'}{2} = \frac{1}{2}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}$ and $\frac{A-A'}{2} = \frac{1}{2}\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$ and proving A = $\frac{A+A'}{2} + \frac{A-A'}{2}$

2. A=$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & x \\ -2 & 2 & -1 \end{bmatrix}$ and AA^T=9I, Find x.

Solution: Applying AA^T=9I and getting $\begin{bmatrix} 9 & 4+2x & 0 \\ 4+2x & 5+x^2 & -2-x \\ 0 & -2-x & 9 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$x = -2$

3. X+Y=$\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and X-Y=$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ Find matrices X and Y.

Solution: 2X=$\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$ X= $\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$ Y= $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

4. Find the matrix X such that:

$X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

Solution: X as 2 x 2 matrix

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$ writing equations and

finding the values of a, b, c and d X=$\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

5. Find the values of x, y, a and b when

$$\begin{bmatrix} 2x+3y & a-2b \\ 2a+b & 3x-2y \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 6 & 11 \end{bmatrix}$$

Solution: $2x + 3y = 3, 3x - 2y = 11$

x=3 and y=-1

$2a + b = 6$, $a - 2b = 8$, a=4 and b=-2

6. If $\begin{bmatrix} x & 2 & -3 \\ 5 & y & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{bmatrix}$, find x and y

Solution: $\begin{bmatrix} 3x+8-6 & -x+4 & 2x+10-9 \\ 15+4y+4 & -5+2y & 2-5+3 \\ 3-4+2 & -1-2 & 2-5+3 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{bmatrix}$

3x+8-6=5 and -5+2y+0=-5 , x=1 and y=0

7. If A = $\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$ abd B = $[1 \quad 3 \quad -6]$ verify that (AB)' = B' A'

Solution: AB = $\begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$ and (AB)' = $\begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$

A' = $[-2 \quad 4 \quad 5]$ $B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$ finding B' A' = $\begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$

8. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

Solution: A & B are symmetric if A= A' and B = B'

Let C = AB – BA C' = (AB – BA) '

C' = (AB)' – (BA)' = B'A' – A'B'=BA – AB = -(AB – BA)= -C

LONG ANSWER TYPE QUESTIONS

1. Find a matrix A such that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$

Solution: Let A=$\begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix}$

Getting $\begin{bmatrix} 2x-a & 2y-b & 2z-c \\ x & y & z \\ -3x+4a & -3y+4b & -3z+4c \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$

Equating corresponding elements, getting equations in variable, x=1,y=-2,z=-5

a=3,b=4,c=0 A=$\begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$

2. Find the value of x, . if $[x \quad -5 \quad -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = 0$. Hence find$[x \quad -5 \quad -1] \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}$

Solution: $[x \quad -5 \quad -1] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$=$[x-2 \quad -10 \quad 2x-8]$

$[x-2 \quad -10 \quad 2x-8] \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}$ =$[x^2-48]$ $[x^2-48]=0$ $x=\pm 4\sqrt{3}$

Then find the product $[x \quad -5 \quad -1] \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}$ by putting $x = 4\sqrt{3}$, we get [27]

and x = $-4\sqrt{3}$, we get [27]

3. If A=$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and$A^3 - 6A^2$+7A+K$I = 0$. Find K.

Solution: $A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$

Sub in eqn $A^3 - 6A^2$+7A+K$I = 0$

$A^3 - 6A^2$+7A+K$I = 0$

$$\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} +7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = O$$

On solving, we get K=2

4. A typist charges Rs. 145 for typing 10 English and 3 Hindi pages, while charges for typing 3 English and 10 Hindi pages are Rs. 180. Using matrices, find the charges of typing one English and one Hindi page separately. However, typist charged only Rs. 2 per page from a poor student Shyam for 5 Hindi pages. How much less was charged from this poor boy?

Solution: let X =$\begin{bmatrix} x \\ y \end{bmatrix}$ A=$\begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix}$ and B = $\begin{bmatrix} 145 \\ 180 \end{bmatrix}$

$$\begin{bmatrix} 10 & 3 \\ 3 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 145 \\ 180 \end{bmatrix}$$

$10x + 3y = 145, 3x + 10y = 180$

$x = 10 \ and \ y = 15 \ , \ x + y = 25$

For Shyam instead of 15 Rs , the typist charged 2 rs . So he charged Rs13 less for poor boy per page , for 5 pages the typist charged 13×5=65 Rs less for poor boy.

5. If A=$\begin{bmatrix} 0 & -tan\frac{\alpha}{2} \\ tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of matrix of order2, show that

$I + A = (I - A)\begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix}$

Solution: Let $tan\frac{\alpha}{2}$ =t, then $cos\alpha = \frac{1-t^2}{1+t^2}$ and $sin\alpha = \frac{2t}{1+t^2}$ by substituting show that

LHS=$\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}$ and RHS=$\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}$

CASE STUDY TYPE QUESTIONS

1. A manufacture produces three types of emergency lambs which he sells in two markets. Their Monthly sales are indicated below

	Type 1	Type 2	Type 3
Market A	100	100	50
Market B	80	100	100

If the unit Sale price of the three types of emergency lights are 2000,3000 and 2500 respectively, and unit cost of the above three commodities are Rs. 1500,2200, and Rs. 2000 respectively, then based on the above information answer the following

a) Find the total revenue of market A

Solution: total revenue of market A

$=100 \times 2000 + 100 \times 3000 + 50 \times 2500$ =6,25,000

b) What is profit of market B

Solution: Total profit of market B is $80\times 500 + 100 \times 800 + 100 \times 500$ =170000

Rs.3,25,000

c) Find gross profit in both market

Solution: Profit of market A=155000, Profit of market B=170000,

Total gross profit = Rs.3,25,000

2. Sushama owns a P.G for girls. One day she went to market purchase the food items. She bought 4kg onion,3 kg wheat, and 2kg rice for Rs 560.Next day she bought 2kg onion,4kgwheat and 6 kg rice. It cost her Rs:780. Another day she bought 6 kg onion,2kg wheat and3 kg rice which cost Rs: 640.

a)Convert the given condition above in matrix equation of the form AX=B

Solution : $\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 560 \\ 780 \\ 640 \end{bmatrix}$

b)Find $A+A^T$. Is it symmetric?

Solution: $\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 6 \\ 3 & 4 & 2 \\ 2 & 6 & 3 \end{bmatrix}=\begin{bmatrix} 8 & 5 & 8 \\ 5 & 8 & 8 \\ 8 & 8 & 6 \end{bmatrix}$.It is symmetric

c) Find a matrix P such that $P = A^2 - 5A$

Solution: $A^2 - 5A=\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}$

$=\begin{bmatrix} 34 & 26 & 32 \\ 52 & 34 & 46 \\ 46 & 32 & 33 \end{bmatrix}$--$\begin{bmatrix} 20 & 15 & 10 \\ 10 & 20 & 30 \\ 30 & 10 & 15 \end{bmatrix}=\begin{bmatrix} 14 & 11 & 22 \\ 42 & 14 & 16 \\ 16 & 22 & 18 \end{bmatrix}$

CHAPTER-04: DETERMINANTS

Definitions and Formulae:

To every square matrix we can assign a number called determinant

- **Determinant:**
  - $Let\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, then\ det(A) = |A| = ad - bc$
  - $Let\ A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}, then\ |A| = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$
  - For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros
  - The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), is

    $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$
  - Since area is a positive quantity, we always take the absolute value of the determinant
  - The area of the triangle formed by three collinear points is zero.
  - Equation of line joining the points (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$
- **Minors:** Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.
- **Co-Factors:** Cofactor of an element a_{ij} , denoted by A_{ij} is defined by

  $A_{ij} = (-1)^{i+j}.M_{ij}$, where M_{ij} is minor of a_{ij}
- The value of a determinant Δ = sum of the product of elements of any row (or column) with their corresponding cofactors.
- If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For example $a_{11}A_{31}+a_{12}A_{32}+a_{13}A_{33}=0$
- **Adjoint of a Matrix:** The adjoint of a square matrix $A = [a_{ij}]$ is defined as the transpose of the matrix $[A_{ij}]$, where A_{ij} is the cofactor of the element a_{ij} . Adjoint of the matrix A is denoted by adj A.
- To find adjoint of a 2×2 matrix interchange the diagonal elements and change the sign of non – diagonal elements.
- **Inverse of a Matrix**: Let A be a square matrix. $A^{-1} = \frac{1}{|A|}adjA$
- **Solution of system of linear equations by using matrix method:**

  Let the system of linear equations be

  $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$

  These equations can be written as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

  $AX = B \Longrightarrow \quad X = A^{-1}B$
  - $A^{-1} exists, if\ |A| \neq 0\ \ i.e\ the\ solution\ exists\ and\ it\ is\ unique.$
  - $The\ system\ of\ equations\ is\ said\ to\ be\ consistent\ if\ the\ solution\ exists.$
  - $if\ |A| = 0\ , then\ we\ calculate\ (adjA)B.$
  - $If\ |A| = 0\ and\ (adjA)B \neq O$, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.
  - $If\ |A| = 0\ and\ (adjA)B = O$, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.
- **Important notes:**

- The matrix A is singular if $|A| = 0$
- A square matrix A is said to be non-singular if $|A| \neq 0$
- If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order
- If A is an invertible matrix, then $|A| \neq 0$ and $(A^{-1})^T = (A^T)^{-1}$
- $|\lambda A| = \lambda^n |A|, where\ n = order\ of\ matrix\ A$
- $A(adjA) = (adjA)A = |A|I$
- $|adjA| = |A|^{n-1}, where\ n = order\ of\ matrix\ A$
- $|A(adjA)| = |A|^n, where\ n = order\ of\ matrix\ A$
- $|AB| = |A||B| \quad (AB)^{-1} = B^{-1}A^{-1} \quad |A^{-1}| = |A|^{-1} \quad |A^T| = |A|$
- If A and B are square matrices of the same order, then adj(AB)=adjB. adj A

MULTIPLE CHOICE QUESTIONS

1. If A is a square matrix of order 3 such that $|A| = -5$, then value of $|-A|$ is

   (a) 125 (b) - 125 (c) 5 (d) – 5

   Solution:- $|-A| = (-1)^3|A| = -(-5) = 5$ Correct option: (c)

2. What positive value of x makes the following pair of determinants equal $\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}, \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$

   (a) 4 (b) 8 (c) 2 (d) ±4

   Solution:- $\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix} = \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$ $2x^2-15 = 32-15$ $2x^2=32$ $x^2=16, x=\pm 4$

   The positive value of x is 4 Correct option: (a)

3. If A is a square matrix of order 3 such that $|adjA|=64$, then what is the value of $|A|$

   (a)64 (b) 8 (c) -8 (d) ± 8

   Solution:-$|adjA|=|A|^2=64, |A|=\pm 8$ Correct option : (d)

4. If $A\ (3,4), B(-7,2), C(x,y)$ are collinear, then which of the following is true?

   (a) $x + 5y + 17 = 0$ (b) $x + 5y + 13 = 0$ (c) $x - 5y + 17 = 0$ (d) $x - 5y - 17 = 0$

   Solution:- If A (3,4), B(-7,2), C(x, y) are collinear, then area of triangle ABC=0

   i.e $\frac{1}{2}\begin{vmatrix} 3 & 4 & 1 \\ -7 & 2 & 1 \\ x & y & 1 \end{vmatrix}=0$, i.e $\begin{vmatrix} 3 & 4 & 1 \\ -7 & 2 & 1 \\ x & y & 1 \end{vmatrix}=0$,

   $i.e\ 3(2-y) - 4(-7-x) + 1(-7y-2x) = 0$,i.e x-5y+17=0 Correct option: (c)

5. If A is an invertible matrix of order 2, then $\det(A^{-1})=$

   a) $\frac{1}{detA}$ (b) 0 (c) 1 (d) det(A)

   Solution:- $\det(A^{-1})=\left|\frac{adj(A)}{IAI}\right|, = \frac{1}{|A|^2}|adj(A)| = \frac{1}{|A|^2}|A|^{2-1} = \frac{1}{|A|}$Correct option: (a)

6. If $\begin{bmatrix} cos15^0 & sin15^0 \\ cos75^0 & sin75^0 \end{bmatrix}$

   a) 0 (b) 1 (c)-1 d)90

   Solution: $\sin(15^0+75^0) = \sin 90^0=1$ Correct option: (b)

7. If A = $\begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & 3 \\ 5 & 1 & -1 \end{bmatrix}$, then the value of | A (adj.A) | is

   (a) 100I (b) 10 I (c) 10 (d) 1000

   Solution: Using $|A(adjA)| = |A|^n$, $|A|^3$, we get 1000 where |A| = 10

   Option : (d)

8. If A = $[a_{ij}] = \begin{bmatrix} 2 & -1 & 5 \\ 1 & 3 & 2 \\ 5 & 0 & 4 \end{bmatrix}$ and C_{ij} is the cofactor of element aij, then find the value of

   $a_{21}.C_{11}+a_{22}.C_{12}+a_{23}.C_{13}$.

(a) -57 (b) 0 (c) 9 (d) 57

Solution: writing the values a_{21}, c_{11}..., finding the value as 0 Option : (b)

9. If A and B are two square matrices each of order 3 with |A| = 3 and |B| = 5, the |2AB| is

(a) 30 (b) 120 (c) 15 (d) 225

Solution: using $|kA| = k^n |A|$, where n is order, we get 2^3 x |A| x |B| = 120.Option: (b)

10. Let A be a square matrix of order 3. If |A| = 5, then |adjA| is

(a) 5 (b) 125 (c) 25 (d) -5

Solution: $|adjA| = |A|^{n-1}, where\ n = order\ of\ matrix\ A$, $|A|^2$= 25 Option: (c)

ASSERTION REASONING BASED QUESTIONS

Two statements are given, one labelled Assertion(A) and the other labelled Reason(R).Select the correct answer from the codes (a),(b),(c) and (d) as given below

(a) Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of the Assertion(A)
(b) Both Assertion (A) and Reason(R) are true but Reason(R) is *not* the correct explanation of the Assertion(A)
(c) Assertion (A) is true and Reason (R) is false.
(d) Assertion (A) is false and Reason (R) is true

1) **Assertion(A):** If A=$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ then Adj(A)= $\begin{bmatrix} -2 & 1 \\ 3 & -4 \end{bmatrix}$

Reason(R): If A=$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then Adjoint(A) can be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}

Solution:- Adj(A)= $\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$.

Hence A is false but R is true   Correct option: (d)

2) **Assertion(A):** If A is a square natrix of order 3, then $|2A|$=8$|A|$

Reason(R): Let A be a square matrix of order n. Then | *adj* A| = $|A|^{n-1}$

Solution:-A is true since $|2A|=2^3|A|=8|A|$

R is also true, but R is not the correct explanation of A   Correct option:(b)

3) **Assertion(A):** if A=$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and B=$\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ then $(AB)^{-1}$= $\begin{bmatrix} 61 & 87 \\ 47 & 67 \end{bmatrix}$

Reason(R): For any 2 matrix A and B , $(AB)^{-1}=B^{-1}A^{-1}$

Solution:- $(AB)^{-1}= \frac{adj(AB)}{|AB|} = \begin{bmatrix} 61 & 87 \\ 47 & 67 \end{bmatrix} = B^{-1}A^{-1}$

$(AB)^{-1}=B^{-1}A^{-1}$

Hence Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of the Assertion(A)   Correct option:(a)

4) **Assertion(A):** For two matrices A and B of order 3, $|A|$=2 $|B|$= - 3, then$|AB|$=-6

Reason(R): The determinant of the product of matrices is equal to product of their respective determinants, that is, $|AB| = |A||B|$ where A and B are square matrices of the same order

Solution:- $|AB| = |A||B|$ = 2(-3) = - 6

Hence A is true but R is true.   Correct option(a)

5) **Assertion(A):** The system of equations 2x + 5y = 1; 3x + 2y = 7 are consistent

Reason(R): A system of equations is said to be consistent if they have one or more.

Solution:- The system of equations can be written in the form AX = B, where

A=$\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$,X=$\begin{bmatrix} x \\ y \end{bmatrix}$ B=$\begin{bmatrix} 1 \\ 7 \end{bmatrix}$

On solving these system of equations by matrix method$|A|$= -11 ≠ 0, Hence, A is non-singular matrix and so has a unique solution. Hence they are consistent.

Correct option(a)

6) **Assertion(A):** In a square matrix of order 3 the minor of an element a_{22} is 3 then cofactor of a_{22} is -3.

Reason(R): Cofactor an element $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$

Solution:- Cofactor an element aij = Aij = (-1)i+j Mij

Cofactor an element a22 = A22 = (-1)2+2(3)= 3

Hence Assertion (A) is false but Reason (R) is true Correct option (d)

7) **Assertion(A):** If A is an invertible matrix of order 2, and det A= 3 then det(A^{-1})is equal to $\frac{1}{3}$

Reason(R): If A is an invertible matrix of order 2 then det (A^{-1}) =det(A)

Solution:-Since det(A^{-1})=$\frac{1}{detA} = \frac{1}{3}$

Hence A is true but R is false. Correct option: (c)

8) **Assertion(A):**The value of $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$ is equal to 1

Reason(R): The value of the determinant of a matrix A of order 2x2, where A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is ad-bc

Solution: $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$ = $x(x) - (x+1)(x-1) = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1$

So Assertion A is true. $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$= ad-bc

So Reason R is true. Hence Reason(R) is the correct explanation of the Assertion(A)

Correct option: (a)

9) Assertion(A)]: For any square matrix A, $|A|^2 = 25$, then $|A| = \pm\frac{1}{5}$

Reason(R): $|AB| = |A||B|$

Solution: Assertion is false and Reason is true Correct Option: (d)

10) Assertion(A):A=$\begin{bmatrix} 2 & x & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$, then A^{-1} exists if x = $\frac{-8}{5}$

Reason(R): A square matrix A has inverse if and only if A is non- singular

Solution: |A| = 8 + 5x , A^{-1} exists only when $|A| \neq 0$

So Assertion is False. And Reason is true Correct option: (d)

VERY SHORT ANSWER TYPE QUESTIONS

1. If A=$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and B=$\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$ find $|AB|$

Solution:- AB=$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -8 \\ 0 & -10 \end{bmatrix}$

$|AB|$= -70

2. Find the equation of the line joining A (1 , 3) and B (0, 0) using determinants

Solution:- Let p(x,y) be any point on the line AB

Then area of ΔPAB=0

$\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ x & y & 1 \end{vmatrix}$=0, Equation of line AB is y = 3 x

3. If A = $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, find the value of $| A^2 - 2A |$

Solution:- A^2-2A=$\begin{bmatrix} 7 & 6 \\ 4 & 7 \end{bmatrix}$-$\begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix}$

=$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $| A^2 - 2A |$=25

4. If for any 2x2 square matrix A, A(AdjA)= $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, then write the value of $|A|$

Solution:- A (adj A) = |A| I

$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$=|A|$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$=$\begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix}$

$|A|$=8

5. If A=$\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ write A^{-1} in terms of A

   Solution:- $A^{-1}=\frac{Adj(A)}{|A|}=\frac{-1}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}=\frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}=\frac{1}{19}A$

6. Find minors and cofactors of the elements of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and verify that $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$

   Solution: A_{31}=-12, A_{32}=22, A_{33} = 18 , then $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$

7. Find maximum value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$ (θ is real).

   Solution: Finding the determinant and equating to zero, $\sin\theta\cos\theta = 0$, $\theta = 0$.

8. If |A| =3, $A^{-1}=\begin{bmatrix} 3 & -1 \\ -5/3 & 2/3 \end{bmatrix}$ find A Solution: A= $\begin{bmatrix} 9 & 3 \\ 5 & 2 \end{bmatrix}$

9. Find k if the matrix $\begin{bmatrix} 1 & k & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3x 3 matrix A and $|A| = 4$.

   Solution: Using $|adjA| = |A|^{n-1}$, 2k-6 = 16, k = 11.

SHORT ANSWER TYPE QUESTIONS

1. If A.(adjA)=$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,then find the value of $|A| + |adjA|$

   Solution:-

   $|A.adjA|=|A||adjA|=|A||A|^2=|A|^3$ $27=|A|^3$ $|A|=3$ $|adjA|=3^2=9$ $|A|+|adjA|=12$

2. If A is a skew symmetric matrix of order 3, then prove that det A=0

   Solution:- If A is a skew symmetric matrix of order 3,then A= - A^T

   $|A| = |-A^T| = -|A^T| = -|A|$ (Since $|A^T| = |A|$)

   $2|A| = 0$, Hence $|A|=0$

3. If A=$\begin{bmatrix} 1 & sinx & 1 \\ -sinx & 1 & sinx \\ -1 & -sinx & 1 \end{bmatrix}$, where $0 \le x \le 2\pi$. Then prove that $|A| \epsilon [2,4]$

   Solution:- $|A|=2+2\sin^2x$

   We know that $0 \le \sin^2x \le 1$ i.e. $0 \le 2\sin^2x \le 2$ i.e $2 \le 2+2\sin^2x \le 4$

   i.e. $2 \le |A| \le 4$ Hence $|A| \epsilon [2,4]$

4. If the points (a_1, b_1), (a_2, b_2) and $(a_1 + a_2, b_1 + b_2)$ are collinear, then prove that $a_1b_2=a_2b_1$

   Solution:- If the points (a_1, b_1), (a_2, b_2) and $(a_1 + a_2, b_1 + b_2)$ are collinear,

   then $\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_1+a_2 & b_1+b_2 & 1 \end{vmatrix} = 0$

   On expanding we get $a_2b_1-a_1b_2=0$ Hence $a_2b_1 = a_1b_2$

5. If A=$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & x \\ 2 & 3 & 1 \end{bmatrix}$ and $A^{-1}=\begin{bmatrix} \frac{-1}{2} & -4 & \frac{5}{2} \\ \frac{-1}{2} & 3 & \frac{-3}{2} \\ \frac{1}{2} & y & \frac{1}{2} \end{bmatrix}$ find the values of x and y.

   Solution:- Using $AA^{-1} = I$

   $\begin{bmatrix} 1 & 3+3y & 0 \\ \frac{-1}{2}+\frac{x}{2} & 2+xy & \frac{-1}{2}+\frac{x}{2} \\ 0 & 1+y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, equating and solving $x = 1$ and $y = -1$

6. x = -9 is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$=0 then find the other 2 roots

Solution:-_ $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$=0 x^3 - 67x+126=0 (x+9)(x-7)(x-2)=0 $\Rightarrow x = -9,7,2$

Hence the other two roots are 7 and 2

7. If A=$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and B=$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ then verify that $(AB)^{-1} = B^{-1}A^{-1}$

Solution:-_AB=$\begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$ $|AB|$== -11≠0 $(AB)^{-1}$ exists

$(AB)^{-1} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$ $A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix} = (AB)^{-1}$ Hence proved

LONG ANSWER TYPE QUESTIONS

1. Solve the following system of equations by matrix method

   3x-2y+3z=8, 2x+y-z=1 and 4x-3y+2z=4

   Solution:-_The given system of equations can be written as AX=B, $|A|$= -17≠0 and

   adjA=$\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$, A^{-1}= -1/17$\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$

   X= $A^{-1}B$=1/17$\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$, $\Rightarrow x = 1, y = 2, z = 3$

2. Use the product $\begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$ to solve the system of equations:-

   x+2y-3z=6, 3x+2y-2z=3 and 2x-y+z=2

   Solution: the product of two matrices is 7 I

   And the inverse of $\begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}$ is 1/7 of $\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$

   Let X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$ using X= $A^{-1}B$, we get x=1,y=-5 and z=-5.

3. Using the matrix method, solve the following system of linear equations :

   $\frac{2}{X} + \frac{3}{Y} + \frac{10}{Z} = 4$, $\frac{4}{X} - \frac{6}{Y} + \frac{5}{Z} = 1$, $\frac{6}{X} + \frac{9}{Y} - \frac{20}{Z} = 2$

   Solution:-_The given system of equations can be written in the form AX=B,

   Where A=$\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$ $X = \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix}$ and B=$\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$, $|A|$= 1200 ≠ 0 , A^{-1} exists

   Adj(A)=$\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$ Hence $A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$, Using X=$A^{-1}B$

   $\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$, $= \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$, Hence $x = 2, y = 3, z = 5$

CASE STUDY BASED QUESTIONS

1. Ram buys 5 pens, 3 bags 1 instrument box and pays a sum of Rs. 160. From the same shop, Madhav buys 2 pens, 1 bag and 3 instrument boxes and pays a sum of Rs.190.Also Ankit buys 1 pen, 2 bags and 4 instrument boxes and pays a sum of Rs.250.

   Based on the above information, answer the following questions:

I. Convert the given above situation into a matrix equation of the form AX=B

II. Find $|A|$ 3) Find A^{-1} OR Determine $P=A^2-5A$

Solution:-i) Matrix equation AX=B, where A=$\begin{bmatrix}5 & 3 & 1\\ 2 & 1 & 3\\ 1 & 2 & 4\end{bmatrix}$ X=$\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ B=$\begin{bmatrix}160\\ 190\\ 250\end{bmatrix}$ where x is the number of pens bought, y the number of bags and z the number of instrument boxes.

ii) $|A|$= -22

3) Adj(A)=$\begin{bmatrix}-2 & -10 & 8\\ -5 & 19 & -13\\ 3 & -7 & -1\end{bmatrix}$, $A^{-1}=\frac{1}{-22}\begin{bmatrix}-2 & -10 & 8\\ -5 & 19 & -13\\ 3 & -7 & -1\end{bmatrix}$ OR $P=A^2-5A=\begin{bmatrix}7 & 5 & 13\\ 5 & 8 & 2\\ 8 & 3 & 3\end{bmatrix}$

2. Manjit wants to donate a rectangular plot of land for a school in his village.

When he was asked to give dimensions of the plot, he told that if its length is decreased by 50 m and breadth is increased by 50m, then its area will remain same, but if length is decreased by 10m and breadth is decreased by 20m, then its area will decrease by 5300 m^2

i). Based on the information given above, form equations in terms of x and y

ii). Write down matrix equation represented by the given information

iii). How much is the area of rectangular field?

Solution: (i) $(x-50)(y+50)=xy, x-y=50, (x-10)(y-20)=xy-5300, 2x+y=550$

(ii) $\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}50\\ 550\end{bmatrix}$

(iii) On solving, We get, x=200 m, y=150 m, Area =200 × 150 = 30, 000 sq .m

3. The management committee of a residential colony decided to award some of the students of their colony (say x) for honesty, some (say y) for helping others and some others (say z)for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12.Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33.If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others.

   Based on the above information, answer the following questions:

   1)Convert the given above situation into a matrix equation of the form AX=B

   2)Find A^{-1}

   Find the number of awardees of each category

   Solution:- $x+y+z=12, 3(y+z)+2x=33 \text{ and } x+z=2y$

   i.e, $x+y+z=12, 2x+3y+3z=33 \text{ and } x-2y+z=0$

   Matrix equation AX=B, where A=$\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 3\\ 1 & -2 & 1\end{bmatrix}$ X=$\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ B=$\begin{bmatrix}12\\ 33\\ 0\end{bmatrix}$

   2) $|A|$ =3, $A^{-1}=\frac{1}{3}\begin{bmatrix}9 & -3 & 0\\ 1 & 0 & -1\\ -7 & 3 & 1\end{bmatrix}$ 3) $X=A^{-1}B$, Hence x=3,y=4,z=5

CHAPTER – 5: CONTINUITY AND DIFFERENTIABILITY

GIST / SUMMARY OF THE LESSON

- A real valued function is continuous at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if it is continuous on the whole of its domain.
- Sum, difference, product and quotient of continuous functions are continuous. i.e, f and g are continuous functions, then
- $(f \pm g)(x) = f(x) \pm g(x)$ is continuous.
  $(f.g)(x) = f(x).g(x)$ is continuous.
  $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ where g(x) ≠ 0 is continuous.
- Every differentiable function is continuous. But the converse is not true.
- Chain rule is rule to differentiate composites of functions. If f = v ° u, t = u (x) and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist then $\frac{df}{dx} = \frac{dv}{dt}.\frac{dt}{dx}$
- Logarithmic differentiation is a powerful technique to differentiate functions of the form *f(x)* =$[u(x)]^{v(x)}$. Here *f(x)* and *u(x)* need to be positive for this technique to make sense.

DEFINITIONS AND FORMULAE

- Continuity of a function at a point:
  Let f be a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$
  More elaborately, if the left hand limit, right hand limit and the value of the function at x = c exist and are equal to each other,
  i.e $\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$ then f is said to be continuous at x = c.
- Continuity in an interval:
  (i) f is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.
  (ii) f is said to be continuous in an closed interval $[a, b]$ if
  a) f is continuous in (a, b)
  b) $\lim_{x \to a^+} f(x) = f(a)$
  c) $\lim_{x \to a^-} f(x) = f(b)$
- Geometrical meaning of continuity:
  (i) Function f will be continuous at x = c if there is no break in the graph of the function at the point (c, f(c)).
  (ii) In an interval, function is said to be continuous if there is no break in the graph of the function in the entire interval.
- Discontinuity:
  The function f will be discontinuous at x = a in any of the following cases:
  (i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.
  (ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are equal but not equal to f(a).
  (iii) f(a) is not defined.
- Continuity of some of the common functions
  Function f(x)
  Interval in the which f is continuous
  1. The constant function, i.e. f(x) = c R
  2. The identity function, i.e. f(x) = x R
  3. The polynomial function, i.e.
     $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$ R

4. $|x-a|$ $(-\infty,\infty)$
5. x^{-n}, n is a positive integer $(-\infty,\infty)-\{0\}$
6. $\frac{p(x)}{q(x)}$, *p(x)* and *q(x)* are polynomials in *x* R – {*x* : *q(x)* = 0}
7. $sinx, cosx$ R
8. $tanx, secx$ R – $\{(2n+1)\frac{\pi}{2}$: n$\in Z$
9. $cotx, cosecx$ R - $\{n\pi : n\in Z\}$
10. e^x R
11. $logx$ $(0,\infty)$
12. The inverse of trigonometric functions.
    In their respect domains. $i.e, \sin^{-1}x, \cos^{-1}x\ etc.$

- Let f and g be real valued functions such that (fog) is defined at a. If g is continuous at a and f is continuous at $g(a)$, then (fog) is continuous at a.
- Differentiability

  The function defined by $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, wherever the limit exists, is defined to be the derivative of f at x. In other words, we say that a function f is differentiable at a point c in its domain if both $\lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$, called left hand derivative, denoted by Lf ′(c) and $\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$, called right hand derivative, denoted by Rf ′(c) , are finite and equal.

  (i) The function y = f(x) is said to be differentiable in an open interval (a,b) if it is differentiable at every point of (a,b).

  (ii) The function y = f(x) is said to be differentiable in a closed interval $[a,b]$ if Rf ′(a) and Lf ′(b) exist and f'(x) exists for every point of (a,b).

  (iii) Every differentiable function is continuous, but the converse is not true.
- Algebra of derivatives

  If u, v are functions of x, then

  i) $\frac{d(u\pm v)}{dx} = \frac{du}{dx}\pm\frac{dv}{dx}$

  ii) $\frac{d}{dx}(uv) = u\frac{dv}{dx}+v\frac{du}{dx}$

  iii) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
- Following are some of the standard derivatives (in appropriate domain)
- Following are some of the standard derivatives (in appropriate domain):

  $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

  $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

  $\frac{d}{dx}(\sec^{-1}x) == \frac{1}{|x|\sqrt{x^2-1}}, |x|>1$ $\frac{d}{dx}(\csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}, |x|>1$

  $\frac{d}{dx}(e^x) = e^x$ $\frac{d}{dx}(logx) = \frac{1}{x}$

- Exponential and logarithmic functions

  i) The exponential function with positive base b > 1 is the function y = f(x) = b^x. Its domain is R, the set of all real numbers and range is the set of all positive real numbers. Exponential function with base 10 is called the common exponential function and with base e is called the natural exponential function.

  ii) Let b > 1 be a real number. Then we say logarithm of a to base b is x if b^x = a, Logarithm of a to the base b is denoted by $\log_b a$. If the base b = 10, we say it is common logarithm and if b = e, then we say it is natural logarithms. Logx denotes the logarithm function to base e. The domain of logarithm function is R^+, the set of all positive real numbers and the range is the set of all real numbers.

  iii) The properties of logarithmic function to any base b > 1 are listed below:

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b x^n = n \log_b x$
4. $\log_b x = \frac{\log_c x}{\log_c b}$, where c > 1
5. $\log_b x = \frac{1}{\log_x b}$
6. $\log_b b = 1$ *and* $\log_b 1 = 0$.

iv) The derivative of e^x w.r.t is e^x, i.e.$\frac{d(e^x)}{dx} = e^x$.

The derivative of $\log x$ w.r.t., is $\frac{1}{x}$; $i.e. \frac{d(\log x)}{dx} = \frac{1}{x}$.

- Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = (u(x))^{v(x)}$, where both f and u need to be positive functions for this technique to make sense.
- Differentiable of a function with respect to another function

  Let u=f(x) and v=g(x) be two functions of x, then to find derivative of f(x) w.r.t g(x), i.e., find $\frac{du}{dv}$, we use the formula $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$.
- Second order derivative

  $\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$ is called the second derivative of y w.r.t x. it is denoted by y'' or y_2, if y=f(x).

MULTIPLE CHOICE QUESTIONS

1. The function f (*x*) = $\begin{cases} \frac{sinx}{x} + cosx, if\ x \neq 0 \\ k \quad\quad ,if\ x = 0 \end{cases}$ is continuous at *x* = 0, then the value of *k* is

   A) 3 B) 2 C) 1 D) 1.5

   Solution: If f (*x*) is continuous at x = 0, then

   $\lim_{x \to 0} f(x) = f(0)$,

   $\lim_{x \to 0} \frac{\sin x}{x} + \cos x = k$

   $\lim_{x \to 0} \frac{\sin x}{x} + \lim_{x \to 0} \cos x = k$

   1 + 1 = 2 *k* = 2 Answer: B

2. The function f (*x*) = [*x*], where [*x*] denotes the greatest integer function, is continuous at

   A) 4 B) -2 C) 1 D) 1,5

   Solution: f (*x*) = [*x*] is everywhere continuous except at integer points.

   Hence, it is continuous at *x* = 1.5 Answer: D

3. The number of points at which the function f (*x*) = $\frac{1}{x-[x]}$ is not continuous is

   A) 1 B) 2 C) 3 D) none of these

   Solution: We know that *x* – [*x*] = 0 when *x* is an integer.

   Therefore, f (*x*) is discontinuous at every integer point. Answer: D

4. The function f(x) = tan *x* is discontinuous on the set

   A) $\{n\pi: n \in Z\}$ B) $\{2n\pi: n \in Z\}$ C) $\{(2n+1)\frac{\pi}{2}: n \in Z\}$ D) $\{\frac{n\pi}{2}: n \in Z\}$

   Answer: C

5. If f : R→R defined by f (x) = $\begin{cases} \frac{\cos 3x - \cos x}{x^2} & ,for\ x \neq 0 \\ \alpha & ,for\ x = 0 \end{cases}$ is continuous at x = 0, then α is equal to

   A) -2 B) -4 C) -6 D) -8

Answer: B

6. The set of points where the function f (x) = |3x - 2| is differentiable, is

A) R B) R - $\left\{\frac{3}{2}\right\}$ C) R - $\left\{\frac{2}{3}\right\}$ D) none of these

Answer: C

7. The function f (x) = e $^{|x|}$ is

A) continuous everywhere but not differentiable at x = 0

B) not continuous at x = 0

C) continuous and differentiable everywhere

D) none of these

Answer: A

8. The value of k for which the function f (x) = $\begin{cases} x^2 & ,x \geq 0 \\ kx & ,x < 0 \end{cases}$ is differentiable at x = 0

A) 1 B) 2 C) any real number D) 0

Solution: For f (x) to be differentiable at x = 0, we must have

(LHD of f (x) at x = 0) = (RHD of f (x) at x = 0)

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$\lim_{x \to 0^-} \frac{kx}{x} = \lim_{x \to 0^+} \frac{x^2}{x}$$

$$k = \lim_{x \to 0} x = 0$$

Answer: D

9. Let y = f $\left(\frac{1}{x}\right)$ and f' (x) = x^3. What is the value of $\frac{dy}{dx}$ at x = $\frac{1}{2}$?

A) $-\frac{1}{64}$ B) $-\frac{1}{32}$ C) -32 D) -64

Solution: We have,

$$y = f\left(\frac{1}{x}\right) \qquad \frac{dy}{dx} = f'\left(\frac{1}{x}\right) \times -\frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^3} \times -\frac{1}{x^2} = -\frac{1}{x^5}$$

$$\frac{dy}{dx}_{x=\frac{1}{2}} = -\frac{1}{\left(\frac{1}{2}\right)^5} = -32$$

Answer: C

10. If y = $sin^2(x^3)$, then $\frac{dy}{dx}$ is equal to

A) $2\, sinx^3 cosx^3$ B) $3x^3 sinx^3\, cos\, x^3$ C) $6x^2 sinx^3\, cos\, x^3$ D) $2x^2 sin^2(x)^3$

Solution: We have, y = $sin^2(x)^3$

$$\frac{dy}{dx} = \frac{d}{dx} sin^2 x^3 = \frac{d}{dx}(sinx^3)^2 = 2(sinx^3)^{2-1} \frac{d}{dx}(sinx^3)$$

$$= 2\, sinx^3 cosx^3 \times 3x^2$$

$$= 6x^2 sinx^3\, cos\, x^3$$

Answer: C

ASSERTION - REASON BASED QUESTIONS

In the following questions, a statement of assertion (A) is followed by a statement of reason (R). Choose the correct answer out of the following answers

A) Both A and R are true and R is the correct explanation of A.

B) Both A and R are true, but R is not correct explanation of A.

C) A is true but R is false.

D) A is false and R is true.

1) Assertion (A): If f (x) = $\begin{cases} ax+1, & if\ x \geq 1 \\ x+2, & if\ x < 1 \end{cases}$ is continuous at x = 1, then a = 2.

Reason (R): A function f (x) is continuous at a point α in its domain, if $\lim_{x \to \alpha}$ f (x) exists.

Solution: f (x) is continuous at x = α iff $\lim_{x \to \alpha}$ f (x) exists and is equal to f (α).

So, Reason is not true.

If f (x) is continuous at x = 1, then

$\lim_{x \to 1^-}$ f (x) = $\lim_{x \to 1^+}$ f (x) = f (1)

$\lim_{x \to 1}$ x + 2 = $\lim_{x \to 1}$ ax + 1 = a × 1 + 1

1 + 2 = a + 1

a = 2

So. Assertion is true. Answer: C) A is true but R is false

2) Assertion (A): The function f (x) = $\begin{cases} \frac{e^{\frac{1}{x}}+1}{e^{\frac{1}{x}}-1}, & x \neq 0 \\ , & x = 0 \end{cases}$ is discontinuous at x = 0.

Reason (R): If 'a' is a point in the domain of a function f (x) such that

$\lim_{x \to a^-} \neq \lim_{x \to a^+}$ f (x), then f (x) is discontinuous at x = a.

Solution: We know that a function f(x) is continuous at a point 'a' in its domain iff

$\lim_{x \to a^-}$ f (x) = $\lim_{x \to a^+}$ f (x) = f (a)

Otherwise, f (x) is discontinuous at **x** = a.

So, reason is true.

Now, $\lim_{x \to 0^-}$ f (x) = $\lim_{h \to 0}$ f (0 - h) = $\lim_{h \to 0} \frac{e^{-\frac{1}{h}}+1}{e^{-\frac{1}{h}}-1} = \frac{0+1}{0-1}$ = -1

$\lim_{x \to 0^+}$ f (x) = $\lim_{h \to 0}$ f (0 + h) = $\lim_{h \to 0} \frac{e^{\frac{1}{h}}+1}{e^{\frac{1}{h}}-1} = \frac{1+0}{1-0}$ = 1

$\lim_{x \to 0^-}$ f (x) ≠ $\lim_{x \to 0^+}$ f (x)

f (x) is discontinuous at x = 0

Answer: A) Both A and R are true and R is the correct explanation of A.

3) Assertion (A): If f (x) = $\begin{cases} \frac{sin5x}{x}, & if\ \neq 0 \\ \frac{k}{3}, & if\ x = 0 \end{cases}$ is continuous at x = 0, then k =15

Reason (R): If f (x) is continuous at a point x = a in its domain, then $\lim_{x \to a}$ f (x) = f (a)

Solution: Reason, being the definition of continuity is true.

Function f (x) in Assertion is continuous at x = 0. Therefore, by using assertion, we

have: $\lim_{x \to 0}$ f (x) = f (0)

$\lim_{x \to 0} \frac{sin5x}{x} = \frac{k}{3}$

5 $\lim_{x \to 0} \frac{sin5x}{5x} = \frac{k}{3}$, k=15 Answer: A

4) Define F(x) as the product of two real functions f_1(x)= x, x∈ R, and

$f_2(x) = \begin{cases} sin\frac{1}{x}, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$ as follows: $F(x) = \begin{cases} f_1(x).f_2(x), if\ x \neq 0 \\ 0, \quad if\ if\ x = 0 \end{cases}$

Assertion (A): F(x) is continuous on R.

Reason (R): $f_1(x)\ and\ f_2(x)$ are continuous on R.

Solution: Clearly, $F(x) = \begin{cases} xsin\frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$

$\lim_{x \to 0} F(x) = \lim_{x \to 0} xsin\frac{1}{x} = 0 = F(0)$

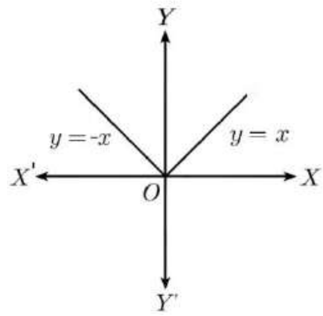
So, F(x) is continuous at x = 0. Answer: C

5) Assertion (A): The function f (x) = $|x|$ is everywhere continuous.

Reason (R): Every differentiable function is continuous.

Solution: Since, if a function f is differentiable at a point c, then it is also continuous at that point. So, Reason is correct.

Assertion is also true as is evident from the given graph of f (x) = $|x|$



However, Reason is not the correct explanation of assertion.

Answer: B

6) Assertion (A): Let y = $sint\ \ then$, $\frac{d^2y}{dt^2}$=y

Reason (R): $\frac{d^2y}{dt^2} = \frac{d}{dt}(\frac{dy}{dt})$

Solution: $\frac{d^2y}{dt^2}$=$-y$ *so* Assertion is false but Reason is true Answer: D

7) Assertion (A): The function f (x) = $|x - 6|cosx$ is differentiable in R - $\{6\}$.

Reason (R): If a function f is continuous at a point c, then it is also differentiable at that point.

Solution: We know that $\emptyset\ (x) = |x - 6|$ is not differentiable at x = 6.

Therefore, f (x) = $|x - 6|\ cosx$ is not differentiable at x = 6.

Thus, f (x) = $|x - 6|cosx$ is differentiable in R -$\{6\}$.

Assertion is true. Reason is false, because $\emptyset\ (x) = |x - 6|$ is continuous at x = 6 but it is not differentiable there at. Answer C

8) Assertion (A): The derivative of a real valued even function is an odd function.

Reason (R): The derivative of a real valued odd function is an even function.

Solution: Let f (x) be a real valued even function.

Then, $f(-x)\ =\ f(x)$

$\frac{d}{dx}\left(f(-x)\right) = \frac{d}{dx}\left(f(x)\right)$

$f'(x)\frac{d}{dx}(-x) = f'(x)$

$-f'(-x) = f'(x)$

$f'(-x) = -f'(x)$, $f'(x)$ is an odd function.

So, Assertion is true.

Let f (x) be a real valued odd function. Then,

f (-x) = - f (x)

$\frac{d}{dx}\left(f(-x)\right) = -\frac{d}{dx}(f(x))$

f' (-x) $\frac{d}{dx}$ $(-x)$ = - f'(x)

- f' (x) = - f' (x)

f'(-x) = f' (x)

Thus, f' (x) is an even function. Reason is true.

However, Reason is not correct explanation of Assertion. Answer: B

9) Assertion (A): If $f(x) = \frac{x^2 - x}{x^2 + 2x}$, then $\frac{d}{dx}(f^{-1}(x)) = \frac{3}{(1-x)^2}$

Reason (R): $fof^{-1}(x) = x$ for all x ∈ Domain (f^{-1}).

Solution: Reason is true as f^{-1} is the inverse of function *f*.

Now, $f(x) = \frac{x^2 - x}{x^2 + 2x} = \frac{(x^2+2x) - 3x}{x^2 + 2x} = 1 - \frac{3}{x + 2}$

Clearly, $f(x)$ is defined for all $x \in R - \{0,2\}$. Given, that $f^{-1}(x)$ is the inverse function of $f(x)$.

$(fof)^{-1}(x) = x$ for all x ∈ Domain (f^{-1})

$f(f^{-1}(x)) = x \quad 1 - \frac{3}{f^{-1}(x)+2} = x$

$1 - x = \frac{3}{f^{-1}(x)+2} \quad f^{-1}(x) + 2 = \frac{3}{1-x}$

$f^{-1}(x) = \frac{3}{1-x} - 2 \quad \frac{d}{dx}(f^{-1}(x)) = \frac{3}{(1-x)^2}$

Thus, Assertion is also true and the reason is the correct explanation of Assertion.

Answer: A

10) Assertion (A): If $f(x)$ is an even differentiable function, then $f'(x)$ is an odd function.

Reason (R): If $f(x)$ and $g(x)$ are real differentiable functions, then

$$\frac{d}{dx}\{(f(g(x))\} = \frac{d}{d\,g(x)}\{f(g(x))\} \times \frac{d}{dx} g(x)$$

$$\text{i.e. } \frac{d}{dx}\{f(g(x))\} = f'(g(x)) \times g'(x)$$

Solution: Reason is the standard chain rule and hence it is true.

If $f(x)$ is an even function, then $f(-x) = f(x)$

$\frac{d}{dx}(f(-x)) = \frac{d}{dx}(f(x)) \qquad f'(-x)\frac{d}{dx}(-x) = f'(x)$

$-f'(-x) = f'(x) \qquad f'(-x) = -f'(x)$

So, $f'(x)$ is an odd function. Hence, Reason is true. Thus, both assertion and reason are true and reason is the correct explanation for Assertion. Answer: A

VERY SHORT ANSWER TYPE QUESTIONS

1. Discuss the continuity of the function $f(x)$ given by $f(x) = \begin{cases} 2 - x, & x < 2 \\ 2 + x, & x \geq 2 \end{cases}$ at $x = 2$.

Solution: We observe that,

$$(\text{LHL at } x = 2) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} (2 - x) = 2 - 2 = 0$$

[since, $f(x) = 2 - x$ for $x < 2$]

$$(\text{RHL at } x = 2) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} (2 + x) = 2 + 2 = 4$$

[since, $f(x) = 2 + x$ for $x \geq 2$]

$$\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$$

Hence, $f(x)$ is not continuous at $x = 2$.

2. Show that the function $f(x) = 2x - |x|$ is continuous at $x = 0$.

Solution: We have,

$f(x) = 2x - |x| = \begin{cases} 2x - x, & if \ \geq 0 \\ 2x-(-x), & if\ x<0 \end{cases} = \begin{cases} x, & if\ x \geq 0 \\ 3x, & if\ x < 0 \end{cases}$

Now,

(LHL at $x = 0$) = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 3x = 3 \times 0 = 0$

(RHL at $x = 0$) = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$

And, $f(0) = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

So, $f(x)$ is continuous at x = 0.

3. Show that the function $f(x)$ given by $f(x) = \begin{cases} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & when\ x \neq 0 \\ 0, & when\ x = 0 \end{cases}$

is discontinuous at $x = 0$.

Solution: We observe that,

(LHL at $x = 0$) = $\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$

$= \lim_{h \to 0} \frac{e^{-\frac{1}{h}} - 1}{e^{-\frac{1}{h}} + 1} = \lim_{h \to 0} \frac{\frac{1}{e^{\frac{1}{h}}} - 1}{\frac{1}{e^{\frac{1}{h}}} + 1} = \frac{0-1}{0+1} = -1 \qquad \left[since, \lim_{h \to 0} \frac{1}{e^{\frac{1}{h}}} = 0\right]$

And, RHL at $x = 0$) = $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$= \lim_{h \to 0} \frac{e^{\frac{1}{h}} - 1}{e^{\frac{1}{h}} + 1} = \lim_{h \to 0} \frac{1 - \frac{1}{e^{\frac{1}{h}}}}{1 + \frac{1}{e^{\frac{1}{h}}}} = \frac{1-0}{1+0} = 1$

(LHL at $x = 0$) ≠ (RHL at $x = 0$)

So, $f(x)$ is not continuous at $x = 0$ and has a discontinuity of first kind at $x = 0$.

4. Show that $f(x) = |x|$ is not differentiable at $x = 0$

Solution: We observe that:

(LHD at $x = 0$) = $\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{0 - h - 0} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$

$= \lim_{h \to 0} \frac{|-h| - |0|}{-h} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1$

(RHD at $x = 0$) = $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$

(LHD at $x = 0$) ≠ (RHD at $x = 0$)

So, $f(x)$ is not differentiable at $x = 0$.

5. If $f(x) = |\cos x|$, find $f'\left(\frac{\pi}{4}\right)$ and $f'\left(\frac{3\pi}{4}\right)$.

Solution: We have, $f(x) = |\cos x| = \begin{cases} \cos x, if\ 0 < x \leq \frac{\pi}{2} \\ -\cos x, if\ \frac{\pi}{2} < x < \pi \end{cases}$

$$f'(x) = \begin{cases} -\sin x, & if\ 0 < x < \frac{\pi}{2} \\ \sin x, & if\ \frac{\pi}{2} < x < \pi \end{cases}$$

Note that $f(x)$ is not differentiable at $x = \frac{\pi}{2}$

Thus, $f'\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$ and $f'\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

6. If y = cosec $(cot^{-1}x)$, then prove that $\sqrt{1+\ x^2}\ \frac{dy}{dx} - x = 0$

Solution: We have,

y = cosec $(cot^{-1}x)$

y = cosec $(cosec^{-1}\sqrt{1+\ x^2}) = \sqrt{1+\ x^2}$

Now, $y = \sqrt{1+\ x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(1+\ x^2)^{-\frac{1}{2}}\ \frac{d}{dx}(1+\ x^2)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} \times 2x = \frac{x}{\sqrt{1+x^2}}$$

$$\sqrt{1+\ x^2}\ \frac{dy}{dx} = x$$

$$\sqrt{1+\ x^2}\frac{dy}{dx} - x = 0$$

7. Differentiate the given function with respect to x: $\log\left(x + \sqrt{a^2+\ x^2}\right)$

Solution: Let $y = \log\left(x+\ \sqrt{a^2+\ x^2}\right)$. Then,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\log\left(x+\ \sqrt{a^2+\ x^2}\right)\right\} = \frac{1}{x+\sqrt{a^2+x^2}} \times \frac{d}{dx}\ \left\{x+\ \sqrt{a^2+\ x^2}\right\}$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{a^2+x^2}}\left\{1+\ \frac{1}{2}(a^2+x^2)^{-\frac{1}{2}} \times \frac{d}{dx}\ (a^2+x^2)\right\}$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{a^2+x^2}}\left\{1+\ \frac{1}{2\sqrt{a^2+\ x^2}} \times 2x\right\} = \frac{1}{x+\sqrt{a^2+x^2}} \times \frac{\sqrt{a^2+\ x^2}+x}{\sqrt{a^2+\ x^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2+x^2}}$$

8. If $y = f(x^2)$ and $f'(x) =\ (e^{\sqrt{x}})$, find $\frac{dy}{dx}$.

Solution: We have, $y = f(x^2)$ and $f'(x) =\ (e^{\sqrt{x}})$

$$\frac{dy}{dx} = \frac{d}{dx}(f\ (x^2))$$

$$= f'\ (x^2)\frac{d}{dx}(x^2)$$

$$= f'(x^2)\ 2x = e^{|x|}2x$$

$$= 2xe^{|x|}$$

9. If sin $y = x$ sin (a + y), prove that $\frac{dy}{dx} = \frac{sin^2\ (a+y)}{\sin a}$

Solution: We have,

sin $y = x$ sin $(a + y)$

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating both sides with respect to y, we get

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - siny\ cos(a+y)}{sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin(a+y-y)}{sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{sin^2(a+y)}{\sin a}$$

10. If $y = \tan x + \sec x$, prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$

Solution: We have, $y = \tan x + \sec x$

$$\frac{dy}{dx} = sec^2x + \sec x \tan x = \frac{1}{cos^2x} + \frac{sinx}{cos^2x} = \frac{1+sinx}{cos^2x}$$

$$\frac{dy}{dx} = \frac{1+sinx}{1-sin^2x} = \frac{1}{1-sinx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{1}{1-sinx}\right\} = \frac{d}{dx}\{(1-\sin x)^{-1}\}$$

$$\frac{d^2y}{dx^2} = (-1)(1-\sin x)^{-2}\frac{d}{dx}(1-\sin x)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{(1-sinx)^2}(-\cos x) = \frac{\cos x}{(1-sinx)^2}$$

SHORT ANSWER TYPE QUESTIONS

1. Determine the value of the constant m so that the function $f(x) = \begin{cases} m(x^2-2x), & if\ x < 0 \\ \cos x, & if\ x \geq 0 \end{cases}$ is continuous.

Solution: When $x < 0$, we have f (x) = $m(x^2 - 2x)$, which being a polynomial is continuous at each $x < 0$. When $x > 0$, we have $f(x) = \cos x$ which being a cosine function is continuous at each $x > 0$. Let us now consider a point $x = 0$. At $x = 0$, we have

(LHL at $x = 0$) = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} m(x^2-2x) = 0$ for all values of m.

(RHL at $x = 0$) = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \cos x = 1$

Clearly, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$ for any value of m.

So, $f(x)$ cannot be made continuous for any value of m.

Thus, the value of m does not exist for which $f(x)$ can be made continuous.

2. If f(x)=$\begin{cases} \frac{|x|}{x}, & if\ x \neq 0, \\ 0, if\ x = 0 \end{cases}$

Solution:.

LHL= $\lim_{x\to0^-} \frac{-x}{x} = -1$

RHL= $\lim_{x\to0^+} \frac{x}{x} = 1$

LHL$\neq RHL$, f(x) is discontinuous at x=0

3. Discuss the continuity of the function $f(x)$ at $x = \frac{1}{2}$, where $f(x) = \begin{cases} \frac{1}{2} - x ; 0 \leq x < \frac{1}{2} \\ 1 ; x = \frac{1}{2} \\ \frac{3}{2} - x ; \frac{1}{2} < x \leq 1 \end{cases}$

Solution: We observe that:

(LHL at $x = \frac{1}{2}$) = $\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}} \left(\frac{1}{2} - x\right) = \frac{1}{2} - \frac{1}{2} = 0$

(RHL at $x = \frac{1}{2}$) = $\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}} \left(\frac{3}{2} - x\right) = \frac{3}{2} - \frac{1}{2} = 1$

$\lim_{x \to \frac{1}{2}^-} f(x) \neq \lim_{x \to \frac{1}{2}^+} f(x)$

Hence, $f(x)$ is not continuous at x = $\frac{1}{2}$.

4. Show that the function $f(x) = \begin{cases} x^2 sin\left(\frac{1}{x}\right), if\ x \neq 0 \\ 0, if\ x = 0 \end{cases}$ is differentiable at x = 0 and $f'(0) = 0$.

Solution: We observe that,

(LHD at $x = 0$) = $\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0 - h - f(0))}{0 - h - 0} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$ =
$\lim_{h \to 0} \frac{(-h)^2 \sin\left(\frac{1}{-h}\right) - 0}{-h} = \lim_{h \to 0} h \sin\frac{1}{h}$

= 0 × (an oscillating number between -1 and 1) = 0

(RHD at $x = 0$) = $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0 + h - f(0))}{0 + h - 0} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$
$= \lim_{h \to 0} \frac{(h)^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0} h \sin\frac{1}{h}$

= 0 × (an oscillating number between -1 and 1) = 0

∴ (LHD at $x = 0$) = (RHD at $x = 0$) = 0

So, $f(x)$ is differentiable at $x = 0$ & $f'(0) = 0$

5. Show that the function $f(x) = |x + 1| + |x - 1|$ for all $x \in$ R, is not differentiable at $x = -1$ and $x = 1$.

Solution: We have,

$$f(x) = |x - 1| + |x + 1| = \begin{cases} -(x + 1) - (x - 1) = -2x,\ if\ x < -1 \\ x + 1 - (x - 1) = 2,\ if -1 \leq x < 1 \\ x + 1 + x - 1 = 2x,\ if\ x \geq 1 \end{cases}$$

Differentiability at $x = -1$: We find that

(LHD at $x = -1$) = $\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x - (-1)}$

$= \lim_{x \to -1^-} \frac{-2x - 2}{x + 1} = \lim_{x \to -1^-} \frac{-2(x + 1)}{x + 1} = \lim_{x \to -1^-} (-2) = -2$

(RHD at $x = -1$) = $\lim_{x \to -1^+} \frac{f(x) - f(-1)}{x - (-1)}$

$= \lim_{x \to -1^+} \frac{2 - 2}{x + 1} = \lim_{x \to -1^+} \frac{0}{x + 1} = \lim_{x \to -1^+} (0) = 0$

∴ (LHD at $x = -1$) ≠ (RHD at $x = -1$)

So, $f(x)$ is not differentiable at $x = -1$

Differentiability at $x = 1$: We find that

(LHD at $x = 1$) = $\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{2 - 2}{x - 1} = \lim_{x \to 1^-} \frac{0}{x - 1}$

$= \lim_{x \to 1^-} 0 = 0$

(LHD at $x = 1$) = $\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{2 - 2}{x - 1} = \lim_{x \to 1^-} \frac{0}{x - 1}$

$= \lim_{x \to 1^-} 0 = 0$

(RHD at $x = 1$) = $\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{2x - 2}{x - 1} = \lim_{x \to 1^+} 2\left(\frac{x-1}{x-1}\right)$

$= \lim_{x \to 1^+} 2 = 2$

∴ (LHD at $x = 1$) ≠ (RHD at $x = 1$)

So, $f(x)$ is not differentiable at $x = 1$

Hence, $f(x)$ is not differentiable at $x = -1$ and $x = 1$

6. If $f(x) = |\cos x - \sin x|$, find $f'\left(\frac{\pi}{6}\right)$ and $f'\left(\frac{\pi}{3}\right)$

Solution: We have,

$$f(x) = |\cos x - \sin x| = \begin{cases} \cos x - \sin x, & if\ 0 < x < \frac{\pi}{4} \\ -(\cos x - \sin x), & if\ \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

$$f'(x) = \begin{cases} -\sin x - \cos x, & if\ 0 < x < \frac{\pi}{4} \\ \cos x + \sin x, & if\ \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

$f'\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - \cos\frac{\pi}{6} = -\frac{\sqrt{3}+1}{2}$ and $f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} + \sin\frac{\pi}{3} = \frac{\sqrt{3}+1}{2}$

7. If $y = \left\{x + \sqrt{x^2 + a^2}\right\}^n$, then prove that $\frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$

Solution: We have, $y = \left\{x + \sqrt{x^2 + a^2}\right\}^n$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left\{x + \sqrt{x^2 + a^2}\right\} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \times \frac{d}{dx}\left\{x + \sqrt{x^2 + a^2}\right\}$$

$$\frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \times \left\{\frac{d}{dx}(x) + \frac{d}{dx}\sqrt{x^2 + a^2}\right\}$$

$$\frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \times \left\{1 + \frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}} \times \frac{d}{dx}(x^2 + a^2)\right\}$$

$$\frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \times \left\{1 + \frac{1}{2\sqrt{x^2+a^2}} \times 2x\right\}$$

$$\frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \times \left\{1 + \frac{x}{\sqrt{x^2+a^2}}\right\}$$

$$\frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \left\{\frac{\sqrt{x^2+a^2} + x}{\sqrt{x^2+a^2}}\right\}$$

$$= \frac{n\left\{x + \sqrt{x^2+a^2}\right\}^n}{\sqrt{x^2+a^2}} = \frac{ny}{\sqrt{x^2+a^2}}$$

8. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$

Solution: We have,

$x\sqrt{1+y} + y\sqrt{1+x} = 0$

$x\sqrt{1+y} = -y\sqrt{1+x}$

$x^2(1+y) = y^2(1+x)$

$x^2 - y^2 = y^2x - x^2y$

$(x+y)(x-y) = -xy(x-y)$

$x + y = -xy$; $\quad x = -y - xy$

$y(1+x) = -x$; $\quad y = -\frac{x}{1+x}$ $\quad [since, x \neq y]$

$$\frac{dy}{dx} = -\left\{\frac{(1+x) \times 1 - x(0+1)}{(1+x)^2}\right\}$$

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

LONG ANSWER QUESTIONS

1. If $f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$, $x \neq \frac{\pi}{4}$. Find the value of $f\left(\frac{\pi}{4}\right)$ so that $f(x)$ becomes continuous at x $= \frac{\pi}{4}$.

Solution: For $f(x)$ to be continuous at $x = \frac{\pi}{4}$, we must have

$$\lim_{x \to \frac{\pi}{4}} f(x) = f\left(\frac{\pi}{4}\right)$$

$$f\left(\frac{\pi}{4}\right) = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

$$f\left(\frac{\pi}{4}\right) = \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\left(\cos x - \cos\frac{\pi}{4}\right)}{\cot x - \cot\frac{\pi}{4}}$$

$$= -\sqrt{2} \lim_{x \to \frac{\pi}{4}} \frac{2\sin\left(\frac{x}{2} - \frac{\pi}{8}\right)\sin\left(\frac{x}{2} + \frac{\pi}{8}\right)}{\cos x \sin\frac{\pi}{4} - \sin x \cos\frac{\pi}{4}} \times \sin x \, sin\frac{\pi}{4}$$

$$f\left(\frac{\pi}{4}\right) = -2\sqrt{2} \lim_{x \to \frac{\pi}{4}} \frac{\sin\left(\frac{x}{2} - \frac{\pi}{8}\right)\sin\left(\frac{x}{2} + \frac{\pi}{8}\right)}{-\sin\left(x - \frac{\pi}{4}\right)} \times \sin x \, sin\frac{\pi}{4}$$

$$f\left(\frac{\pi}{4}\right) = -2\sqrt{2} \lim_{x \to \frac{\pi}{4}} \frac{\sin\left(\frac{x}{2} - \frac{\pi}{8}\right)\sin\left(\frac{x}{2} + \frac{\pi}{8}\right)}{-2\sin\left(\frac{x}{2} - \frac{\pi}{8}\right)\cos\left(\frac{x}{2} - \frac{\pi}{8}\right)} \times \sin x \, sin\frac{\pi}{4}$$

$$f\left(\frac{\pi}{4}\right) = \sqrt{2} \lim_{x \to \frac{\pi}{4}} \frac{\sin\left(\frac{x}{2} + \frac{\pi}{8}\right)}{\cos\left(\frac{x}{2} - \frac{\pi}{8}\right)} \times \sin x \, sin\frac{\pi}{4}$$

$$= \sqrt{2} \frac{\sin\left(\frac{\pi}{8} + \frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8} - \frac{\pi}{8}\right)} \times \sin x \, sin\frac{\pi}{4} \quad = \left(sin\frac{\pi}{4}\right)^2 \quad = \frac{1}{2}$$

2. If $x = sin^{-1}\left(\frac{2t}{1+t^2}\right)$ and $y = tan^{-1}\left(\frac{2t}{1-t^2}\right)$, t > 1. Prove that $\frac{dy}{dx} = -1$.

Solution: Let $t = \tan\theta$. Then,

$$t > 1 \Rightarrow \tan\theta > 1 \;\Rightarrow\; \frac{\pi}{4} < \theta < 2\theta < \pi$$

$$\therefore x = sin^{-1}\left\{\frac{2t}{1+t^2}\right\} = sin^{-1}\left\{\frac{2\tan\theta}{1+tan^2\theta}\right\}$$

$$x = sin^{-1}(\sin 2\theta) = sin^{-1}\{sin(\pi - 2\theta)\} = \pi - 2\theta = \pi - 2\,tan^{-1}t$$

$$\frac{dx}{dt} = 0 - \frac{2}{1+t^2} = -\frac{2}{1+t^2}$$

and, $y = tan^{-1}\left\{\frac{2t}{1-t^2}\right\}$

$$y = tan^{-1}\left\{\frac{2\tan\theta}{1-tan^2\theta}\right\} = tan^{-1}(\tan 2\theta) = tan^{-1}\{-tan(\pi - 2\theta)\}$$

$$y = -tan^{-1}\{tan(\pi - 2\theta)\} = -(\pi - 2\theta) = -\pi + 2\,tan^{-1}t$$

$$\frac{dy}{dt} = 0 + \frac{2}{1+t^2} = \frac{2}{1+t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+t^2}}{-\frac{2}{1+t^2}} = -1$$

3. If $y = \left\{x + \sqrt{x^2+1}\right\}^m$, show that $(x^2+1)y_2 + xy_1 - m^2y = 0$

Solution: We have, $y = \left\{x + \sqrt{x^2+1}\right\}^m$. differentiating with respect to *x*, we get

$$\frac{dy}{dx} = m\left\{x + \sqrt{x^2+1}\right\}^{m-1} \times \frac{d}{dx}\left\{x + \sqrt{x^2+1}\right\}$$

$$\frac{dy}{dx} = m\left\{x + \sqrt{x^2+1}\right\}^{m-1} \times \left\{1 + \frac{2x}{2\sqrt{x^2+1}}\right\} = \frac{m\left\{\sqrt{x^2+1}+x\right\}^m}{\sqrt{x^2+1}}$$

$$\frac{dy}{dx} = \frac{my}{\sqrt{x^2+1}}; \qquad y_1 = \frac{my}{\sqrt{x^2+1}}; \qquad y_1\sqrt{x^2+1} = my$$

$${y_1}^2\{x^2+1\} = m^2y^2$$

Differentiating with respect to x, we get,

$$2y_1y_2(1+x^2) + y_1{}^2\,(2x) = 2m^2yy_1$$

$$y_2(1+x^2) + xy_1 - m^2y = 0$$

4. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

solution: We have, $x^m.\ y^n = (x+y)^{m+n}$

Taking log on both sides, we get

$$m\log x + n\log y = (m+n)\log(x+y)$$

Differentiating both sides with respect to x, we get

$$m\times\frac{1}{x} + n\times\frac{1}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\frac{d}{dx}(x+y)$$

$$\frac{m}{x} + \frac{n}{y}\times\frac{dy}{dx} = \frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\left\{\frac{n}{y} - \frac{m+n}{x+y}\right\}\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\left\{\frac{nx+ny-my-ny}{y(x+y)}\right\}\frac{dy}{dx} = \left\{\frac{mx+nx-mx-my}{(x+y)x}\right\}$$

$$\frac{nx-my}{y\,(x+y)}.\frac{dy}{dx} = \frac{nx-my}{(x+y)\,x};\qquad \frac{dy}{dx} = \frac{y}{x}$$

5. If $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$, find $\frac{dy}{dx}$

Solution: We have, $y = (\sin x)^{\tan x} + (\cos x)^{\sec x} = e^{\tan x.\log\sin x} + e^{\sec x.\log\cos x}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan x.\log\sin x}\right) + \frac{d}{dx}\left(e^{\sec x.\log\cos x}\right)$$

$$\frac{dy}{dx} = e^{\tan x.\log\sin x}\frac{d}{dx}(\tan x\log\sin x) + e^{\sec x.\log\cos x}\frac{d}{dx}(\sec x\log\cos x)$$

$$\frac{dy}{dx} = (\sin x)^{\tan x}\left\{\frac{d}{dx}(\tan x)\times\log\sin x + \tan x\times\frac{d}{dx}(\log\sin x)\right\} + (\cos x)^{\sec x}\left\{\frac{d}{dx}(\sec x)\times\log\cos x + \sec x\times\frac{d}{dx}(\log\cos x)\right\}$$

$$\frac{dy}{dx} = (\sin x)^{\tan x}\left\{sec^2 x\log\sin x + \tan x\times\frac{1}{\sin x}\times\cos x\right\} + (\cos x)^{\sec x}\left\{\sec x\tan x\log\cos x + \sec x\left(\frac{1}{\cos x}\right)(-\sin x)\right\}$$

$$\frac{dy}{dx} = (\sin x)^{\tan x}\{sec^2x\log\sin x + 1\} + (\cos x)^{\sec x}\{\sec x\tan x\log\cos x - \sec x\tan x\}$$

CASE STUDY BASED QUESTIONS

1. Let $f(x)$ be a real valued function. Then its

Left Hand Derivative (L.H.D.) : $Lf'(a) = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$

Right Hand Derivative (R.H.D.) : $Rf'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its L.H.D. and R.H.D. at $x = a$exist and both are equal.

For the function $f(x) = \begin{cases} |x-3|, & x\ge 1 \\ \frac{x^3}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1\end{cases}$ answer the following questions:

i) What is R.H.D. of $f(x)$ at $x = 1$?

ii) Check if the function $f(x)$ is differentiable at $x = 1$.

Solution: i) We have, $f(x) = \begin{cases} |x-3|, & x\ge 1 \\ \frac{x^3}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1\end{cases}$

$$f(x) = \begin{cases} \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \\ 3 - x, & 1 \le x < 3 \\ x - 3\,, & x \ge 3 \end{cases}$$

(RHD of $f\ (x)$ at $= 1$)

$$= \lim_{h \to 0} \frac{f\,(1+h) - f\,(1)}{h} = \lim_{h \to 0} \frac{3 - (1+h) - (3-1)}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = -1$$

ii) We find that: (LHD of $f(x)$ $at\ x = 1$) = (RHD of $f(x)$ at x =1)

So, $f\ (x)$ is differentiable at $x = 1$

2. A function $f(x)$ is differentiable at a point c in its domain iff $\lim_{x \to c} \frac{f\,(x) - f\,(c)}{x - c}$ exists finitely. This limit, if it exists, is called the derivative of $f(x)$ at $x = c$ and is denoted by $f'(c)$. Based on the above information answer the following questions:

   The function $f(x) = |x + 1|$,then

   (i) check the continuity of f(x) at x= -1

   (ii) Check the differentiability of f(x) at $x = -1$

   Solution: i) We have, $f(x) = |x + 1| = \begin{cases} -x - 1, x < -1 \\ x + 1, x \ge 1 \end{cases}$. Clearly, $f(x)$, being a polynomial function, is everywhere continuous , at $x = -1$ also.

   (ii) (LHD at $x = 1$) = $\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1^-} \frac{-x - 1 - 0}{x + 1} = -1$

   (RHD at $x = 1$) = $\lim_{x \to -1^+} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1^+} \frac{-x + 1 - 0}{x + 1} = 1$

   ∴ (LHD at $x = 1$) ≠ (RHD at $x = 1$).

   So, $f(x)$ is not differentiable at $x = -1$.

3. A function $f(x)$ is said to be continuous at $x = a$, if the function is defined at $x = a$, $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$. Otherwise $f(x)$ is discontinuous at $x = a$ and a is called the point of discontinuity of $f(x)$.

   Based on the information given above, answer the following questions:

   If f(x)=[x] in [3, 7]

   i) Find the number of points of discontinuity of f(x).

   ii) If $f(x) = \begin{cases} \frac{e^{3x} - 1}{In(1+2x)}, x \ne 0 \\ k, \quad x = 0 \end{cases}$ is continuous at $x = 0$, then find the value of k

   Solution: i) The function $f(x) = [x]$ is discontinuous at $x = 4, 5, 6$ in$[3, 7]$.

   It is right continuous at $x = 3$ and left discontinuous at $x = 7$.

   Hence, there are four points of discontinuity at x ∈{4,5,6,7}

   ii) If f(x) is continuous at x=0, then

$$\lim_{x \to 0} f(x) = f(0) = \lim_{x \to 0} \frac{e^{3x} - 1}{In(1+2x)} = k = \lim_{x \to 0} \frac{e^{3x} - 1}{3x} \times \frac{2x}{In(1+2x)} \times \frac{3}{2} = k$$

$$\Rightarrow k = \frac{3}{2}$$

CHAPTER – 6 : APPLICATIONS OF DERIVATIVES

Gist/Summary of the lesson:

1. **Rate of change in quantities:** the rate of change is the ratio of change in a quantity with comparison to change in another quantity.

   To find the rate of change in a quantity P with respect to change in another quantity Q is $\frac{dP}{dQ}$

   If rate of change in P with respect to Q is positive, the quantity P increases as Q increase OR P decreases as Q decrease. (P is directly proportional to Q)
   If rate of change in P with respect to Q is negative, the quantity P decreases as Q increase OR P increases as Q decrease (P is inversely proportional to Q)
2. **Increasing and decreasing functions:**
   a) A function is increasing over an interval [*a,b*] if for each $x_1 and\ x_2 \in [a, b]$ such that $x_1 > x_2$, $f(x_1) \geq f(x_2)$.
   b) A function is decreasing over an interval [*a,b*] if for each $x_1 and\ x_2 \in [a, b]$ such that $x_1 > x_2$, $f(x_1) \leq f(x_2)$.
   c) A function is strictly increasing over an interval [*a,b*] if for each $x_1 and\ x_2 \in [a, b]$ such that $x_1 > x_2, f(x_1) > f(x_2)$.
   d) A function is strictly decreasing over an interval [*a,b*] if for each $x_1 and\ x_2 \in [a, b]$ such that $x_1 > x_2, f(x_1) < f(x_2)$.
   e) A function is increasing at any point if its derivative is positive or zero at that point. (As $f(x + h) - f(x)$ is positive)
   f) A function is decreasing at any point if its derivative is negative or zero at that point. (As $f(x + h) - f(x)$ is negative)
   g) A function is increasing over an interval if its derivative is positive at each point of that interval
   h) A function is decreasing over an interval if its derivative is negative or zero at each point of that interval
   i) To find whether a function is increasing/decreasing at a point, find derivative of the function and its value at that point, if the value is positive, it is increasing; and if the value is negative the function is decreasing.
   j) To find the interval on which a function is increasing/decreasing find the derivative of the function and find the values of x for which the derivative is positive/negative.
3. **Maxima and Minima:**
   a) The maximum/Minimum value of a function over a closed interval is called Absolute maximum/ minimum of the function.
   b) The maximum/minimum value of a function in the neighbourhood of a point, is called local maximum/minimum of the function.
   c) A point in the domain of a function is called a turning point/critical point, if the derivative of the function becomes zero at that point.
   d) At each stationery point (other than point of inflection) the function attains a maximum or minimum value.
   e) The value attains at stationery point may be maximum or minimum. It can be decided by the help of first derivative of the function (FIRST DERIVATIVE TEST) or Second derivative of the function (SECOND DERIVATIVE TEST).
   f) First derivative test: Function attains maximum value at a stationery point if the value of first derivative is negative at any point to the right of the stationery point.
   g) Second derivative test: Function attains maximum value at a stationery point if the value of second derivative at the stationery point is negative.

Definitions and Formulae:

- **Rate of change**: Rate of change in quantity **A** with reference to another quantity **B** is $\frac{dA}{dB}$

  If x denotes the number of items produced, and P is Profit function, $\frac{dP}{dx}$ is known as marginal profit.

  If x denotes the number of items produced, and C is Cost function, $\frac{dC}{dx}$ is known as marginal cost.

If x denotes the number of items produced, and R is Revenue function, $\frac{dR}{dx}$ is known as marginal Revenue.

- **Increasing decreasing functions**: A function is increasing at a point $x = a$, if $f'(a) \geq 0$
  A function is decreasing at a point $x = a$, if $f'(a) \leq 0$
  A function is strictly increasing at a point $x = a$, if $f'(a) > 0$
  A function is strictly decreasing at a point $x = a$, if $f'(a) < 0$
  WORKING RULE to find the interval on which a function is increasing/decreasing
  (i) Write the given function
  (ii) Find its derivative
  (iii) Find the values of x (interval) for which the derivative is positive/negative
- **Maxima and minima**: Maximum value of a function in an interval is called absolute maximum.
  Minimum value of a function in an interval is called absolute minimum.
  At every extreme (Maximum/minimum) value, the derivative of the function become zero.
  WORKING RULE to find MAXIMUM/MINIMUM value of a function.
  (i) Identify the function, which is to be maximised or minimised, notice the variables in it.
  (ii) Identify the relation between these variables which is given in the question as a hint.
  (iii) Using above relation, write one variable in terms of other, and express the function which is to be maximised/minimised as a single variable function.
  (iv) Find the first and second derivatives of this function.
  (v) Find the values of x for which the first derivative becomes zero, these points are called critical or turning or stationery points.
  (vi) At each stationery point check the sign of second derivative, if it is positive-Minimum exists at that point, if it is negative-Maximum exists at that point and if it is zero-neither minimum nor maximum exist at that point

MULTIPLE CHOICE QUESTIONS

1) The rate of change in radius of a circle whose area is increasing at the rate 44 cm^2/sec when the radius of circle is 7 cm.

a) 1 cm/sec b) 2 cm/sec c) 3 cm/sec d) 4 cm/sec

Solution: A $= \pi r^2$ and $\frac{dA}{dt} = 2\pi r.\frac{dr}{dt}$ given that $\frac{dA}{dt} = 44$ and r = 7

Substituting all these we get, $44 = 2 \times \frac{22}{7} \times 7 \times \frac{dr}{dt}$, gives $\frac{dr}{dt} = 1$ **Answer: a**

2) The minimum value of $y = x^{\frac{1}{x}}$ $(x \neq 0)$ is

a) e b) $\frac{1}{e}$ c) $e^{\frac{1}{e}}$ d) $\left(\frac{1}{e}\right)^e$

Solution: $y = x^{\frac{1}{x}}$ and $\frac{dy}{dx} = x^{\frac{1}{x}}\left[\frac{1-\log x}{x^2}\right]$

$\frac{dy}{dx} = 0$ gives that either $x^{\frac{1}{x}}$=0 (impossible) or $\log x = 1$ and $x \neq 0$

So, $x = e$ is the turning point, and at x=3, $\frac{dy}{dx}$ is positive (First derivative test used ; $e <$ 3), hence y attains minimum value at x=e. **Answer: c**

3) The Absolute maximum value of the function $f(x) = \sin x + \cos x$in the interval $[0, \pi]$ is

a) 1 b) -1 c) $\frac{1}{\sqrt{2}}$ d) $\sqrt{2}$

Solution: $y = \sin x + \cos x$, and $\frac{dy}{dx} = \cos x - \sin x$, stationery points in $[0, \pi]$ are $\frac{\pi}{4}$ only

$f(0) = 1, f(\pi) = -1, and\ f\left(\frac{\pi}{4}\right) = \sqrt{2}$

Hence absolute maximum value is $\sqrt{2}$ **Answer: d**

4) The side of an equilateral triangle is increasing at the rate of 5cm/sec. The rate at which its area increases, when side is $10\sqrt{3}$ cm is

a) $300\sqrt{3}$ b) $100\sqrt{3}$ c) $25\sqrt{3}$ d) 75

Solution: A $=\frac{\sqrt{3}}{4}s^2$ and $\frac{dA}{dt}=\frac{\sqrt{3}}{4}.2s\frac{ds}{dt}$, given that $\frac{dA}{dt}=\frac{\sqrt{3}}{4}\times 2\times 10\sqrt{3}\times 5$ =75

Answer: d

5) Total revenue in rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is

a) 96 b) 126 c) 116 d) 90

Solution: $R'(x) = 6x + 36$ and $R'(15) = 126$. **Answer: b**

6) A particle is moving along a curve $y = 3x^2 - 5x + 6$. The point on the curve where y-coordinate is changes 7 times as fast as the x coordinate is

a) (2, 8) b) $(3, -4)$ c) $(\frac{1}{3}, \frac{14}{3})$ d) (0, 0)

Solution: $\frac{dy}{dx} = 6x - 5$ and given that $\frac{dy}{dx} = 7$ and $6x - 5 = 7, x = 2, y = 8$ **Answer: a**

7) The rate of change in volume of a cube with respect to its side is,

a) its lateral surface area b) its total surface area

c) half of its total surface area d) half of its lateral surface area

Solution: $V = a^3$ and $\frac{dV}{da} = 3\,a^2$ and $\frac{dV}{da} = \frac{1}{2}(6\,a^2)$= half the Total Surface area of cube **Answer: c**

8) The slope (Slope of a curve $y = f(x)$ is $\frac{dy}{dx}$) of the curve $y = x^3 - 6x^2 + 19x - 20$ is minimum at

a) (2, 2) b) $(2, -2)$ c) $(-2, 2)$ d) (2, 0)

Solution: $y = x^3 - 6x^2 + 19x - 20$ and $slope = \frac{dy}{dx} = 3x^2 - 12x + 19$

To find minimum value of slope, Consider $f(x) = 3x^2 - 12x + 19$

$f'(x) = 6x - 12, and\ f''(x) = 6 > 0$ $f'(x) = 0$ implies $x = 2$, at (2, 2) **Answer: a**

9) The function $f(x) = 2x^3 - 15x^2 + 36x + 6$ is increasing in the interval,

a) $(-\infty, 2) \cup (3, \infty)$ b) $(-\infty, 2)$ c) $(-\infty, 2] \cup [3, \infty)$ d) $[3, \infty)$

Solution: $f'(x) = 6x^2 - 30x + 36$ and $f'(x) > 0$ if $6(x-3)(x-2) > 0$ is possible when x does not lie between 2 to 3, i.e. $x \in (-\infty, 2] \cup [3, \infty)$ **Answer: c**

10) The interval in which $y = x^2e^{-x}$ $(x \neq 0)$ increasing is

a) $(-\infty, \infty)$ b) $(-2, 0)$ c) $(2, \infty)$ d) $(0, 2)$

Solution: $y = x^2e^{-x}$ and $\frac{dy}{dx} = e^{-x}.x.(2 - x)$

$\frac{dy}{dx} > 0$ gives that $x > 0$ and $x < 2$ as $e^{-x} > 0$ for all x. So, $x \in (0,2)$ **Answer: d**

ASSERTION - REASON BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason(R). Pick the correct option:

A) Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).

B) Both Assertion (A) and Reason(R) are true but Reason(R) is NOT the correct explanation of Assertion (A).

C) Assertion (A) is true but Reason(R) is false.

D) Assertion (A) is false but Reason(R) is true.

1) **Assertion (A):** The function $f(x) = 3 + \frac{5}{x}, x \neq 0$ is strictly decreasing

Reason(R) : For strictly decreasing function $f'(x) < 0$.

Solution: $f(x) = 3 + \frac{5}{x}$, and $f'(x) = -\frac{5}{x^2}$ is negative for all $x \in R$

Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A). **Answer: A**

2) **Assertion (A):** The rate of change in area of a circle with respect to its radius is its circumference.

Reason(R) : The rate of change in area of a plane figure is always its perimeter.

Solution: $Area\ of\ circle = \pi r^2$, and

rate of change in Area with respect to r $= \frac{dA}{dr} = 2\pi r$ (circumference of circle)

But it is true in case of circles, not always. so, Assertion (A) is true but Reason(R) is false

Answer: C

3) **Assertion (A):** The total cost $C(x)$ in rupees with the production of x units of an item is given by $C(x) = 3x^2 + 5x - 9$, and then the marginal cost is -9 when x= 2.

Reason(R): Marginal cost is the rate of change in total cost with respect to the number of items produced. **Answer: D**

Solution: $C(x) = 3x^2 + 5x - 9$, and $C'(x) = 6x + 5$, at $x = 2$, the marginal cost is 17

Assertion (A) is false, but Reason(R) is true.

4) **Assertion (A):** The rate of change of the function $f(x) = \sin x$, with respect to x is $\cos x$

Reason(R) : $\cos x$ is the complementary trigonometric function of $\sin x$. **Answer: B**

Solution: $f(x) = \sin x$, and $f'(x) = \cos x$ is true

Both Assertion (A) and Reason(R) are true, but Reason(R) is not the correct explanation of Assertion (A).

5) **Assertion (A):** At every critical point (except at point of inflection), a function attains its extreme value.

Reason(R): A function $f(x)$ is decreasing function if $f'(x) < 0$ and increasing function if $f'(x) > 0$ **Answer: A**

Solution: A point is called critical point if the derivative of the function at that point is zero, so, at this point the function is neither increasing nor decreasing.

So have a peak.

Both Assertion (A) and Reason(R) are true, and Reason(R) is the correct explanation of Assertion (A).

6) **Assertion (A):** The minimum value of $f(x) = (2x - 1)^2 + 3$ is 3

Reason(R) : Square of a number is always non-negative. **Answer: A**

Solution: $f(x) = (2x - 1)^2 + 3$, and least value of $(2x - 1)^2$ is zero for all $x \in R$

Hence the minimum value of $f(x)$is 3

Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).

7) **Assertion (A):** Maximum value of the function $f(x) = (\sin x + \cos x)$, on the interval $[0, \frac{\pi}{2}]$ is $\sqrt{2}$

Reason(R) : Maximum value of $[f(x) + g(x)]$= Max$f(x)$+Max$g(x)$. **Answer: C**

Solution: $f(x) = \sin x + \cos x$, and $f'(x) = \cos x - \sin x$, $x = \frac{\pi}{4}$ is the critical point

Max value of $f(x) = \sqrt{2}$ but Max $(\sin x)$+Max$(\cos x)$=2

Assertion (A) is true but Reason(R) is false.

8) **Assertion (A):** The function $f(x) = \log x$, $x > 0$ is strictly increasing

Reason(R) : For strictly increasing function $f'(x) > 0$. **Answer: A**

Solution: $f(x) = \log x$, and $f'(x) = \frac{1}{x}$ is positive for all $x > 0$

Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).

9) **Assertion (A):** The absolute maximum value of $f(x) = 4 - x^2$ on [1, 2] is 4.

Reason(R) : If $x_1, x_2, \dots, x_n$ are critical points of the function $f(x)$ then the absolute max of $f(x)$ on the interval $[a, b]$ is Abs Max = Max$\{ f(a), f(x_1), f(x_2), \dots f(x_n), f(b)\}$

Answer: D

Solution: $f(x) = 4 - x^2$, and $f'(x) = -2x$ only critical point is 0, which do not lie between 1 and 2. So maximum is $f(1) = 3$ but the reason is correct

Assertion (A) is false but Reason(R) is true

10) **Assertion (A):** The function $f(x) = \cot x, \{x: x \in R, x \neq n\pi, n \in Z\}$ is strictly decreasing

Reason(R) : For strictly decreasing function $f'(x) < 0$. **Answer: A**

Solution: $f(x) = \cot x$, and $f'(x) = -cosec^2 x$ is negative for all $x \in R$

Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).

VERY SHORT ANSWER TYPE QUESTIONS

1) Question: Show that the function given by $f(x) = \cos x$ is strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$

Solution: A function is strictly decreasing over an interval, if its derivative is negative (non-zero) in that interval.

given $f(x) = \cos x$, and $f'(x) = -\sin x$, and $-\sin x < 0$ for all $x \in \left(\frac{\pi}{2}, \pi\right)$

2) Question: Find the interval on which $f(x) = xe^x$ is increasing.

Solution: A function is increasing over an interval, if its derivative is positive in that interval.

given $f(x) = xe^x$, and $f'(x) = e^x(x+1)$, and $e^x(x+1) \geq 0$ for all $x \in [-1, \infty)$ as e^x is a non-negative function.

3) Question: Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

Solution: A function is strictly increasing over an interval, if its derivative is non-zero positive in that interval.

given $f(x) = \log x$, and $f'(x) = \frac{1}{x}$, and $\frac{1}{x} > 0$ for all $x \in (0, \infty)$

4) Question: Find the values of 'k', for which $f(x) = x^2 - 2kx + c$ is an increasing function on $[1, 2]$.

Solution: A function is increasing over an interval, if its derivative is positive in that interval.

given $f(x) = x^2 - 2kx + c$, and $f'(x) = 2x - 2k = 2(x - k)$, and the function is increasing, if $x - k \geq 0$, *or* $k \leq x$ for all $x \in [1,2]$, hence $k \leq 1$.

5) Question: The volume of a cube is increasing at the rate of 9 cm^3/sec. How fast is the surface area increasing when the length of an edge is 10 cm.

Solution: Let volume of cube V = a^3 and $\frac{dV}{dt} = 3a^2 \frac{da}{dt}$.

$$\frac{da}{dt} = \frac{1}{3a^2}\frac{dV}{dt}, \text{ we get } \frac{da}{dt} = \frac{3}{100}$$

Surface area of Cube = S = 6 a^2 and $\frac{dS}{dt} = 6.2a.\frac{da}{dt} = 12 \times 10 \times \frac{3}{100} = 3.6\ cm^2$/sec

6) Question: A particle is moving along the curve $x^2 = 2y$. At what point, ordinate increases at the same rate as abscissa increases?

Solution: given $x^2 = 2y$, and $2x = 2\frac{dy}{dx}$, and $x = \frac{dy}{dx}$ and $\frac{dy}{dx} = 1$ implies x=1.

Hence the required point is $(1, \frac{1}{2})$.

7) Question: A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the latter is 10 cm.
Solution: Let Volume of Sphere be V, and radius be r.

$$V = \frac{4}{3}\pi r^3 \text{ and } \frac{dV}{dr} = \frac{4}{3}\pi.3r^2 = \frac{4}{3}\pi.300 = 400\pi\ cm^3/sec$$

8) Question: Surface area of a balloon (spherical), when air is blown into it, increases at a rate of 5 mm^2/s. When the radius of the balloon is 8mm, find the rate at which the volume of the balloon is increasing?
Solution: Let Volume of Sphere be V, Surface area be S and radius be r.

$$\text{Given } \frac{dS}{dt} = 5;\ S = 4\pi r^2 \text{ and } \frac{dS}{dt} = 4\,\pi.2r.\frac{dr}{dt} \text{ implies } \frac{dr}{dt} = \frac{5}{64\pi}$$

$$V = \frac{4}{3}\pi r^3\ ;\ \frac{dV}{dr} = \frac{4}{3}\times\pi\times 3r^2\times\frac{dr}{dt}, \text{ substituting values we get } \frac{dV}{dt} = 20\text{cm}^3/\text{sec}$$

9) Question: The radius of a cylinder is decreasing at a rate of 2 cm/sec and the altitude is increasing at the rate of 3 cm/sec. Find the rate of change of volume of this cylinder when its radius is 4 cm and altitude is 6 cm.
Solution: Let Volume of cylinder be V, radius be r and altitude be h.

$$\text{Given } \frac{dr}{dt} = -2 \ and\ \frac{dh}{dt} = 3;\ \text{ and } V = \pi r^2 h$$

$$\text{implies } \frac{dV}{dt} = \pi\left[r^2.\frac{dh}{dt} + h.2r.\frac{dr}{dt}\right]$$

$$\text{substituting values, we get } \frac{dV}{dt} = \pi[48-96] = -48\pi$$

10) Question: Find the maximum and minimum values, if any, of the function given by $f(x) = 2\sin 3x + 5$.
Solution: the maximum and minimum values of $\sin x$ are -1 and 1.

$$-1 \le \sin 3x \le 1 \text{ implies } -2 \le 2\sin 3x \le 2$$

$-2+5 \le 2\sin 3x + 5 \le 2+5 \Longrightarrow 3 \le 2\sin 3x + 5 \le 7$ Max 7, Min 3

SHORT ANSWER TYPE QUESTIONS

1) Question: Find the intervals in which the function f given by $f(x) = \tan x - 4x,\ x \in \left(0, \frac{\pi}{2}\right)$ is (a) Strictly increasing (b) Strictly decreasing

Solution: (a) $f(x) = \tan x - 4x$ and $f'(x) = \sec^2 x - 4$

$f(x)$ is increasing if $f'(x) > 0$ i.e. $\sec^2 x > 4$ or $\sec x < -2$ or $\sec x > 2$

$\sec x < \sec\frac{2\pi}{3},\ and\ \sec x > \sec\frac{\pi}{3}$ so, f(x) is strictly increasing on $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

(b) $f(x)$ is decreasing if $f'(x) < 0$ i.e. $\sec^2 x < 4$

$-2 < \sec x < -1$ or $1 < \sec x < 2$

$\frac{2\pi}{3} < x < \pi$ (out of domain) or $0 < x < \frac{\pi}{3}$,

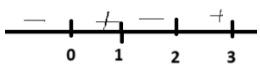
so, f(x) is strictly decreasing on $\left(0, \frac{\pi}{3}\right)$

2) Question: Find the intervals in which $y = [x(x-2)]^2$ is
(a) increasing function (b) decreasing function

Solution: $f(x) = [x(x-2)]^2$ and $f'(x) = 4x(x-1)(x-2)$

$f(x)$ is increasing if $f'(x) > 0$

i.e. $4x(x-1)(x-2) \ge 0$



so, f(x) is strictly increasing on $(0,\ 1) \cup (2, \infty)$
and so, f(x) is strictly decreasing on $(-\infty,\ 0) \cup (1, 2)$

3) Question: Prove that $f(x) = \log|\sin x|$ is increasing on $(0, \frac{\pi}{2}]$ and decreasing on $[\frac{\pi}{2}, \pi)$

Solution: $f(x) = \log|\sin x|$

For $x \in (0, \pi]$, $|\sin x| = \sin x$ hence $f(x) = \log\sin x$

$f'(x) = \cot x$

$f(x)$ is increasing if $f'(x) > 0$

For all $x \in (0, \frac{\pi}{2}]$ $\cot x \geq 0$ (quadrant 1), so $f(x)$ is increasing

For all $x \in [\frac{\pi}{2}, \pi)$ $\cot x \leq 0$ (quadrant 2), so $f(x)$ is increasing

4) **Question:** Prove that $f(x) = \frac{4\sin x}{2+\cos x} - x$ is increasing on $[0, \frac{\pi}{2}]$

Solution: $f(x) = \frac{4\sin x}{2+\cos x} - x$

$$f'(x) = \frac{(2+\cos x)4\cos x - 4\,sinx\,(-\sin x)}{(2+\cos x)^2} - 1$$

$$f'(x) = \frac{8\cos x + 4\cos^2 x + 4\sin^2 x \; - \; 4 - \cos^2 x - 4\cos x}{(2+\cos x)^2}$$

$$f'(x) = \frac{\cos x(4 - cos\,x)}{(2+\cos x)^2}$$

For all $x \in (0, \frac{\pi}{2}]$, $\cos x \geq 0$ and $4 - \cos x \geq 0$ $for\ all\ x \in R$

$f'(x)$ is positive

$f(x)$ is increasing for all $x \in [0, \frac{\pi}{2}]$

5) **Question:** A ladder 25m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall, at the rate of 2 cm/sec. How fast is its height on the wall decreasing when the foot of the ladder is 24 m away from the wall.

Solution: Let the foot of the ladder is x cm away from the wall, and the top of ladder touches the wall at y cm height from ground at any instant time 't'.

As per given data, $x^2 + y^2 = 625$ and so, $y = \sqrt{625 - x^2}$

Given $\frac{dx}{dt} = 2$, and $y = 7$ $when\ x = 24$

Also, $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$, substituting values, we get, $\frac{dy}{dt} = -\frac{48}{7}$

The height is decreasing at the rate $\frac{48}{7}$ m/sec.

6) **Question:** sand is pouring from a pipe at the rate of 12 cm^3/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always three times the radius of the base. How fast is the radius of the sand cone increasing when the height is 6 cm.

Solution: given $\frac{dv}{dt} = 12$, and $h = 3r$; Volume of cone $= \frac{1}{3}\pi r^2 h = \pi r^3$

$\frac{dv}{dt} = 3\pi r^2 . \frac{dr}{dt}$ substituting values we get $\frac{dr}{dt} = \frac{1}{\pi}$

7) **Question:** The length x of a rectangle is decreasing at the rate of 2cm/min and the width y is increasing at the rate of 5 cm/min. When $x = 7\ cm$ and $y = 9cm$. Find the rate of change of (a) the perimeter, and (b) the area of the triangle.

Solution: given $\frac{dx}{dt} = -2, \frac{dy}{dt} = 5$

Perimeter of the Rectangle = $2(x + y)$

Rate of change in perimeter = $2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$ = $6\ cm/min$

Area of rectangle = xy

Rate of change in Area = $x.\frac{dy}{dt} + y.\frac{dx}{dt}$ = 17 cm^2/min

8) **Question:** Find both the maximum value and the minimum values of

$3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval [0, 3]

Solution: given $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$

$f'(x) = 12x^3 - 24x^2 + 24x - 48 = 12(x^3 - 2x^2 + 2x - 4)$

$f'(x) = 12(x^2 + 2)(x - 2)$

Turning points are given by $f'(x) = 0$ i.e. $x = 2$ is the only turning point

$f(0) = 25, f(2) = -39, f(3) = 16$ Max value: 25, Min Value: -39

9) **Question:** Prove that the product of squares of two numbers whose sum is 15 is maximum when they are equal.

Solution: Let the numbers be x, and y

Given that $x + y = 15$ and required $x^2.y^2$ is maximum.

$f(x) = x^2(15 - x)^2$

$f'(x) = 2x.(15 - x)^2 + x^2.2(15 - x)(-1)$

$f'(x) = 2x(15 - x)(15 - 2x)$

$f''(x) = -4x(15 - x) - 2x(15 - 2x) + 2(15 - x)(15 - 2x)$

Critical or turning points are 0, 15 and $\frac{15}{2}$

$f''(0) > 0, f''(15) > 0, f''\left(\frac{15}{2}\right) < 0$

So, f(x) is maximum when $x = \frac{15}{2}$, i.e. x=y

10) **Question:** Prove that the function given by $f(x) = \frac{log\, x}{x}$ has maximum at $x = e$

Solution: The given function is $f(x) = \frac{\log x}{x}$

$f'(x) = \frac{1 - log\, x}{x^2}$, critical points are given by $\log x = 1$ or $x = e$

$f''^{(x)} = \frac{\log x - 3}{x^3}$ $f''(e) < 0$ as $\log e - 3 = -2 < 0$

So, f attains a maximum value at 'e'.

LONG ANSWER TYPE QUESTIONS

1) **Question:** Show that the right circular cylinder of given surface area and maximum volume is such that its height is equal to the diameter of the base.

Solution: Let the height of cylinder be h, radius of base be r, &Volume is V,Surface area S.

Given that S = K (constant), means $2\pi r(h + r) = K$, or $h = \frac{K}{2\pi r} - r$

And $V = \pi r^2 h$ or $V = \pi r^2 \left(\frac{K}{2\pi r} - r\right) = \frac{Kr}{2} - \pi r^3$ and required V to be maximum

V is considered as $f(r)$

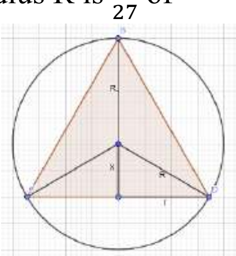
$f'(r) = \frac{K}{2} - 3\,\pi r^2$ and $f''(r) = -6\pi r$ is negative for all positive r

Critical points are given by $\frac{K}{2} - 3\,\pi r^2 = 0$ or $K = 6\pi r^2$

So, at $K = 6\pi r^2$ or $r = \sqrt{\frac{K}{6\pi}}$ Volume is maximum.

When $K = 6\pi r^2$ the value of $h = 3r - r$=2r = diameter of base.

2) **Question:** Prove that the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.



Solution: Let the height of cone be h, radius of its base be r, x be the perpendicular distance of the centre of sphere from centre of cone base and Volume of cone is V, and radius of sphere be R.

$h = R + X$ and $r^2 = R^2 - X^2$

And $V = \frac{1}{3}\,\pi r^2 h$ or $3V = \pi([R^2 - X^2][R + X]) = \pi(R^3 + R^2X - RX^2 - X^3)$ and required V to be maximum

V is considered as $f(X)$

$f'(X) = R^2 - 2RX - 3X^2$ and $f''(X) = -2R - 6X$

Critical points are given by $X = -R$ or $X = \frac{R}{3}$ (as X is length, $X = -R$ is impossible)

So, at $X = \frac{R}{3}$ and $f''\left(\frac{R}{3}\right) = -2R - 6\left(\frac{R}{3}\right)$ which is negative, so Volume is maximum.

When $X = \frac{R}{3}$ then $h = \frac{4R}{3}$ Volume of cone$= \frac{1}{3}\pi \frac{R^2}{9}\frac{4R}{3} = \frac{1}{27}\left(\frac{4}{3}\pi R^3\right)$

3) **Question:** An open tank with a square base and vertical sides is to be constructed from a metal sheet to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width.

Solution: Let the side of the base be x cm and height of vertical side be y cm.

Given that V = K (constant), means $x^2y = K$, or $y = \frac{K}{x^2}$

And $S = 4xy + x^2$ or $S = 4x.\frac{K}{x^2} + x^2 = \frac{4K}{x} + x^2$ and required S to be minimum

S is considered as $f(x)$

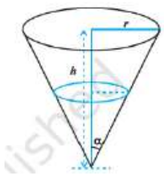
$f'(x) = -\frac{4K}{x^2} + 2x$ and $f''(x) = \frac{8K}{x^3} + 2$ is positive for all positive x

Critical points are given by $-\frac{4K}{x^2} + 2x = 0$ or $2K = x^3$

So, at $K = \frac{x^3}{2}$ or $x = \sqrt[3]{2K}$ Volume is minimum.

When $K = \frac{x^3}{2}$ the value of $y = \frac{x}{2}$

4) **Question:** A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic meters per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.



Solution: Let the semi vertical angle of inverted cone be α, radius of top is r and vertical height be h

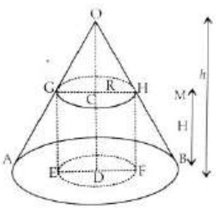
Given that $\alpha = \tan^{-1}\frac{1}{2}$, means $\tan\alpha = \frac{1}{2} = \frac{r}{h}$, or $h = 2r$

And $\frac{dv}{dt} = 5$, $V = \frac{1}{3}\pi r^2 h$ or $V = \frac{1}{3}\frac{\pi h^3}{4}$ as $r = \frac{h}{2}$

$\frac{dV}{dt} = \frac{\pi}{12}.3h^2.\frac{dh}{dt}$ So, $5 = \frac{\pi}{12}.3(4)^2.\frac{dh}{dt}$.

Rate of increase in water level is $\frac{5}{4\pi}$ or $\frac{35}{88}$ meter/hour

5) **Question:** Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2\alpha$.



Solution: Let the height of cylinder be H, radius of its base be R, and Volume is V, and the height of Cone be h, radius of the base of cone is r, semi vertical angle α

Given that $\frac{r}{h} = \tan\alpha = \frac{R}{h-H}$ so $R = \frac{r(h-H)}{h}$

And volume of cylinder $V = \pi R^2 H$ or $V = \frac{\pi r^2(h-H)^2 H}{h} = \frac{\pi r^2}{h}.(h^2H + H^3 - 2hH^2)$

and required V to be maximum

V is considered as $f(H)$

$f'(H) = \frac{\pi r^2}{h}.(h^2 + 3H^2 - 4hH)$ and $f''(H) = 2\frac{\pi r^2}{h}.(3H - 2h)$

Critical points are given by $h^2 + 3H^2 - 4hH = 0$ implies $H = h$ or $H = \frac{h}{3}$

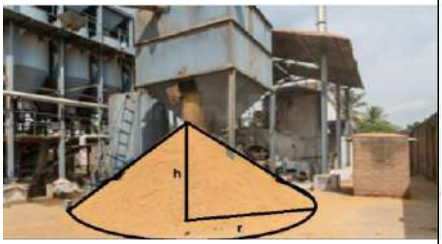
So, at $H = \frac{h}{3}$, $f''(H) < 0$ Volume is maximum.

When $h = 3H$ the Volume of cylinder is $\frac{\pi r^2(h-H)^2 H}{h} = \frac{\pi h^2\tan^2\alpha\left(h-\frac{h}{3}\right)^2\frac{h}{3}}{h} = \frac{4\pi}{27}h^4\tan^2\alpha$

CASE BASED QUESTIONS

1) **Question:** In a rice mill husk is being poured in the shape of a cone so that the height of the cone is always one third the base diameter of the cone.

Using the above information answer the following questions.(2+2)



(i) Find the rate of change in the base area when the amount of husk poured at the rate 100 cc per minute and radius of the base is 5 cm.

(ii) Find the rate of change in the volume of the cone when the height is 7cm and increasing at the rate $\frac{4}{11}$cm/min

Solution:

(i) Given that $\frac{dv}{dt} = 100$ and $v = \frac{1}{3}\pi r^2 h$ or $v = \frac{2}{9}\pi r^3$

So, $\frac{dv}{dt} = 100 = \frac{2}{3}\pi r^2 \frac{dr}{dt}$ implies $\frac{dr}{dt} = \frac{150}{\pi r^2}$

Now area of the base is $A = \pi r^2$ and $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

Therefore, rate of change in area of base at radius = 5, is $\frac{dA}{dt} = 60\ cm^2$ /minute

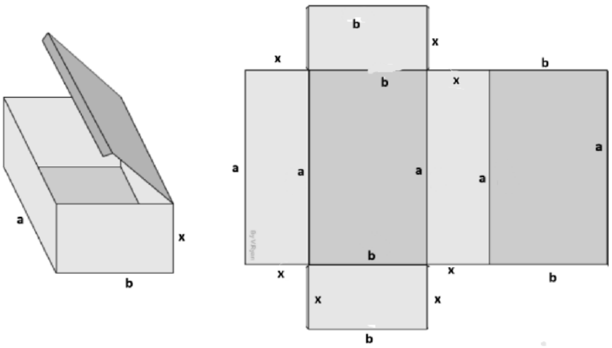
(ii) Given that $\frac{dh}{dt} = \frac{4}{11}$ and $h = 7$

Then $v = \frac{1}{3}\pi r^2 h = \frac{3\pi h^3}{4}$and hence, $\frac{dv}{dt} = \frac{9}{4}\pi h^2 . \frac{dh}{dt}$

$\frac{dv}{dt} = 2 \times 7 \times 9 = 126\ cc/min$

2) **Question:** A sweet shop owner wishes to order the carton boxes in cuboid form such that the length of the box is double the width to hold 72 cc halwa by using least amount of carton.

Assume the length of the box as a , breadth as b and the height as x as shown the figure given below.



Using the above information answer the following questions

(i) Find the volume of the box in terms of b and x, and write x in terms of b

(ii) Write the formula for total surface area of the box and express it as a function of b

(ii) Find the dimensions of the box for which the total surface area is least.

Solution:

(i) Volume of C = $l\ b\ h = a.b.x = 2b.b.x$ and given V=72, $x = \frac{36}{b^2}$

(ii) Total Surface area of the box = $2(ab + bx + ax) = 2(2b^2 + 3bx)$

Hence TSA = $f(b) = 4b^2 + \frac{216}{b}$

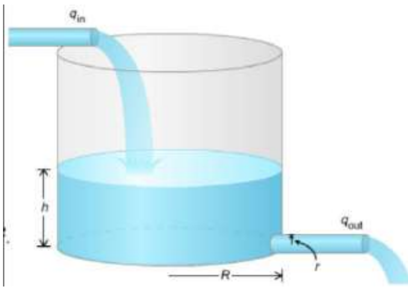
(iii) $f'(b) = 8b - \frac{216}{b^2}$ and $f''(b) = 8 + \frac{432}{b^3}$

Critical points are given by $f'(b) = 0$ i.e. $b^3 = \frac{216}{8} = 27$ so b=3

And at b=3, $f''(b)$ is positive, surface area is minimum

So the dimensions are $a = 6, b = 3, and\ x = 4$

3) **Question:** A cylindrical tank is attached with two pipes q_{in} and q_{out}. q_{in} fills the tank at the rate 5 cc/sec with water and q_{out} empties the tank at the rate 3cc/sec. the radius of the tank is given as 7 cm.



Based on the information given above, answer the following questions.(1+1+2)

(i) If both the pipes are running simultaneously, find the rate at which the height decreases.

(ii) If the area of cross section of the q_{out} pipe is 0.3 cm^2 find the speed of the water flown out. (speed the rate of change in distance)

(iii) If the water flown out from cylindrical tank is stored in tank whose base is a square of side 30 cm, find the rate at which the water level is increasing.

Solution:

(i) If both pipes are running, net increase in the volume = 5 − 3= 2 cc/sec

So, $\frac{dv}{dt} = 2$ and $v = \pi r^2 h, \frac{dv}{dt} = \pi r^2 \frac{dh}{dt}$ (r is constant) $\frac{dh}{dt} = \frac{2}{\pi r^2} = \frac{1}{77}$cm/sec

(ii) Area of cross section of pipe is given as 0.3 cm^2. The water flowing at the rate 3cc per second. The water stagnated in the pipe in 1 second is in the shape of cylinder.

So $\frac{dv}{dt} = 3$, and $v = \pi r^2 h = 0.3h$

Hence $\frac{dv}{dt} = 0.3\frac{dh}{dt}$ implies speed of water = $\frac{dh}{dt} = \frac{3}{0.3} = 10$cm/sec

(iii) Water flowing out at the rate 3cc/sec

This water is stored in cuboid shaped tank whose base area is 900 cm^2 and height h (assume) $v = lbh = 900h$ and $\frac{dh}{dt} = \frac{1}{900}\frac{dv}{dt} = \frac{3}{900} = \frac{1}{300}$ cm/sec.

CHAPTER – 7 : INTEGRALS

Gist of the Lesson/ summary

- Introduction
- Integration as an inverse process of Differentiation
- Methods of Integration
- Integrals of some particular Functions
- Integration by Partial fractions
- Integration by Parts
- Definite integral
- Fundamental Theorem of Calculus
- Evaluation of Definite integrals by substitution
- Some properties of Definite integrals

Definitions and Formulae

1. Introduction

   Integrals are a fundamental concept in calculus that involve finding the anti-derivative of a function, effectively reversing the process of differentiation. They can be either indefinite or definite, with indefinite integrals having no limits and representing family of functions, while definite integrals have limits and will be numerical value.

2. Some properties of indefinite integrals

   Let $\frac{d}{dx}F(x) = f(x)$. Then ,we write $\int f(x)dx$ = F(x) +C. these integrals are called indefinite integrals or general integrals, C is called a constant of integration. All these integrals differ by a constant.

   If two functions differ by a constant , they have the same derivative.

   i) $\frac{d}{dx}\int f(x)\,dx = f(x)$ and $\int f'(x)dx = f(x) + C$,

   where C is any arbitrary constant.

   ii) $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$, then $\int f(x)\,dx$ and $\int g(x)\,dx$ are equivalent

   iii) $\int(f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

   iv) $\int kf(x)\,dx$ = k$\int f(x)\,dx$, k is any constant

   v) $\int(k_1f_1(x) + k_2f_2(x) + k_3f_3(x) + \cdots \ldots.. k_nf_n(x))\,dx$
   $= k_1\int f_1(x)dx + \ldots\ldots k_n\int f_n(x)dx.$

3. **Methods of integration**

   a) Integration by substitution

   b) Integration by trigonometric formula

   c) Method of partial fraction.

S.No.	Form of the rational function	Form of the partial fractions
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)},$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ **Where,** x^2+bx+c **cannot be factorised further**	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

d) **Integration by parts**

$\int f(x)g(x)dx = f(x)\int g(x)\,dx - \int[f'(x)\int g(x)dx]dx$

If we take f as the first function , easily differentiable and g as the second function, easily integrable then this formula may be stated as follows:

Note:- For choosing the first function f (x) use the abbreviate as ILATE ,

ILATE- inverse, log, arithmetic, trigonometric function, exponential function.

4. **Definite integrals**

$\int_a^b f(x)dx = F(b) - F(a)$, if F is an antiderivative of f (x)

5. **Fundamental theorem of calculus**

(i) Area function: The function A (x) denotes the area function and is given by $A(x) = \int_a^x f(x)dx$

6. **Properties of definite integrals**

$p_0 : \int_a^b f(x)dx = \int_a^b f(t)dt$

$p_1 : \int_a^b f(x)dx = -\int_b^a f(x)dx$, in particular, $\int_a^a f(x)dx = 0$

$p_2 : \int_a^b f(x)dx = \int_b^c f(x)dx + \int_c^b f(x)dx$, $a < c < b$

$p_3 : \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$p_4 : \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$p_5 : \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$p_6 : \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, if\ f(2a-x) = f(x), \\ 0\ if \quad f(2a-x) = -f(x), \end{cases}$

p_7 : (i) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$, if f is an even function i.e, $f(-x) = f(x)$,

(ii) $\int_{-a}^a f(x)dx = 0$, if f is an odd function i.e., $f(-x) = -f(x)$

Formulae:

* $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$	* $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert + C$, if $x > a$
* $\int 1.dx = x + C$	* $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert + C$, if $x > a$
* $\int \frac{1}{\sqrt{x}} = 2\sqrt{x} + C$	* $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C$
* $\int \frac{1}{x}dx = \log_e x + C$	* $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a} + C$
* $\int e^x dx = e^x + C$	

$*\int a^x dx = \frac{a^x}{\log_e a} + C$	$*\int \frac{dx}{\sqrt{a^2+x^2}} = \log\mid x+\sqrt{x^2+a^2}\mid + C$
$*\int \sin x dx = -\cos x + C$	$*\int \frac{dx}{\sqrt{x^2-a^2}} = \log\mid x+\sqrt{x^2-a^2}\mid + C$
$*\int \cos x dx = \sin x + C$	$*\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left\|x+\sqrt{x^2+a^2}\right\| + C$
$*\int \sec^2 x\ dx = \tan x + C$	$*\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left\|x+\sqrt{x^2-a^2}\right\| + C$
$*\int \operatorname{cosec}^2 x\ dx = -\cot x + C$	$*\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$
$*\int \sec x.\tan x\ dx = \sec x + C$	$*\int \{f_1(x) \pm f_2(x) \pm ....f_n(x)\}dx$ $= \int f_1(x)dx \pm \int f_2(x)dx \pm .... \pm \int f_n(x)dx$
$*\int \operatorname{cosec} x.\cot x\ dx = -\operatorname{cosec} x + C$	$*\int \lambda f(x)dx = \lambda \int f(x)dx + C$
$*\int \tan x\, dx = -\log\left\|\cos x\right\| + C = \log\left\|\sec x\right\| + C$	$*\int f(x)g(x)dx =$ $f(x)\int g(x)\,dx - \int[f'(x)\ \int g(x)dx]dx$
$*\int \cot x\ dx = \log\mid \sin x\mid + C$	
$*\int \sec x\ dx = \log\mid \sec x + \tan x\mid + C$ $= \log\left\|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right\| + C$	
$\int \operatorname{cosec} x\ dx = \log\mid \operatorname{cosec} x - \cot x\mid + C$ $= -\log\mid \operatorname{cosec} x + \cot x\mid + C = \log$ $\left\|\tan\frac{x}{2}\right\| + C$	

MULTIPLE CHOICE QUESTIONS

1. The value of $\int_{-\pi}^{\pi}(sin^{83}x + x^{123})dx$ is

   a) 1 b) 0 c) $-\pi$ d) π

   Solution: $\int_{-\pi}^{\pi}(x^{123})dx + \int_{-\pi}^{\pi} sin^{83}x\ dx = 0 + 0 = 0$, both are odd functions and using properties $\int_{-a}^{a} f(x)dx = 0\ if\ f(x) is\ odd$ **Answer**: option (b)

2. If $\int_0^a \frac{dx}{1+4x^2} dx = \frac{\pi}{8}$, then a is

   a) $\frac{1}{2}$ b) $\frac{1}{3}$ c)2 d) 3

   Solution:

   $\frac{1}{4}\int_0^a \frac{dx}{\frac{1}{4}+x^2} = \frac{\pi}{8}$; $\frac{1}{4}\frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{x}{1/2}\right)\right]_0^a = \frac{\pi}{8}$

$\left[\tan^{-1} 2x\right]_0^a = \frac{\pi}{4}$;$\tan^{-1} 2a = \frac{\pi}{4}$

2a =1 ;a = $\frac{1}{2}$ **Answer:** option is (a)

3. The value of $\int e^x(cosx - sinx)dx$ is

a) $e^x cosx$ +C b) $e^x sinx$ + C c) $-e^x cosx$ + C d)$-e^x sinx$ + C

Solution : since$\int e^x(f(x) + f^1(x))dx = e^x f(x)$ +C

here take f(x) = cosx, $f^1(x)$ = -sinx **Answer :** (a) is correct

4. $\int \frac{dx}{sin^2xcos^2x}$ is equal to

$a)$ $tanx + cotx + c$ b) $(tanx + cotx)^2 + C$

c) $tanx - cotx$ + C d) $(tanx - cotx)^2 + C$

Solution : $\int \frac{dx}{sin^2xcos^2x} = \int \frac{sin^2x+cos^2x}{sin^2xcos^2x}$dx $=\int \frac{sin^2x}{sin^2xcos^2x}$dx $+\int \frac{cos^2x}{sin^2xcos^2x}$dx

$=\int sec^2 x\, dx + \int cosec^2 x\, dx = tanx - cotx + C$

Answer - c) is correct

5. The solution of $\int \frac{3ax}{b^2+c^2x^2}$dx

a) $\frac{3a}{2c^2}\ln|b^2 + c^2x^2|$ + C b) $\frac{3a}{2c^2}\ln|b^2 - c^2x^2|$ + C

c) $\frac{3a}{2c^2}\ln|c^2x^2 - b^2|$ + C d) $\frac{3a}{2c^2}\ln|b^2 + c^2x^2|$

Solution : let v =$b^2 + c^2x^2$,dv = $2c^2x$dx **Answer:** a) is correct

6. $\int_{-1}^{1} \frac{|x|}{x}\, dx$, x≠ 0 is

a) -1 b) 0 c) 1 d) 2

Solution: First split the integral as $\int_{-1}^{0} \frac{|x|}{x} dx + \int_{0}^{1} \frac{|x|}{x} dx$

$\int_{-1}^{0} \frac{-x}{x}$ dx + $\int_{0}^{1} \frac{x}{x}$ dx and integrate and find the value as 2;**Answer:** d) is correct

7. If $\int \frac{2^{\frac{1}{x}}}{x^2}$ dx = k $2^{1/x}$ +C , then k is

a) - $\frac{1}{log2}$ b) – log2 c) -1 d) 1

Solution: put $\frac{1}{x}$ = t , dt = $-\frac{1}{x^2}dx$, **Answer:** a) is correct

8. $\int \frac{1-2sinx}{cos^2x} dx$ is

a) tanx – 2 secx +C b) - tanx + 2 secx +C c) - tanx – 2 secx +C d) tanx + 2 secx +C

Solution : $\int \frac{1}{cos^2x} dx$ - 2 $\int \frac{sinx}{cos^2x} dx = \int sec^2x\, dx$ - 2 $\int tanx\, secx\, dx$

= tanx -2 secx + C **Answer :** a) is correct

9. If $\int \frac{3e^x-5e^{-x}}{4e^x+5e^{-x}}\, dx$ = ax +b log$|4e^x + 5e^{-x}|$ + C , then

a) a= $-\frac{1}{8}$, b = $\frac{7}{8}$ b) a= $\frac{1}{8}$, b = $\frac{7}{8}$ c) a= $-\frac{1}{8}$, b = $-\frac{7}{8}$ d) a= $\frac{1}{8}$, b = $-\frac{7}{8}$

Solution: Differentiate on both sides,

$\frac{3e^{x}-5e^{-x}}{4e^{x}+5e^{-x}}$ = a + b $\frac{4e^{x}-5e^{-x}}{4e^{x}+5e^{-x}}$ and compare we get the **Answer: c) is correct**

10. $\int_{a+c}^{b+c} f(x)dx$ *is equal to*

a) $\int_a^b f(x-c)dx$ b) $\int_a^b f(x+c)dx$ c) $\int_a^b f(x)dx$ d) $\int_{a-c}^{b-c} f(x)dx$

Solution: b) is the correct option, since by putting x =t + c we get,

$\int_a^b f(x-c)dx = \int_a^b f(x+c)dx$

ASSERTION - REASON BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason(R). Pick the correct option:

a) Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).

b) Both Assertion (A) and Reason(R) are true but Reason(R) is NOT the correct explanation of Assertion (A).

c)Assertion (A) is true but Reason(R) is false.

d)Assertion (A) is false but Reason(R) is true.

1. Assertion: The value of $\int e^{x}(tanx + sec^{2}x)dx$ is $e^{x}tanx + C$

   Reason: The integral value of $e^{x}(f(x) + f'(x))$ is $e^{x}f(x) + C$

   Answer: a), *apply* $f(x) = tanx$

2. Assertion: $\int_{-2}^{2} log\left(\frac{1+x}{1-x}\right)$dx = 0

   Reason: If f is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

   Answer: a) since f(x) is odd function.

3. Assertion: If the derivative of function x is $\frac{d}{dx}(x) = 1$, then $\int(1)dx = x + c$.

   Reason: If $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^{n}$, then the corresponding integral of the function is $\int x^{n}\ dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1.$

   Answer: a) since $\frac{d}{dx}(x) = 1$ *and* $\int dx = x + C$

4. Assertion: $\int e^{5logx}\ dx = \frac{x^{6}}{6}$ + C

   Reason: e^{logx} = x . Answer a), reason both true and correct explanation

5. Assertion: $\int[\sin(\log x) + \cos(logx)]dx = x\sin\ (\log x) + C$

   Reason: $\frac{d}{dx}$ [$x\sin(\log x)$] = sin (log x) + cos(log x). Answer: a)

   $\frac{d}{dx}\{xsin(\log x)\} = x\frac{d}{dx}\sin(logx) + \sin\ (logx)\frac{d}{dx}(x)$

   =xcos(logx)x$\frac{1}{x}$ + sin(logx) = cos(logx) + sin(logx)

6. Assertion: $\int_0^{\pi}\cos x\ dx = 2$

   Reason: The function f(x)=cos x is decreasing in [0,π]

   Answer b) $\int_0^{\pi}\cos x\ dx = \int_0^{\frac{\pi}{2}}\cos x\ dx + \int_{\frac{\pi}{2}}^{\pi}\cos x\ dx$ = 1 + 1 = 2

7. Assertion: $\int_0^{\pi/2}\cos 2x\ dx = 1$

   Reason: The function cos2x is decreasing in [0,$\frac{\pi}{2}$]

   Answer b) $\int_0^{\pi/2}\cos 2x\ dx = \int_0^{\frac{\pi}{4}}\cos 2x\ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos 2x\ dx = \frac{1}{2} + \frac{1}{2}$ =1.

8. Assertion: $\int_{-\frac{\pi}{2}}^{\pi/2} sin^7 x dx = 0$

Reason: $\int_{-a}^{a} f(x)dx = 0$ if f(x) is an odd function.

Answer a) since f(x) is an odd function

9. Assertion: $\int e^x(sinx - cosx\,)\,dx$ = - $e^x cosx + C$

Reason: $\int e^x(f(x) + f'(x))\,dx = e^x f(x) + C$

Answer a) take f(x) = - cosx , f '(x) =sinx

10. Assertion: $\int_0^{10} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}}$ dx = 10

Reason: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ Answer d) since $\int_0^{10} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{10-x}} = 5$

VERY SHORT ANSWER TYPE QUESTIONS

1. Integrate ,$\left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right)$ w.r.t.x

Answer $\int \left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right) dx = \int 2a\,x^{-\frac{1}{2}}\,dx$ -$\int bx^{-2}\,dx$ +$\int 3cx^{\frac{2}{3}}$ dx

=4a$\sqrt{x}$ + $\frac{b}{x}$ + $\frac{9cx^{\frac{5}{3}}}{5}$ + C.

2. Evaluate,$\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$

Answer ,

Let I = $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx = \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1}\,dx + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}\,dx$,{ 0dd + even}

$= 0 + 2\int_0^1 \frac{|x|+1}{(|x|+1)^2}\,dx = 2\int_0^1 \frac{x+1}{(x+1)^2}\,dx = 2\int_0^1 \frac{1}{x+1}\,dx$

$= [2log|x+1|]_0^1 = 2log2$

3. Verify the following using the concept of integration as an anti-derivative

$\int \frac{x^3}{x+1}\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - log|x+1| + C$

Answer $\frac{d}{dx}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - log|x+1| + C\right)$

$= 1 - x + x^2 - \frac{1}{x+1} = \frac{x^3}{x+1}$

Then, $\left(x - \frac{x^2}{2} + \frac{x^3}{3} - log|x+1| + C\right) = \int \frac{x^3}{x+1}\,dx$

4. Evaluate: $\int \frac{dx}{2sin^2x+5\,cos^2x}$,

Answer, Dividing Nr and Dr by cos^2x

I = $\int \frac{sec^2x}{2tan^2x+5}\,dx$

= $\int \frac{dt}{2t^2+5}$, by putting t= tanx, dt = $sec^2x\,dx$

$=\frac{1}{2}\int \frac{dt}{t^2+\left(\sqrt{\frac{5}{2}}\right)^2} = \frac{1}{2}\frac{\sqrt{2}}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{2}tanx}{\sqrt{5}}\right) + C = \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{2}tanx}{\sqrt{5}}\right) + C.$

5. Evaluate : $\int \sqrt{10-4x+4x^2}$ dx.

Answer: I = $\int \sqrt{10-4x+4x^2}$ dx = $\int \sqrt{(2x-1)^2+3^2}\,dx$

$=\frac{1}{2}\int \sqrt{(t)^2+3^2}\,dx$, by putting t= 2x-1, then dt=2dx

$=\frac{1}{2}t\frac{\sqrt{t^2+9}}{2} + \frac{9}{4}\,log\left|t+\sqrt{t^2+9}\right| + C$

$= \frac{1}{4}(2x-1)\sqrt{(2x-1)^2+9} + \frac{9}{4}log\left|(2x-1)+\sqrt{(2x-1)^2+9}\right| + C$

6. Evaluate : $\int_0^{\frac{\pi}{4}} \sqrt{1+sin2x}\, dx$

Answer : We have, I = $\int_0^{\frac{\pi}{4}} \sqrt{(sinx+cosx)^2}\ dx \quad = \int_0^{\frac{\pi}{4}} (sinx+cosx)\, dx$

$= (-cosx + sinx)_0^{\frac{\pi}{4}} = 1$

7. Evaluate : $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\, dx$

Answer, I = $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\, dx$ ………………(1)

$= \int_2^8 \frac{\sqrt{10-(10-x)}}{\sqrt{10-x}+\sqrt{10-(10-x)}}\, dx$

I $= \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}\, dx$ …………..(2)

Adding (1) and (2) We get 2I = $\int_2^8 1\ dx$ =x= 8-2 =6, Hence, I= 3.

8. Evaluate, $\int_0^{\frac{\pi}{2}} \frac{tan^7x}{cot^7x+tan^7x}\, dx$

Answer, we have, I = $\int_0^{\frac{\pi}{2}} \frac{tan^7x}{cot^7x+tan^7x}\, dx$ ……………(1)

I = $\int_0^{\frac{\pi}{2}} \frac{tan^7(\frac{\pi}{2}-x)}{cot^7(\frac{\pi}{2}-x)+tan^7(\frac{\pi}{2}-x)}\, dx$, since $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

I = $\int_0^{\frac{\pi}{2}} \frac{cot^7x}{tan^7x+cot^7x}\, dx$ …………..(2)

Adding (1) and (2)

We get 2I = $\int_0^{\frac{\pi}{2}} \frac{tan^7x+cot^7x}{cot^7x+tan^7x}\, dx \quad = \int_0^{\frac{\pi}{2}} 1\, dx \quad = [x]_0^{\pi/2} = \frac{\pi}{2}$ Hence, I = $\frac{\pi}{4}$.

9. Evaluate $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}, \beta > \alpha$

Answer: Put x-$\alpha = t^2$. Then β – x = $\beta - (t^2 + \alpha) = \beta - t^2 - \alpha = -t^2 - \alpha + \beta$

And dx=2tdt. Now

I = $\int \frac{2tdt}{\sqrt{t^2\,(\beta-\alpha-t^2}} \ = \int \frac{2dt}{\sqrt{(\beta-\alpha-t^2)}} = 2\int \frac{dt}{\sqrt{k^2-t^2}}$, where $k^2 = \beta - \alpha$

$= 2sin^{-1}\frac{t}{k} + C = 2sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C$.

10. Evaluate $\int tan^8 x\, sec^4 x\, dx$

Answer : I = $\int tan^8 x\, sec^4 x\, dx \quad = \int tan^8$ x$(sec^2 x)\, sec^2 xdx$

$= \int tan^8 x\,(tan^2 x + 1)\, sec^2 xdx$

$= \int tan^{10} x\, sec^2\ xdx + \int tan^8 xsec^2 xdx$

$= \frac{tan^{11}x}{11} + \frac{tan^9 x}{9}$ + C

SHORT ANSWER TYPE QUESTIONS

1. Evaluate : $\int \frac{x^3}{x^4+3x^2+2}$

Answer: Put x^2= t. Then 2x dx =dt

Now I = $\int \frac{x^3\,dx}{x^4+3x^2+2} = \frac{1}{2}\int \frac{tdt}{t^2+3t+2}$

Consider $\frac{t}{t^2+3t+2} = \frac{A}{t+1} + \frac{B}{t+2}$

Comparing coefficient, we get A= -1, B=2.

Then $1 = \frac{1}{2}\left[\, 2\int \frac{dt}{t+2} - \int \frac{dt}{t+1} \,\right]$

$= \frac{1}{2}$ [2log | t+2|-log|t+1|]

$= \log \left|\frac{x^2+2}{\sqrt{x^2+1}}\right| + C$

2. Evaluate : $\int x^2\, tan^{-1}\, x\, dx$

Answer : $\int x^2\, tan^{-1}\, x\, dx$

$= tan^{-1}$ x $\int x^2$ dx - $\int \frac{1}{1+x^2} \frac{x^3}{3} dx$

$= \frac{x^3}{3}\ tan^{-1}$ - $\frac{1}{3} \int \left(x - \frac{x}{1+x^2}\right)$dx

$= \frac{x^3}{3}\ tan^{-1}$ x - $\frac{x^2}{6} + \frac{1}{6}$ log $|1+x^2|$ +C

3. Evaluate: $\int \frac{x^3+x}{x^4-9}\, dx$

Answer : We have Type equation here.

$I = \int \frac{x^3+x}{x^4-9}\, dx = \int \frac{x^3}{x^4-9}$ dx + $\int \frac{xdx}{x^4-9} = I_1 + I_2$

Now $I = \int \frac{x^3}{x^4-9}$

Put t = $x^4 - 9$ so that $4x^3$ dx = dt. Therefore

$I_1 = \frac{1}{4} \int \frac{dt}{t} = \frac{1}{4}$log |t|+$c_1$

Again, $I_2 = \int \frac{xdx}{x^4-9}$

Put x^2 = u so that 2x dx=du . then

$I_2 = \frac{1}{2} \int \frac{du}{u^2-(3)^2} = \frac{1}{2\times 6} log \left|\frac{u-3}{u+3}\right| + c_2$

$= \frac{1}{12} log \left|\frac{x^2-3}{x^2+3}\right| + c_2$

Thus = I= $I_1 + I_2$

$\frac{1}{4} log|x^4 - 9| + \frac{1}{12} log\left|\frac{x^2-3}{x^2+3}\right|$ + C .

4. Evaluate : $\int_0^1 x\,(tan^{-1}x)^2\, dx$

Answer $\int_0^1 x\,(tan^{-1}x)^2\, dx$

Integrating by parts, we have

I $= \frac{x^2}{2}\, [\, (tan^{-1}\, x)^2\,]_0^1 - \frac{1}{2} \int_0^1 x^2 . \, 2\, \frac{tan^{-1}\, x}{1+x^2}\, dx$

$= \frac{\pi^2}{32} - \int_0^1 \frac{x^2}{1+x^2}\, tan^{-1}xdx$

$= \frac{\pi^2}{32}$ - I_1, where $I_1 = \int_0^1 \frac{x^2}{1+x^2}\ tan^{-1}xdx$

Now $I_1 = \int_0^1 \frac{x^2+1-1}{1+x^2}\ tan^{-1}xdx$

$= \int_0^1 tan^{-1}xdx - \int_0^1 \frac{1}{1+x^2}\ tan^{-1}xdx$

$= I_2$ - $\frac{1}{2}(tan^{-1}x)^2)_0^1 = I_2 - \frac{\pi^2}{32}$

Here $I_2 = \int_0^1 tan^{-1}xdx = (x\, tan^{-1}\, x)_0^1 - \int_0^1 \frac{x}{1+x^2} dx$

$= \frac{\pi}{4} - \frac{1}{2}\ (log|1+x^2|)_0^1 = \frac{\pi}{4}$ - $\frac{1}{2}$ log2.

Thus $I_1 = \frac{\pi}{4} - \frac{1}{2}$ log2 - $\frac{\pi^2}{32}$

Therefore I = $\frac{\pi^2}{32}$ - $\frac{\pi}{4}+\frac{1}{2}$log2+ $\frac{\pi^2}{32}=\frac{\pi^2}{16}$ - $\frac{\pi}{4}+\frac{1}{2}$log2

$\frac{\pi^2-4\pi}{16}$ +log$\sqrt{2}$.

5. Evaluate $\int_0^1 f(x)dx$, where $f(x)$ = |x +1|+|x|+|x-1|

We can redefined f as $f(x) = \begin{Bmatrix} 2-x, if -1 < x \le 0 \\ x+2, \ if\ 0 < x \le 1 \\ 3x, if\ 1 < x \le 2) \end{Bmatrix}$

Therefore $\int_{-1}^2 f(x)dx = \int_{-1}^0 (2-x)dx + \int_0^1 (2+x)dx + \int_1^2 3xdx$ (by p_2)

= $(2x - \frac{x^2}{2})_{-1}^0 + (\frac{x^2}{2}+2x)_0^1 + (\frac{3x^2}{2})_1^2$

= 0 –(2– $\frac{1}{2}$) + ($\frac{1}{2}$ + 2) + 3 $(\frac{4}{2}-\frac{1}{2}) = \frac{5}{2}+\frac{5}{2}+\frac{9}{2}=\frac{19}{2}$.

6 . $Evaluate, \int_0^{\frac{\pi}{4}} \log(1+tanx)\,dx$

Sol : $I = \int_0^{\frac{\pi}{4}} \log\,(1+tanx)$dx

$I = \int_0^{\frac{\pi}{4}} \log\,(1+\tan\left(\frac{\pi}{4}-x\right))dx$

$I = \int_0^{\frac{\pi}{4}} \log\,(1+\frac{tan\frac{\pi}{4}-tanx}{1+tan\frac{\pi}{4}tanx})$dx

I = $\int_0^{\frac{\pi}{4}} \log\,(\frac{2}{1+tanx})$dx $=\int_0^{\frac{\pi}{4}} \log(2)\,dx - I$

2I $=\int_0^{\frac{\pi}{4}} \log(2)\,dx$=log2$(\frac{\pi}{4}-0) = \frac{\pi}{4}log2$; $I = \frac{\pi}{8}log2$

7. Evaluate : $\int \frac{2}{(1-x)(1+x^2)}dx$

Sol $\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x}+\frac{Bx+C}{1+x^2}$

$\Rightarrow A = 1, B = 1, C = 1$

$hence\ I = \int \frac{2}{(1-x)(1+x^2)}dx = \int \left(\frac{1}{(1-x)}+\frac{x+1}{1+x^2}\right)dx$

$= \int \frac{1}{1-x}dx + \int \frac{x+1}{1+x^2}dx$

$= \int \frac{1}{1-x}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$

$= -log|1-x| + \frac{1}{2}\log(x^2+1) + tan^{-1}x + C$

8. Evaluate: $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}$dx.

Sol: $I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx \ldots\ldots\ldots\ldots\ldots(1)$

$I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx \ldots\ldots\ldots.(2)$

$$(by\ using\ property \int_0^a f(x)dx = \int_0^a f(a-x)dx)$$

Adding (1) and (2)

$$I + I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx + \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx$$

$$2I = \int_2^8 \frac{\sqrt{10-x}+\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx = \int_2^8 1\,dx = (8-2) = 6$$

Hence, I = 3

9. Evaluate: $\int_0^{\pi} \frac{x\ \tan x}{\sec x+\tan x}$dx.

SOL : $I = \int_0^{\Pi} \frac{x \tan x}{\sec x+\tan x}dx$ ……………………….(1) (by removal of x property)

$I = \int_0^{\pi} \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}dx$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi-x)\tan x}{\sec x+\tan x}dx$ ------------(2)

Adding (1) & (2)

$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x+\tan x}dx = \quad \pi\int_0^{\pi} \frac{\tan x}{\sec x+\tan x}dx = \pi\int_0^{\pi} \frac{\tan x}{\sec x+\tan x} X \frac{\sec x-\tan x}{\sec x-\tan x}dx$

$= \pi\int_0^{\pi}(\tan x\sec x - \tan^2 x)dx = \pi\int_0^{\pi}(\tan x\sec x - \sec^2 x+1)dx$

$I = \frac{1}{2}\{\pi(\sec x - \tan x + x)]_0^{\pi} = \frac{1}{2}\pi(\pi-2)$

10.Evaluate : $\int \frac{cosx}{(1-sinx)(2-cosx)}dx$

Sol $\int \frac{dt}{(1-t)(2-t)}$

$\Rightarrow A = 1, B = -1$

$Hence\ I = \int \frac{1}{(1-t)(2-t)}dt = \int\left(\frac{1}{(1-t)} - \frac{1}{2-t}\right)dx$

$= \int \frac{1}{1-t}dx - \int \frac{dt}{2-t} = -log|1-t| + log|2-t|$ + C= $log\left|\frac{2-t}{1-t}\right|$ + C

$= log\left|\frac{2-sinx}{1-sinx}\right|$ + C

LONG ANSWER TYPE QUESTIONS

1. Evaluate: $\int \frac{x^2+x+1}{(x^2+1)(x+2)}$dx

Sol $\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$

$\Rightarrow x^2 + x + 1 = A(x^2 + 1) + (B x + C) (x + 2)$

$\Rightarrow x^2 + x + 1 = x^2 (A + B) + x(2B + C) + (A + 2C)$

On comparing the coefficients of x^2, x and constant terms both sides, we get

A + B = 1 ……. (ii)

2B + C = 1 …….. (iii)

and A + 2C = 1 ……. (iv)

On substituting the value of B from q. (ii) in Eq. (iii), we get

2(1 – A) + C = 1⇒ 2 – 2A + C = 1⇒ 2A – C = 1 ……. (v)

From above equations we get

$$\Rightarrow A = \frac{3}{5}, B = \frac{2}{5} \text{ and } C = \frac{1}{5}$$

$$\Rightarrow \frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \frac{3}{5}\int \frac{dx}{x+2} + \frac{1}{5}\int \frac{2x+1}{x^2+1}dx$$
$$\Rightarrow \frac{3}{5}\int \frac{dx}{x+2} + \frac{1}{5}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$$
$$\Rightarrow \frac{3}{5}\log(x+2) + \frac{1}{5}\log(x^2+1) + \frac{1}{5}\tan^{-1}(x) + c$$

2. Evaluate : $\int_{-1}^{2}|x^3 - x|\,dx$

Sol $\int_{-1}^{2}|x^3 - x|\,dx = \int_{-1}^{0}|x^3 - x|\,dx + \int_{0}^{1}|x^3 - x|\,dx + \int_{1}^{2}|x^3 - x|\,dx$

$= \int_{-1}^{0}(x^3 - x)\,dx - \int_{0}^{1}(x^3 - x)\,dx + \int_{1}^{2}(x^3 - x)\,dx$

$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{-1}^{2} - \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{0}^{1} + \left(\frac{x^4}{4} - \frac{x^2}{2}\right)_{1}^{2} = \frac{3}{2} + \frac{-3}{4} + 2 = \frac{11}{4}$

3. Evaluate : $\int \frac{(3sin\theta-2)cos\theta}{5-cos^2\theta-4sin\theta}\,d\theta$

Sol $\int \frac{(3sin\theta-2)cos\theta}{5-cos^2\theta-4sin\theta}\,d\theta = \int \frac{(3sin\theta-2)cos\theta}{sin^2\theta-4sin\theta+4}\,d\theta$

Let $sin\theta = y$

$cos\theta d\theta = dy$

$\int \frac{(3y-2)}{y^2-4y+4}\,dy = \int \frac{(3y-2)}{(y-2)^2}\,dy$

$let\ \frac{(3y-2)}{(y-2)^2} = \frac{A}{y-2} + \frac{B}{(y-2)^2}$

On comparing or equating corresponding coefficients, we have A=3 , B =4

$\int \frac{(3y-2)}{(y-2)^2}\,dy = \int \frac{3}{y-2} + \frac{4}{(y-2)^2}\,dy$=3log|y-2|$-\frac{4}{y-2}$ +c

=3log(2- $sin\theta) + \frac{4}{2-sin\theta}$ + c

4. Evaluate: $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx$

`

Sol: we have $x + 2 = A\frac{d}{dx}(x^2 + 2x + 3) + B$

$x + 2 = A(2x + 2) + B$

On comparing or equating corresponding coefficients, we have

$A = \frac{1}{2}$, $B = \frac{1}{2}$

$I = \frac{1}{2}\int \frac{(2x+2)dx}{\sqrt{x^2+2x+3}} + \frac{1}{2}\int \frac{2\,dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}}dx$

I = $\sqrt{x^2 + 2x + 3}$ +log$\left|x + 1 + \sqrt{x^2 + 2x + 3}\right|$+C

5. Prove $\int_0^{\pi} \log \sin x\,dx = -\pi\log 2$ using properties of definite integration.

Sol: Since log [sin $(\pi - x)$] = log sin x

∴ $I = 2\int_0^{\pi/2} \log sin\,x dx$

⇒ $I = 2\int_0^{\pi/2} \log\ \sin\left(\frac{\pi}{2} - x\right)dx$

⇒ $I = 2\int_0^{\pi/2} \log\ \cos\ x dx$

Adding (3) and (4)

$$2I = 2\int_0^{\pi/2} \log\sin x\cos x dx$$

$$\Rightarrow I = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log\sin 2x dx - \log 2\int_0^{\pi/2} 1\cdot dx$$

$$\Rightarrow I = I_1 - \log 2[x]_0^{\pi/2} = I_1 - \frac{\pi}{2}\log 2$$

Where $I_1 = \int_0^{\pi/2} \log\sin 2x dx$

Let $2x = t \Rightarrow 2dx = dt$

$$= \frac{1}{2}\int_0^{\pi} \log\sin t dt = \frac{1}{2}\int_0^{\pi} \log sinx\, dx$$

$$= \frac{1}{2}\times 2\int_0^{\pi/2} \log\sin x dx$$

$$\Rightarrow I_1 = \frac{1}{2}I$$

$$\text{From (5)} \Rightarrow I = \frac{1}{2}I - \frac{\pi}{2}\log 2$$

$$\Rightarrow \frac{1}{2}I = -\frac{\pi}{2}\log 2 \quad ;$$

$$I = -\pi\log 2$$

CASE STUDY BASED QUESTIONS

1. The given integral $\int f(x)dx$ can be transformed in to another form by changing the independent variable x to t by substituting x = g(t)

Consider I = $\int f(x)dx$

Put x = g(t), $\frac{dx}{dt} = g'(t)$, we write dx = $g'(t)dt$

Thus I = $\int f(g(t))g'(t)$ dt.

This change of variable is very important tools available to us in the name of integration by substitution.

Based on the above information answer the following questions

Evaluate, $\int 2xsin(x^2 + 2)dx$

i) To evaluate the above question which quantity we assume as another new variable.

ii) Solve the given question using fundamental integrals.

iii) Consider y is the solution then find y|.

Solution $\int 2xsin(x^2 + 2)dx$

i) Let $x^2 + 2$ = t

ii) Differentiate with respect to t

we get, 2xdx = dt,

Now, we will evaluate, $\int sint$ dt = - cost +C = - cos($x^2 + 2$) + C

Hence, y=- cos($x^2 + 2$) + C

iii) y|= $2xsin(x^2 + 2)$

2. Let f be a continuous function and differentiable function defined in a closed interval [a,b] and F

be an anti-derivative of f then, $\int_a^b f(x)dx = F(x)$ + C = F(b) – F(a) called definite integral.

It is very useful because it gives us a method of calculating the definite integral more easily.

There is no need to keep integration constant as it gets cancel by substituting the upper limit and lower limit

Based on the above information answer the following

Evaluate, $\int_2^3 \frac{xdx}{1+x^2} dx$

i) which technique you will use to evaluate the integral.

ii) To evaluate the above question which quantity we assume as another new variable.

iii) what is the value of definite integral.

Solution $\int_2^3 \frac{xdx}{1+x^2} dx$

i) using substitution method,

ii) let $1 + x^2$ = t

iii) differentiate with respect to t, we get 2xdx = dt , $\frac{1}{2}\int_5^{10} \frac{dt}{t} = [logt]_5^{10}$

= log10-log5 = log2.

3. The rational functions which we shall consider for integration purposes will those denominators can be factorised in to linear and quadratic factors. Assume that we want to evaluate$\int \frac{p(x)}{q(x)}$ dx where, $\frac{p(x)}{q(x)}$ is a proper rational function. It is always possible to write the integrand as a sum of simpler rational function by a method called partial fraction decomposition. After this the integration can be carried out easily using the already known methods.

If the rational function is of the form $\frac{px+q}{(x-a)(x-b)}$ then we write the partial fraction is of the form $\frac{A}{(x-a)} + \frac{B}{(x-b)}$, where A and B are to be determined

Based on the above information answer the following;

Evaluate,$\int \frac{dx}{(x+1)(x+2)}$

i) What are the values of A and B when we use partial fractions

ii) After finding the values of A and B how will you evaluate integral and write the final answer.

Solution.

(i) since it is a proper fraction, we can write partial fraction

$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$, After taking L.C.M and simplifying we get,

1 = A(x+2) + (x+1), on comparing we get A + B = 0, 2A+B =1

On solving these equations, we get A = 1 and B = -1

(ii) A = 1 and B = -1

Therefore $\int \frac{dx}{(x+1)(x+2)} = \int \frac{1}{x+1} dx$ - $\int \frac{1}{x+2} dx$

= log(x+1) – log(x+2)

= log $\left(\frac{x+1}{x+2}\right) + C$

CHAPTER – 8 : APPLICATION OF INTEGRALS

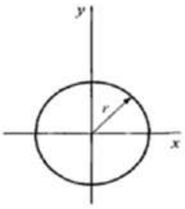
GIST OF THE LESSON

- Integrals are useful in finding areas under simple curves such as lines, circles, parabolas and ellipses.
- Standard Form of Some Important Curves:

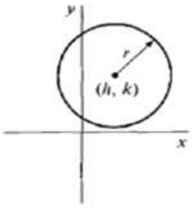
Straight line: $ax + by = c$

Circle: $x^2 + y^2 = r^2$ (A circle with centre at origin and radius r)

$(x-h)^2 + (y-k)^2 = r^2$ (A circle with centre at (h,k) and radius r)

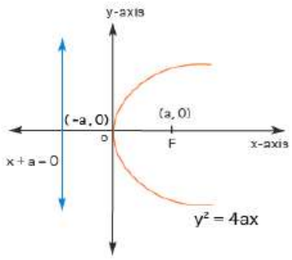


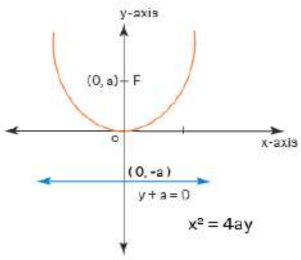
$x^2 + y^2 = r^2$



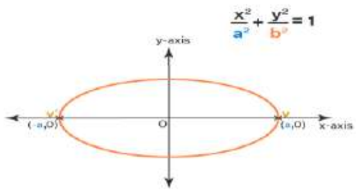
$(x-h)^2 + (y-k)^2 = r^2$

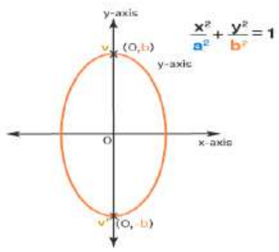
Parabola:





Ellipse:





- Area of the region bounded by the curve y = f(x), x-axis and the lines x = a and x = b (where b>a) is given by

  Area = $\int_a^b y\ dx = \int_a^b f(x)\ dx$

- Area of the region bounded by the curve x = g(y), y-axis and the lines y = c and y = d (where d>c) is given by

  Area = $\int_c^d x\ dy = \int_c^d g(y)\ dy$

MULTIPLE CHOICE TYPE QUESTIONS

1. The area enclosed by the circle $x^2 + y^2 = 2$ is equal to

   a) 4π sq units b) $2\sqrt{2}$ sq units c) $4\pi^2$ sq units d) 2π sq units

   Ans. d

2. The area enclosed by the curve $y = \sqrt{16 - x^2}$ and x-axis is equal to

a) 8π sq units b) 20π sq units c) 16π sq units d) 256π sq units

Ans. a

3. The area of the region bounded by y=x+1 and the lines x=2 and x=3 is

a) $\frac{7}{2}$ sq units b) $\frac{9}{2}$ sq units c) $\frac{11}{2}$ sq units d) $\frac{13}{2}$ sq units

Ans. a

4. The area bounded by the parabola $y^2 = 36x$, line x = 1 and the x-axis is _______ sq units.

a) 2 b) 4 c) 6 d) 8 Ans. B

5. The area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to

a) $\pi^2 ab$ b) πab c) $\pi a^2 b$ d) πab^2

Ans. b

6. The area of the region bounded by the curve $y = x^2$ and the line y = 16 is

a) $\frac{32}{3}$ sq units b) $\frac{256}{3}$ sq units c) $\frac{64}{3}$ sq units d) $\frac{128}{3}$ sq units

Ans. b

7. The area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is

a) 5π sq units b) 20π sq units c) 25π sq units d) 16π sq units

Ans. b

8. The area of the region bounded by the curve y = x^3, x-axis and the lines x=1 and x=4 is

a) $\frac{255}{4}$ sq units b) $\frac{225}{2}$ sq units c) $\frac{125}{3}$ sq units d) $\frac{124}{3}$ sq units Ans. a

9. The area of the region bounded by the curve x = 2y+3, y-axis and the lines y=1 and y= -1 is

a) 4 sq units b) 3/2 sq units c) 6 sq units d) 8 sq units Ans. c

10. The area of the region bounded by the parabola $y^2 = x$ and the straight line 2y=x is

a) $\frac{4}{3}$ sq units b) 1 sq unit c) $\frac{2}{3}$ sq units d) $\frac{1}{3}$ sq units Ans. a

ASSERTION - REASON BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason(R). Pick the correct option:

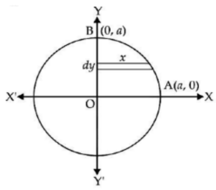
A) Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).

B) Both Assertion (A) and Reason(R) are true but Reason(R) is NOT the correct explanation of Assertion (A).

C) Assertion (A) is true but Reason(R) is false.

D) Assertion (A) is false but Reason(R) is true.

1. Assertion (A): The area enclosed by the circle $x^2 + y^2 = 16$ is 16π sq units.



Reason (R): The area enclosed by the circle $x^2 + y^2 = a^2$ is given by $\int_0^a \sqrt{a^2 - x^2}\, dx$

Solution: A) both assertion and reasoning are correct and reason is the correct explanation

2. Assertion: $\int \sin x\, dx = -\cos x + C$

Reason: sinx is an odd function and the integral of an odd function is -f(x) + C.

Solution: Assertion is true but reason is false. The integral of sin x is -cos x + C

Solution: (C) Assertion is true but reason is false

3. Assertion: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^5 x\, dx = 0$

Reason : If f(x) is odd function $\int_{-a}^{a} f(x) dx = 0$

Solution: A) both assertion and reasoning are correct and reason is the correct explanation

4. Assertion: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos\ x\ dx = 0$

Reason : If f(x) is odd function $\int_{-a}^{a} f(x)dx = 0$

Solution: D) assertion is False and reasoning is correct

5. Assertion: $\int x^2 - 3x + 2\ dx = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c$

Reason: The integral of a polynomial function ax^n is $\frac{a}{n+1}x^{n+1} + c$, where 'C' is the constant of integration.

Solution: A) both assertion and reasoning are correct and reason is the correct explanation

6. Assertion: $\int \frac{2x}{x^2+1}\ dx = \log|x^2 + 1| + c$

Reason: $\int \frac{f'(x)}{f(x)}\ dx = log|f(x)| + c$

Solution: A) both assertion and reasoning are correct and reason is the correct explanation

7. Assertion: $\int \sin 2x\ dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$

Reason: $1 - \cos 2x = 2\sin^2 x$

Solution: A) both assertion and reasoning are correct and reason is the correct explanation

8 Assertion: $\int e^x(cos\ x + sin\ x\ )\ dx = e^x cos\ x + c$

Reason: $\int e^x(f(x) + f'(x)\ )\ dx = e^x f(x) + c$

Solution: D) assertion is False and reasoning is correct

9. Assertion: $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sec^2 x\ dx = -2$

Reason: If f(x) ≥ 0 on [a ,b] then $\int_a^b f(x)\ dx \geq\ 0$

Solution: D) assertion is False and reasoning is correct

$\because$ $sec\ x$ is not defined at x = $\frac{\pi}{2}$ in $[\frac{\pi}{4}, \frac{3\pi}{4}]$

10. Assertion: $\int \log(\log x) + \frac{1}{\log x} dx = x\log(\log x) + c$

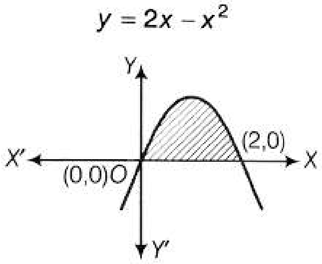
Reason: $\int e^x(f(x) + f'(x)\ )\ dx = e^x f(x) + c$

Solution: A) both assertion and reasoning are correct and reason is the correct explanation

VERY SHORT ANSWER TYPE QUESTIONS

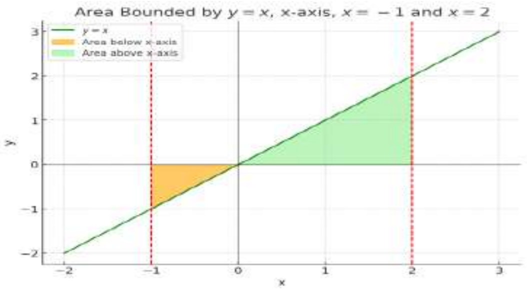
1. Find the area of the region bounded by the curve y = 2x – x^2 and the x-axis

Solution: $\int_0^2 2x - x^2\ dx = \frac{4}{3}$ sq. unit



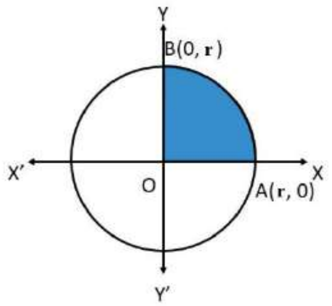
2. Find the area bounded by y = x , the x- axis and the ordinate $x = -1, x = 2$

Solution: A $= \int_0^2 x\,dx + \left|\int_{-1}^0 x\,dx\right| = \frac{5}{2}$.



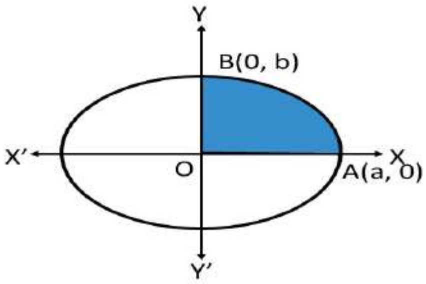
3. Find the area bounded by the circle $x^2 + y^2 = r^2$

   Solution: A= 4x $\int_0^r \sqrt{r^2 - x^2}$ dx = πr^2



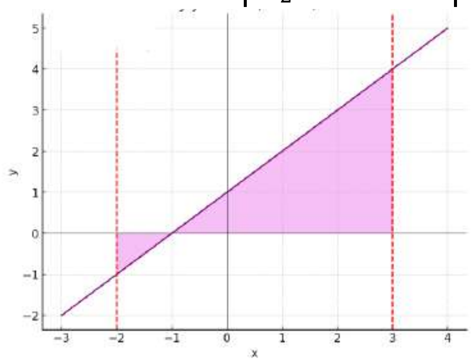
4. Find area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

   Solution: $A = 4 \text{ X } \frac{b}{a} \int_0^a \sqrt{a^2 - x^2}$ dx = πab



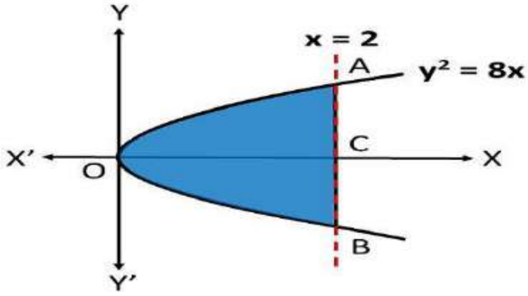
5. Using integration find the area of the region bounded by the line y – 1 = x, x-axis and the ordinates x = -2 and x = 3

   Solution: A = $\left|\int_{-2}^{-1} (x+1)\,dx\right| + \int_{-1}^{3} x + 1\,dx = \frac{17}{2}$



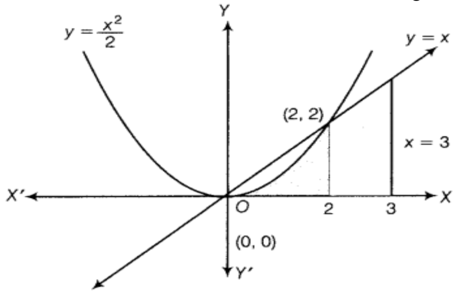
6. Using integration find the area of the region bounded between the line x = 2, and the parabola $y^2 = 8x$

   Solution: A = $2\int_0^2 \sqrt{8x}\,dx = \frac{32}{3}$ sq.units



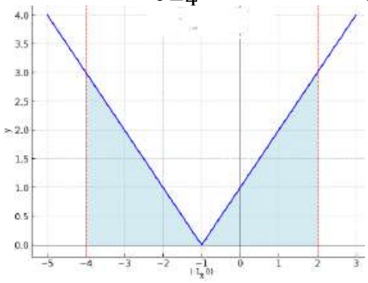
7. Using integration ,find the area of the region bounded by the curve y $=x^2$ and y = x

Solution: A $=\int_0^1 x\,dx \;-\; \int_0^1 x^2\,dx = \frac{1}{6}$ sq. units



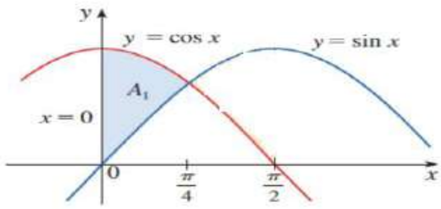
8. Draw the graph of y = $|x + 1|$and find the area between x- axis $x = -4$ and $x = 2$

Solution: A = $-\int_{-4}^{-1} x + 1\,dx + \int_{-1}^{2} x + 1$= 9 sq. units



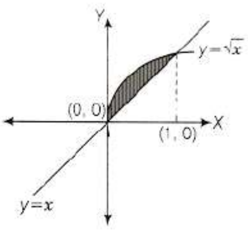
9. Draw the graph of y = sinx , y = cosx between x = 0 and x = $\frac{\pi}{4}$ ans solve the bounded by the region

Solution: A = $\int_0^{\frac{\pi}{4}} cosx - sinx$ dx = $\sqrt{2}$ - 1sq.units



10. Find the area of the region bounded by y = $\sqrt{x}$ and y = x

Solution: A = $\int_0^1 \sqrt{x}$ dx - $\int_0^1 x$ dx = $\frac{2}{3}$ -$\frac{1}{2}$ $=\frac{1}{6}$



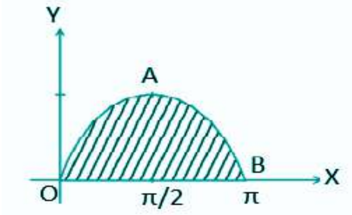
SHORT ANSWER QUESTIONS

1. Find the area of the curve y = sin x between 0 and π.

Solution: y = sin x

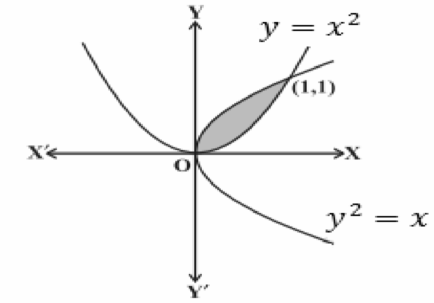
Area of OAB=$\int_0^{\pi} y dx$=$\int_0^{\pi} sinx\, dx$=$[-cosx]_0^{\pi}$

= – [cos π – cos 0] = -(-1 -1) = 2 sq units



2. Find the area of the region bounded by the two parabolas $y = x^2$ and $y^2 = x$.

Solution Given two parabolas are $y = x^2$ and $y^2 = x$. The point of intersection of these two parabolas is O (0, 0) and A (1, 1) as shown in the below figure.



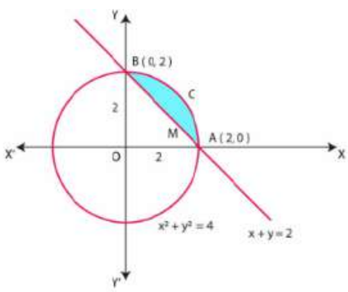
$y^2 = x$, $y = \sqrt{x} = f(x)$ $y = x^2 = g(x)$, where, $f(x) \geq g(x)$ in [0, 1].

Area of the shaded region = $\int_0^1 y dx$=$\int_0^1 \left(\sqrt{x} - x^2\right)dx$=$\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$

= (⅔) – (⅓)= ⅓ sq. units

3. Find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line x + y = 2.

Solution: Equation of circle is $x^2 + y^2 = 4 \Rightarrow y = \sqrt{}\sqrt{(2^2 - x^2}$



Equation of a lines is x + y = 2

Area of OACB, bounded by the circle and the coordinate axes is =$\int_0^2 \sqrt{2^2 - x^2}\, dx$

$= \left[\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}\frac{x}{2}\right]_0^2$

$= \frac{2}{2}\sqrt{2^2 - 2^2} + \frac{2^2}{2}\sin^{-1}\frac{2}{2} - \frac{0}{2}\sqrt{2^2 - 0^2} + \frac{2^2}{2}\sin^{-1}\frac{0}{2}$

$= \frac{1}{2}\sin^{-1} 1 = \frac{1}{2} \times \frac{\pi}{2} = \pi$

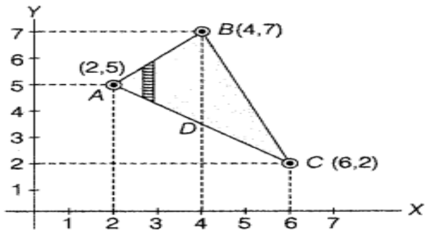
Area of triangle OAB, bounded by the straight line and the coordinate axes is

=$\int_0^2 y dx$=$\int_0^2 (2 - x)dx$=$\left[2x - \frac{x^2}{2}\right]_0^2$=$2\times 2 - \frac{2^2}{2} - [2 \times 0 - 0]$ = 4 – 2 – 0 + 0 = 2 sq. units

The required area = Area of OACB – Area of triangle OAB = (π – 2) sq. units

4.Using integration, find the area of Δ ABC, the coordinates of whose vertices are A (2, 5), B (4, 7) and C (6, 2).

Solution



Equation of the line AB is given by $\frac{y-y_1}{x_2-x_1} = \frac{x-x_1}{x_2-x_1} = \frac{y-5}{7-5} = \frac{x-2}{4-2}$

⇨ $y = x + 3$

Equation of the line BC is given by $\frac{y-7}{2-7} = \frac{x-4}{6-4}$ => $y = \frac{-5x}{2} + 17$

Equation of the line AC is given by $\frac{y-5}{2-5} = \frac{x-2}{6-2}$ => $y = \frac{-3x}{2} + \frac{13}{2}$

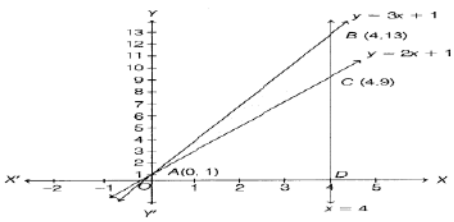
Required area = (Area under line segment AB) + (Area under line segment BC) – (Area under line segment AC)

$= \int_2^4 (x+3)dx + \int_4^6 \left(-\frac{5x}{2} + 17\right) dx - \int_2^6 \left(-\frac{3x}{4} + \frac{13}{2}\right) dx$

$= \left[\frac{x^2}{2} + 3x\right]_0^4 + \left[\frac{-5x^2}{4} + 17x\right]_4^6 dx - \left[\frac{-3x^2}{8} + \frac{13x}{2}\right]_2^4$ =12+9-14=7units

5. Using integration, find the area of the triangular region whose sides have the equations y = 2x + 1, y = 3x + 1 and x = 4

Solution



By solving these equations we get the vertices of triangle as A(0, 1), B(4, 13) and C(4, 9).

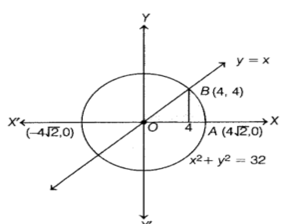
∴ Required area = Area (OABDO) – area (OACDO)

$= \int_0^4 (3x+1)dx - \int_0^4 (2x+1)dx = \left[\frac{3x^2}{2} + x\right]_0^4 - [x^2 + x]_0^4$

$= \frac{3\times4^2}{2} + 4 - 0 - (4^2 + 4 - 0)$

= 24+4-20=8sq units

6. Find the area of the region in the first quadrant enclosed by the x-axis, the line y = x and the circle $x^2 + y^2 = 32$



the required area = Area of shaded region OABO

$$=\int_0^4 x dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2} dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \frac{1}{2}\left[x\sqrt{\left(4\sqrt{2}\right)^2 - x^2} + \left(4\sqrt{2}\right)^2 \sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

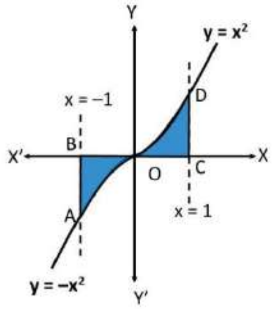
$$=\left[\frac{4^2}{2} - \frac{0^2}{2}\right]+\frac{1}{2}\left[4\sqrt{2}\sqrt{\left(4\sqrt{2}\right)^2 - \left(4\sqrt{2}\right)^2} + \left(4\sqrt{2}\right)^2 \sin^{-1}\frac{4\sqrt{2}}{4\sqrt{2}} - (4\sqrt{\left(4\sqrt{2}\right)^2 - 4^2} + \left(4\sqrt{2}\right)^2 \sin^{-1}\frac{4}{4\sqrt{2}})\right]$$

$$=\frac{1}{2}(16-0)+\frac{1}{2}\left[0 + 32\sin^{-1}(1) - 16 - 32\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]= 8+\frac{1}{2}(32\frac{\pi}{2} - 16 - 32\frac{\pi}{4})$$

$= 8 + 8\pi - 8 - 4\pi = 4\pi$ sq units

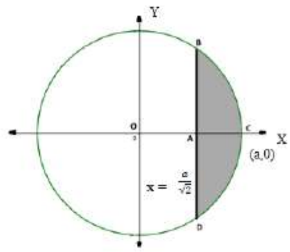
7.The area bounded by the curve y = $x|x|$ and the ordinates x = -1 and x = 1.

Solution: A = $2\int_0^1 x^2\, dx = \frac{2}{3}$



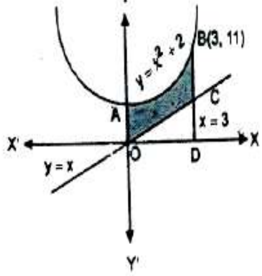
8. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line x = $\frac{a}{\sqrt{2}}$

Solution: A =2 $\int_{\frac{a}{\sqrt{2}}}^{a} \sqrt{a^2 - x^2}$ dx = $\frac{a^2}{2}$ ($\frac{\pi}{2}$ - 1)



9. Area of the region bounded by the curve y =x^2 + 2 , y = x , y = 0 and x = 3

Solution: A = $\int_0^3 (x^2 + 2)dx - \int_0^3 x\, dx = \frac{21}{2}$ sq units

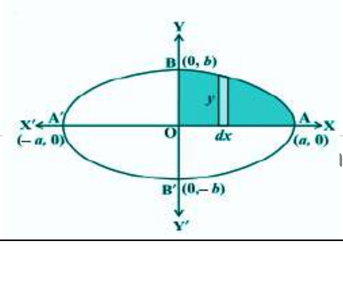


LONG ANSWER TYPE QUESTIONS

1) Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$=>\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$ $=> y = \frac{b}{a}\sqrt{(a^2 - x^2)}$



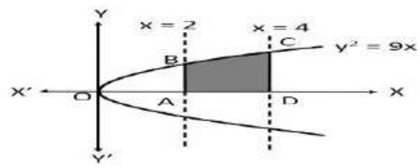
Area of ellipse = 4 × Area of AOB = $4\int_0^a \frac{b}{a}\sqrt{(a^2-x^2)}dx$

$=\frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$=\frac{4b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\frac{a}{a}\right]=\frac{4b}{a}\times\frac{a^2}{2}\sin^{-1}(1)$ = 2ab × $\sin^{-1}(1)$ = 2ab × π/2= πab

2) Find the area of the region bounded by $y^2 = 9x$, x = 2, x = 4 and the x-axis in the first quadrant.

Solution: $y^2 = 9x$, y = ±√(9x), y = ±3√x



:

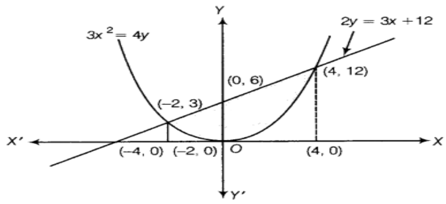
Required area= $\int_2^4 ydx$=3 $\int_2^4 \sqrt{x}\,dx$

$= 3\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_2^4$ $=3\times\frac{2}{3}\left[x^{\frac{2}{3}}\right]_2^4$ $=2\left[2^3-\left(\sqrt{2}\right)^3\right]=$ = 2

$[(2)^3 - (\sqrt{2})^3]$ = 2[8 – 2√2]= 16 – 4√2 sq

3) Find the area enclosed between the parabola $4y = 3x^2$ and the straight line 3x – 2y + 12 = 0

Solution:

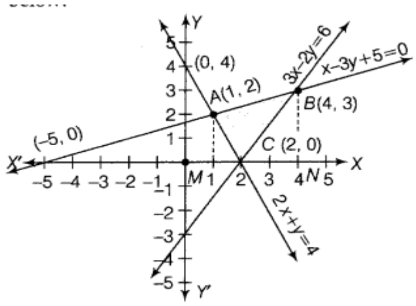


Required are = $\int_{-2}^{4}\left(\frac{3x+12}{2}-\frac{3x^2}{4}\right)dx$

$=\left[\frac{3x^2}{4}+6x-\frac{x^3}{4}\right]_{-2}^{4}=\frac{3\times4^2}{4}+6\times4-\frac{4^3}{4}-\left[\frac{3\times4}{4}-12+\frac{8}{4}\right]$

$=12+24-16-3+12-2=27$ sq units

4) Using integration, find the area of the region bounded by the lines 2x + y = 4, 3x – 2y = 6 and x – 3y + 5 = 0



Now, required area of ΔABC

= Area of region ABNMA – (Area of ΔAMC + Area of ΔBCN)

$=\int_1^4\left(\frac{x+5}{3}\right)dx-\int_1^2(4-2x)dx-\int_2^4\left(\frac{3x-6}{2}\right)dx$

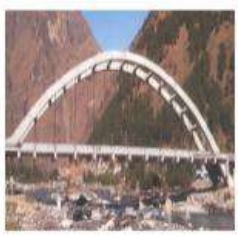
$=\frac{1}{3}\left[\frac{x^2}{2}+5x\right]_1^4-[4x-x^2]_1^2-\frac{1}{2}\left[\frac{3x^2}{2}-6x\right]_2^4$

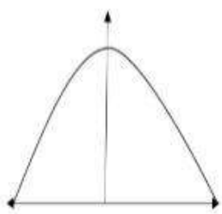
$=\frac{1}{3}\left[\left(\frac{16}{2}+2\right)-\left(\frac{1}{2}+5\right)\right]$-[(8 – 4) – (4 – 1)] $-\frac{1}{2}[(24-24)-(6-12)]$

$=\frac{1}{3}\times\frac{45}{2}$-1-3=$\frac{15}{2}-4$ = $\frac{7}{2}$ sq units

CASE BASED TYPE QUESTIONS

1) the bridge connects two hills 100 feet apart. The arch on on the bridge is in a parabolic form. Th highest point on the bridge is 10 feet above the road at the middle of the bridge as seen in the figure.
Based on the information given above, answer the following questions:





(i) The equation of the parabola designed on the bridge is
Ans)b. $x^2 = -250y$

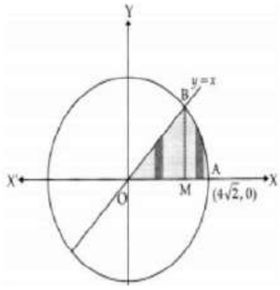
(ii) The value of the integral $\int_{-50}^{50} \frac{x^2}{250}\, dx$ is Ans: 1000/3

iii) The integrand of the integral $\int_{-50}^{50} x^2\, dx$ is either even or odd function. Ans: even

OR

(iv) Find the area formed by the curve $x^2 = 250y$, x-axis , y = 0 and y = 10
Ans: 1000/3 sq units.

2. In the figure given below O(0, 0) is the center of the circle. The line y = x meets the circle in the first quadrant at point B. Answer the following questions based on the given figure



(i) Find the equation of the circle Ans: __$x^2 + y^2 = 32$_.

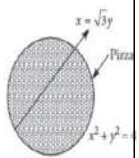
(ii) Find the co-ordinates of B Ans: __(4,4)__

(iii) Find the Area of ΔOBM Ans:8 sq. units.

OR

(iv) Find the Area (BAMB) Ans: $4\pi - 8$ sq. units

3. A child cuts a pizza with a knife. Pizza is circular in shape which is represented by $x^2 + y^2 = 4$ and sharp edge of the knife is represented by $x = \sqrt{3}y$. Based on this information, answer the following questions.



(i) Find the points of intersection of the edge of the knife and the pizza as shown in the figure. Ans: $(\sqrt{3}, 1)$ and $(-\sqrt{3}, -1)$

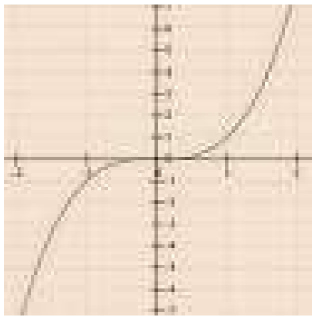
What will the area of each pizza slice if the child divides pizza into four equal slices?

Ans. π sq units

(ii) What is the area of the whole pizza?

Ans. 4π sq units

4. An insect moves on a curve represented by y = x^3 . It started from a point (-2,-8) on the curve and as soon as it reached at a point (2,8) got tired and slept.The path of its movement is given below. Based on this information answer the following questions.



(i) Find the area enclosed by the curve y= x^3 , the lines x=2 and x= -2

(ii) If it would have moved along the line represented by y = x what is the area bounded by the curve y = x^3 and y=6 x

CHAPTER-9-DIFFERENTIAL EQUATIONS

GIST/SUMMARY OF THE CHAPTER

- Definition, order and degree, general and particular solutions of a differential equation.
- Solution of differential equations by method of separation of variables.
- Solutions of homogeneous differential equations of first order and first degree.
- Solutions of linear differential equation of the type:
  (i) $\frac{dy}{dx} + py = q$, where p and q are functions of x or constants.
  (ii) $\frac{dx}{dy} + px = q$, where p and q are functions of y or constants.

DEFINITIONS AND FORMULAE:

- An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.
- Order of a differential equation is the order of the highest order derivative occurring in the differential equation.
- Degree of a differential equation is defined if it is a polynomial equation in its derivatives.
- Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.
- A function which satisfies the given differential equation is called its solution.
- The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.
- Variable separable method is used to solve such an equation in which variables can be separated completely i.e. terms containing y should remain with dy and terms containing x should remain with dx.
- A differential equation which can be expressed in the form $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$ where, f (x, y) and g(x, y) are homogenous functions of degree zero is called a homogeneous differential equation.
- A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only is called a first order linear differential equation. Another form of first order differential equation is $\frac{dx}{dy} + Px = Q$ where P and Q are constants or functions of y.
- (i). The integrating factor (I.F) of the differential equation $\frac{dy}{dx} + Py = Q$ $is\ e^{\int P(x)dx}$.
  Its solution is given by y (I.F) $=\int [Q\,.(I.F)]\,dx + C$.
- (ii). The integrating factor (I.F) of the differential equation $\frac{dx}{dy} + Px = Q$ is $e^{\int P(Y)dy}$
  Its solution is given by x (I.F) $=\int [Q\,.(I.F)]\,dy + C$

MULTIPLE CHOICE QUESTIONS

1. If p and q are respectively the order and degree of the differential equation, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$,then (p-q) is

   (A)0 (B)1 (C)2 (D)3

   $Solution$: $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

   Here p = 2, q=1 p-q=1 $Answer$: B

2.The order and degree of the following differential equation are respectively:

$\frac{d^4y}{dx^4} + 2e^{\frac{dy}{dx}} + y^2 = 0$

(A) -4,1 (B) 4, not defined (C)1,1 (D) 4,1

$Solution$: Here order =4

$\frac{d^4y}{dx^4} + 2e^{\frac{dy}{dx}} + y^2$ is not a polynomial in $\frac{dy}{dx}$.

$So\ its\ degree\ is\ not\ defined.$ $Answer$: B

(A)-4,1 (B) 4, not defined (C)1,1 (D) 4,1

3. The solution for the differential equation $log\left(\frac{dy}{dx}\right) = 3x + 4y\ is$:

(A)$3e^{4y} + 4e^{-3x} + C = 0$ (B) $e^{3x+4y} + C = 0$

(C)$3e^{-3y} + 4e^{4x} + 12C = 0$ (D) $3e^{-4y} + 4e^{3x} + 12C = 0$

$Solution$: $log\left(\frac{dy}{dx}\right) = 3x + 4y$

$$\frac{dy}{dx} = e^{3x+4y}$$

$$\frac{dy}{dx} = e^{3x}\,e^{4y}$$

$$\frac{dy}{e^{4y}} = e^{3x}dx$$

Integrating both sides :

$$\int e^{-4y}\,dy = \int e^{3x}\,dx$$

$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

$$3e^{-4y} = -4\,e^{3x} - 12\,C$$

$$3e^{-4y} + 4\,e^{3x} + 12\,C = 0$$

Answer: D

4.The differential equation $\frac{dy}{dx} = F(x,y)$ $will$ not be a homogeneous differential equation if F(x, y) is

(A) $cos\,x - sin\left(\frac{y}{x}\right)$ (B) $\frac{y}{x}$ (C) $\frac{x^2+y^2}{xy}$ (D) $cos^2\left(\frac{x}{y}\right)$

Solution :

If F (x, y) =$cosx - sin\left(\frac{y}{x}\right), then F(\lambda x, \lambda y) \neq \lambda^0 F(x,y)$

Hence $\frac{dy}{dx} = F(x,y)$ will not be a homogeneous differential equation.Answer : A

5. For what value of n is the following a homogeneous differential equation: $\frac{dy}{dx} = \frac{x^3 - y^n}{x^2y + xy^2}$

(A) 1 (B) 2 (C) 3 (D) 4

Solution : For the given differential equation to be homogeneous, the value of n should be 3.

Answer : C

6.The integrating factor of the differential equation $(x + 2y^2)\frac{dy}{dx} = y\ (y > 0)\ is$:

(A) $\frac{1}{x}$ (B) x (C) y (D) $\frac{1}{y}$

Solution: Given differential equation is $(x + 2y^2)\frac{dy}{dx} = y\ (y > 0)$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{y}x + 2y \quad \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y$$

$$It\ is\ a\ linear\ differential\ equation\ of\ the\ form\ \frac{dx}{dy} + Px = Q\ where$$

$P = \frac{-1}{y}$ and $Q = 2y$

I.F = $e^{\int P\,dy} = e^{\int \frac{-1}{y} dy} = e^{-log\,y} = e^{log\,y^{-1}} = e^{log\frac{1}{y}} = \frac{1}{y}$ $Answer$: D

7. The number of arbitrary constants in the general solution of the differential equation:

$\frac{dy}{dx} + y = 0$

(A)0 (B)1 (C)2 (D)3

$Solution$: Number of arbitrary constants = Order of the differential equation

The order of the given differential equation

$\frac{dy}{dx} + y = 0$ *is one.*

Number of arbitrary constants in the general solution of the differential equation is one.

$Answer$:B

8. A particular solution of the differential equation

$x\frac{dy}{dx} + y = 0, when\ x = 1\ and\ y = 1\ is$

(A) y = x (B)$y = e^x$ (C) $y = \frac{1}{x}$ (D)y = log x

$Solution$: Given differential equation is $x\frac{dy}{dx} + y = 0$

$x\frac{dy}{dx} = -y$

$\frac{dy}{y} = -\frac{dx}{x}$

Integrating both sides

$\int \frac{dy}{y} = \int -\frac{dx}{x}$

$\log y = -\log x + C$

$\log y + \log x = C$

$\log xy = C$ -------(1)

$\Rightarrow C = \log 1 = 0$

$Substituting\ C = 0\ in\ eqn(1)\ we\ get\ \log xy = 0$

$\Rightarrow xy = 1$

$\Rightarrow y = \frac{1}{x}$

$Answer$: C

9. The curve for which the slope of the tangent at any point is equal to the ratio of the abcissa to the ordinate of the point is :

(A) an ellipse (B) parabola (C) circle (D) rectangular hyperbola

$Solution$: Slope of tangent to the curve = $\frac{dy}{dx}$

$$According\ to\ the\ question\frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow y\,dy = x\,dx$$

$$Integrating\ both\ sides\ we\ get$$

$$\int y\,dy = \int x\ dx$$

$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$

$\Rightarrow y^2 - x^2 = 2C$ which is an equation of rectangular hyperbola.

$Answer$:D

10. Family y = A x + $A^3$of curves will correspond to a differential equation of order

(A) 3 (B) 2 (C) 1 (D) not defined

$Solution: Given\ family\ of\ curves\ y = A\,x + A^3$ -----(1)

$\Rightarrow \frac{dy}{dx} = A\ \ Putting\ the\ value\ of\ \frac{dy}{dx}\ in\ eqn(1)\ we\ get$

$$y = \left(\frac{dy}{dx}\right)x + \left(\frac{dy}{dx}\right)^3$$

$Order = 1$ Answer : C

ASSERTION AND REASON BASED QUESTIONS

Question numbers 1 to 10 are Assertion - Reason based questions. Two statements are given, one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer from the codes (A), (B), (C) and (D) as given below.

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false

(D) Assertion (A) is false, but Reason (R) is true.

1) Assertion (A):The degree of the differential equation $\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)$ is one.
Reason(R) :If the differential equation is a polynomial in differential coefficients, then its degree is defined.

$Solution: Given$ differential equation $\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)$ is not a polynomial equation in $\frac{dy}{dx}$. Its order is not defined.

$Here\ A\ is\ false, R\ is\ true. Answer$: D

2) Assertion (A): The general solution of the differential equation $\frac{dy}{dx} = 1 + 2\left(\frac{y}{x}\right)$ is

$$x + y = Cx^2$$

Reason(R) : Correct substitution for the solution of the differential equation

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)\ is\ x = vy.$$

$Solution: Given$ differential equation is $\frac{dy}{dx} = 1 + 2\left(\frac{y}{x}\right)$

$$Put\ y = vx, it\ reduces\ to\ the\ form$$

$$v + x.\frac{dv}{dx} = 1 + 2v$$

$$\Rightarrow x.\frac{dv}{dx} = 1 + v$$

$$\Rightarrow \frac{dv}{1 + v} = \frac{dx}{x}$$

$$Integrating\ both\ sides, we\ get$$

$$\int \frac{dv}{1 + v} = \int \frac{dx}{x}$$

$$\Rightarrow log|1 + v| = log|x| + log|C_1|$$

$\Rightarrow |1 + v| = |C_1 x|$

$$\Rightarrow\ x + y = \pm C_1 x^2$$

$\Rightarrow x + y = C x^2$ where $\pm C_1 = C$

$A\ is\ true.\ \ R\ is\ false.\ \ Answer: C$

3) Assertion (A) : $\frac{dy}{dx} - y \cos x = 5$ is a first order linear differential equation.
Reason (R) :If P and Q are functions of x only or constants, then the differential equation of the form $\frac{dy}{dx} + Py = Q$ is a first order linear differential equation.

Solution :The given differential equation $\frac{dy}{dx} - y \cos x = 5$ is in the form of

$\frac{dy}{dx} + Py = Q.$

$$Here\ P = -\cos x\ is\ a\ function\ of\ x\ and\ Q = 5\ is\ a\ constant.$$

Here A and R are true and R is the correct explanation of A. Answer : A

4) Assertion (A): $The\ differential\ equation\ 2y\, e^{\frac{x}{y}}\, dx + \left(y - 2\, x\, e^{\frac{x}{y}}\right) dy = 0$ can be solved by making the substitution $y = vx.$

Reason(R) : A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ $can\ be\ solved\ by\ making\ the\ substitution\ x = vy\ and\ of\ the\ form$ $\frac{dy}{dx} = h\left(\frac{y}{x}\right)$ by the substitution y = vx.

$Solution$: Here A is false since the given differential equation is of the form

$\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ $and\ can\ be\ solved\ by\ making\ the\ substitution\ x = vy.$

But R is true. $Answer$:D

5) Assertion (A):The number of arbitrary constants in the solution of the differential equation

$$\frac{d^2y}{dx^2} = 0 \ is\ 2.$$

Reason(R) :The general solution of a differential equation contains as many arbitrary constants as the order of the differential equation.

$Solution$: $\frac{d^2y}{dx^2} = 0$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$$

$$On\ integrating\ we\ get\ \frac{dy}{dx} = C$$

⇒ dy = C dx Integrating both sides we get

$$\int dy = \int C\,dx$$

$$y =, C\,x + C'$$

$There$ are two arbitrary constants in the solution of the differential equation $\frac{d^2y}{dx^2} = 0$

We had a differential equation of order 2.

So both A & R are true and R is the correct explanation of A. Answer: A

6) Assertion (A): Integrating factor of the linear differential equation

$$x\frac{dy}{dx} + 2y = 4x + 5x^2 \text{ is } x^2.$$

Reason(R) : Integrating factor of the linear differential equation $\frac{dx}{dy}$ + Rx = S is $e^{\int R\,dy}$

$Solution$: Given differential equation is

$$x\frac{dy}{dx} + 2y = 4x + 5x^2$$

$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = 4 + 5x$

Comparing it with $\frac{dy}{dx} + P\,y = Q$, we get P = $\frac{2}{x}$, $Q = 4 + 5x$

Integrating factor = $e^{\int Pdx} = e^{\int\frac{2}{x}}$ dx = $e^{\log x^2} = x^2$

Assertion (A) is true.

Also Reason (R) is true but R is not the correct explanation of A. $Answer: B$

7) Assertion (A) : A curve passing through the point $\left(1, \frac{\pi}{4}\right)$ $and\ slope\ of\ the\ tangent\ at\ any\ point\ (x, y)\ is\ given\ by$ $\frac{y}{x} - cos^2\left(\frac{y}{x}\right)$, $then\ the\ equation\ of\ the\ curve\ is\ y = x\,tan^{-1}\left(log\left|\frac{e}{x}\right|\right)$

Reason(R) : The integrating factor (I.F) of the differential equation $\frac{dy}{dx} + Py = Q\ \ is\ e^{\int P\,dx}$.

Its solution is given by y (I.F) =$\int[Q\,.(I.F)]dx \quad + C$.

$Solution: Given\ \frac{dy}{dx} = \frac{y}{x} - cos^2\left(\frac{y}{x}\right)$ -----(i)

$$Putting\ y = vx\ we\ get\ \frac{dy}{dx} = v + x.\frac{dv}{dx}$$

Substituting in (i) we get

$$v + x.\frac{dv}{dx} = v - cos^2 v$$

$$\Rightarrow \frac{dv}{cos^2 v} = \frac{-dx}{x}$$

$$\Rightarrow \int sec^2 v\ dv = -\int \frac{1}{x}\,dx$$

$\Rightarrow tan\,v$ = - log | x | $+ C$ −−−−−−− (ii)

Given that the curve passes through the point $\left(1, \frac{\pi}{4}\right)$

So $tan\left(\frac{\pi}{4}\right)$ = - log 1 + C ⇒ $C = 1$

Putting the value of C =1 in (ii) we get

$tan\left(\frac{y}{x}\right)$ = - log | x | $+ 1$

$\Rightarrow tan\left(\frac{y}{x}\right) = log\,|\frac{e}{x}|$ Since $e = 1$

$\Rightarrow \frac{y}{x} = tan^{-1}\left(log\,|\frac{e}{x}|\right)$

Assertion (A) is true.

Also R is true but R is not the correct explanation of A. *Answer*:B

8) Assertion (A): The general solution of the differential equation $\frac{y}{x}\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ is $1 + x^2 = C(1 + y^2)$

Reason(R) : $\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\,log\,(1 + x^2)$ + C

Solution: The given differential equation is $\frac{y}{x}\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$$\Rightarrow \int \frac{y}{1 + y^2}\,dy = \int \frac{x}{1 + x^2}\,dx$$

$\Rightarrow \frac{1}{2}\,log(1 + y^2) = \frac{1}{2}\,log(1 + x^2)$ + log C

$$\Rightarrow (1 + y^2) = (1 + x^2)C$$

$$or \quad (1 + x^2) = C(1 + y^2)$$

Assertion and reason both are true and reason is the correct explanation of assertion.

Answer: A

9) Assertion (A):The equation of the curve passing through (3,9) which satisfies the differential equation $\frac{dy}{dx} = x + \frac{1}{x^2}\ is\ 6xy = 3x^3 + 29x - 6$

Reason(R) : The solution of the differential equation $\frac{dy}{dx} = \sqrt{4 - y^2}, y \in [-2,2]\ is$

$$y = 2\,sin\,(x + C)$$

Solution:Given $\frac{dy}{dx} = x + \frac{1}{x^2}$

$$\int dy = \int \left(x + \frac{1}{x^2}\right) dx$$

$\Rightarrow \quad y = \frac{x^2}{2} - \frac{1}{x}$ + C ----(1)

Given (1) *passes through* (3,9).

$So\ 9 = \frac{9}{2} - \frac{1}{3} + C$⇒ C $= \frac{29}{6}$

On substituting C = $\frac{29}{6}$ in (1) $we\ gety = \frac{x^2}{2} - \frac{1}{x} + \frac{29}{6}$

$\Rightarrow y = \frac{3x^3 - 6 + 29\,x}{6x} \Rightarrow 6xy = 3x^3 - 6 + 29\,x$

$\Rightarrow \quad 6xy = 3\,x^3 + 29\,x - 6$ *Assertion is true.*

$$Given\ \frac{dy}{dx} = \sqrt{4 - y^2}$$

$$\int \frac{dy}{\sqrt{4 - y^2}} = \int dx$$

$\Rightarrow x + C \quad \Rightarrow \frac{y}{2} = sin\,(x + C) \Rightarrow$ y = $2\,sin\,(x + C)$

Both assertion and reason are true but reason is not the correct explanation of assertion. *Answer*:B

10. Assertion (A): Order of the differential equation whose solution is $y = C_1e^{x + C_2} + C_3e^{x + C_4}$ is 4.

Reason(R) : Order of the differential equation is equal to the number of independent arbitrary constant mentioned in the solution of the differential equation.

Solution: *Given* differential equation is $y = C_1e^{x + C_2} + C_3e^{x + C_4}$

$$\Rightarrow y = C_1e^x\,e^{C_2} + C_3e^x\,e^{C_4}$$

$$\Rightarrow \; y = e^{x}\,(A + B)$$
$$\Rightarrow \; y \;\; = P\,e^{x}$$

$A\ is\ false. But\ R\ is\ true.\ \ Answer$:D

VERY SHORT ANSWER QUESTIONS

1) The bottom valve of a conical tank is opened to remove sugarcane juice in a factory. The rate at which the juice pours out from the conical tank is directly proportional to the cube root of the rate of change of height of the juice present in the tank. If K is the constant of proportionality, write a differential equation depicting the scenario:

Solution:

Let 'V' be the volume of the sugar cane juice in the conical tank, 'h' be the height of the sugar cane juice and 't' be the time.

Given $\frac{dV}{dt}\ \alpha \left(\frac{dh}{dt}\right)^{\frac{1}{3}} \;\Rightarrow\; \frac{dV}{dt} = K\left(\frac{dh}{dt}\right)^{\frac{1}{3}}$

$\Rightarrow \left(\frac{dV}{dt}\right)^{3} = K\left(\frac{dh}{dt}\right)$ which is the required differential equation.

2) Solve the differential equation $\log\left(\frac{dy}{dx}\right) = x - y$.

Solution :

Given differential equation is $\log\left(\frac{dy}{dx}\right) = x - y$

$$\Rightarrow \frac{dy}{dx} = e^{x-y} \Rightarrow \; e^{y}\,dy = e^{x}dx$$

Integrating both sides $\int e^{y}\ \ dy = \int e^{x}\,dx$

$\Rightarrow \quad e^{y} = e^{x} + C \Rightarrow \quad e^{x} - e^{y} + C = 0$

3) Solve the differential equation $\frac{dy}{dx} + y = \cos x - \sin x$

Solution :

Given differential equation $\frac{dy}{dx} + y = \cos x - \sin x$ is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q. Here\ P = 1\ , Q = \cos x - \sin x$

Integrating factor = $e^{\int P\,dx} = e^{\int 1.dx} = e^{x}$

The solution of the linear differential equation is given by

y (I.F) = $\int [Q\,.(I.F)]\,dx + C$

i.e., y . $e^{x} = \int (cosx - sinx)e^{x}\,dx\ ) \Rightarrow$. y . $e^{x} = e^{x}\cos x + C$

$\Rightarrow y = \cos x + C\,e^{-x}$ is the solution.

4) If y(x) is a solution of the differential equation

$\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x\ \ and\ y(0) = 1, then\ find\ the\ value\ of\ y\left(\frac{\pi}{2}\right)$

Solution: Given $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$

$\Rightarrow \frac{1}{1+y}\,\frac{dy}{dx}$ = - $\frac{\cos x}{2+\sin x} \Rightarrow \; \frac{dy}{1+y} = -\frac{\cos x}{2+\sin x}dx$

Integrating both sides we get

$$\int \frac{dy}{1 + y} = -\int \frac{\cos x}{2 + \sin x}dx$$

log| 1 + y| = -log $|2 + \sin x| + \log C$

log| 1 + y| +log $|2 + \sin x| = \log C$

log (1+ y)(2 + sin x) = log C

⇒(1+ y)(2 + sin x) =C ;⇒1 + y = $\frac{C}{2+\sin x}$

$\Rightarrow\ y = \frac{C}{2+\sin x} - 1$---(1) $When\ x = 0, y = 1$

Substituting the values in eqn (1) we get

$1 = \frac{C}{2} - 1 \quad \Rightarrow \frac{C}{2} = 2 \quad \Rightarrow C = 4$

Hence eqn (1) reduces to $y = \frac{4}{2+\sin x} - 1$ To find $y\left(\frac{\pi}{2}\right)$

When x = $\frac{\pi}{2}$, $y = \frac{4}{2+\sin\frac{\pi}{2}} - 1$

$\Rightarrow \quad y = \frac{4}{3} - 1 \qquad \Rightarrow y = \frac{1}{3}$

5) Write the integrating factor of the differential equation :

$$(y - x)dy = (1 + y^2)dx$$

Solution: The given differential equation is

$$(y - x)dy = (1 + y^2)dx$$

This can be written in the form $\frac{dx}{dy} = \frac{(y - x)}{(1 + y^2)}$

$$\frac{dx}{dy} = \frac{y}{1 + y^2} - \frac{x}{1 + y^2}$$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{y}{1+y^2}$

$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{y}{1+y^2}$

This is a linear differential equation of the form

$\frac{dx}{dy} + P\,x = Q. Here\ P = \frac{1}{1+y^2}\ , Q = \frac{y}{1+y^2}$

Integrating factor = $e^{\int P dy} = e^{\int \frac{1}{1+y^2}dy} = e^y$

Integrating factor of the given differential equation is e^y .

6) Solve the differential equation $\frac{dy}{dx} + 1 = e^{x+y}$

Solution : The given differential equation is $\frac{dy}{dx} + 1 = e^{x+y}$ -------(1)

Put x + y = t in eqn (1)

Differentiating both sides w.r.t x we get

$1 + \frac{dy}{dx} = \frac{dt}{dx}$ ----- (2)

Substituting eqn (2) in eqn (1) we get $\frac{dt}{dx} = e^t$

Separating the variables we get $\frac{dt}{e^t} = dx$

Integrating both sides $\int e^{-t}\,dt = \int dx$

$\frac{e^{-t}}{-1} = x + C \;\Rightarrow\; \frac{-1}{e^t} = x + C \;\Rightarrow -1 = e^t\,(x + C) \;\Rightarrow\; e^{x+y}(x + C) + 1 = 0$

7) Solve $x\frac{dy}{dx} = y(logy - logx + 1)$

Solution :

The given differential equation is $x\frac{dy}{dx} = y(logy - logx + 1)$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}(logy - logx + 1) \;\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left[log\left(\frac{y}{x}\right) + 1\right]$ ---(1)

Put y = vx

Differentiating both sides w.r.t x we get

$\frac{dy}{dx} = v + x.\frac{dv}{dx}$ ---(2)

Substituting eqn (2) in eqn (1) we get $v + x.\frac{dv}{dx} = v\,log\,v + v$

$\Rightarrow \quad x.\frac{dv}{dx} = v \log v \;\Rightarrow \frac{1}{v\,log\,v}dv = \frac{dx}{x}$

Integrating both sides we get $\int \frac{1}{v\log v} dv = \int \frac{1}{x} dx$

$$\log(\log v) = \log x + \log C$$

$\Rightarrow$ log (log v) = log C x$\Rightarrow$ log v = Cx$\Rightarrow$ log $\left(\frac{y}{x}\right) = C x$

8) In a controlled condition within a laboratory, a spherical balloon is being deflated at a rate proportional to its surface area at that instant. The spherical shape of the balloon is maintained throughout the process. Form a differential equation that represents the rate of change of its radius.

Solution :

Takes V, S and r to be the volume, surface area and radius of the balloon at time t.

Since the volume of the balloon is decreasing with time, the rate of

Change of its radius is negative.

Uses the given information and expresses the relationship between volume and

Surface area as

$\frac{dV}{dt}$ =-k×S, where k is a positive real number.

Differentiates the above equation after substituting the expressions for volume and

Surface area of a sphere to get the required differential equation as:

$$\frac{4}{3}\pi(3r^2)\frac{dr}{dt} = -k \times 4\pi r^2;\ \frac{dr}{dt} = -k$$

SHORT ANSWER QUESTIONS

1) Find the particular solution of the differential equation

$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$, given that y = $\frac{\pi}{4}$, when x =1

Solution: The given differential equation

$$\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + x\,dy = 0$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right)$ ------(1)

Put y = vx

Differentiating both sides w.r.t x we get

$\frac{dy}{dx} = v + x.\frac{dv}{dx}$ ---(2)

Substituting eqn (2) in eqn (1) we get

$v + x.\frac{dv}{dx} = v - \sin^2 v \Rightarrow x.\frac{dv}{dx} = -\sin^2 v$

Integrating both sides we get $-\int \frac{dv}{\sin^2 v} = \int \frac{dx}{x} \Rightarrow$ -$\int \mathrm{cosec}^2 v\ dv = \int \frac{dx}{x}$

$\Rightarrow$ cot v = log |x| + C$\Rightarrow$ cot $\left(\frac{y}{x}\right) = \log|x| + C$, given that y = $\frac{\pi}{4}$, when x =1

Therefore cot $\left(\frac{\pi}{4}\right)$ = log 1 + C$\Rightarrow$ C = 1

Hence the particular solution of the differential equation is cot $\left(\frac{y}{x}\right) = \log|x| + 1$

2) Solve the differential equation $\frac{dy}{dx} - 3y\cot x = \sin 2x$, given that $y = 2$, when $x = \frac{\pi}{2}$.

Solution : The given differential equation is $\frac{dy}{dx} - 3y\cot x = \sin 2x$,

$$\text{which is of the form } \frac{dy}{dx} + Py = Q$$

Here P = - 3 cot x Q= sin 2x

Integrating factor (I.F) = $e^{\int P\,dx} = e^{\int -3\cot x\ dx} = e^{-3\log\sin x} = \frac{1}{\sin^3 x}$

The solution of the linear differential equation is given by y (I.F) =$\int [Q\,.\,(I.F)]\,dx\ + C$; y.$\frac{1}{sin^3x} = \int sin\,2x..\frac{1}{sin^3x}\,dx$

$$\frac{y}{sin^3x} = \int 2\,sinx\,cos\,x.\frac{1}{sin^3x}dx$$

$$\frac{y}{sin^3x} = 2\int cot\,x\,cosecx\ dx\ +\ C$$

$$\frac{y}{sin^3x} = -2\,cosec\,x\ +\ C$$

$y\ =\ -2\,sin^2x$+ C sin^3x---(1) Given When x =$\frac{\pi}{2}$, $y\ = 2$

2 = -2 + C ⇒ C = 4 Substituting C =4 in eqn (1) we get

$y\ =\ -2\,sin^2x$+ C sin^3x which is the required solution.

3) Find the general solution of the differential equation

$$e^x tany\,dx\ +\ (1 - e^x)sec^2y\,dy = 0$$

Solution :

The given differential equation

$$e^x tany\,dx\ +\ (1 - e^x)sec^2y\,dy = 0$$

$\Rightarrow\ e^x tany\,dx\ = (e^x -\ 1)sec^2y\,dy$

Separating the variables

$\frac{e^x}{(e^x - 1)}$dx = $\frac{sec^2y\,dy}{tany}$

On integrating both sides we get

$\int\frac{e^x}{(e^x - 1)}dx\ = \int\frac{sec^2y\,dy}{tany}$ ⇒ log$|e^x -\ 1|$ = $log|tany|\ +\ log\,|C|$

$\Rightarrow(e^x -\ 1) =\ \pm\,C\ tany\ \Rightarrow(e^x -\ 1) =\ C_1 tany, where\ C_1 = \pm C$

4) Find the particular solution of the differential equation given by

$$x^2\frac{dy}{dx} - y\ +\ x\,sin\left(\frac{y}{x}\right) = 0$$

Solution:

The given differential equation is

$$x^2\frac{dy}{dx} - y\ +\ x\,sin\left(\frac{y}{x}\right) = 0$$

$\Rightarrow x\frac{dy}{dx}\ =\ y\ -\ x\,sin\left(\frac{y}{x}\right)\ \Rightarrow\frac{dy}{dx}\ = \frac{y}{x}\ -\ sin\left(\frac{y}{x}\right)$ ----(1)

It is a homogeneous differential equation.

Put y = vx

$\frac{dy}{dx}\ =\ v\ +\ x.\frac{dv}{dx}$ ---(2)

Substituting (2) in (1) we get$v\ +\ x.\frac{dv}{dx}\ =\ v\ -\ sinv;\Rightarrow\ x.\frac{dv}{dx} = -\,sinv$

$$\Rightarrow\ \frac{dv}{sin\,v} = -\ \frac{dx}{x}$$

Integrating both sides we get $\int\frac{dv}{sin\,v} = -\int\frac{dx}{x}$

$$\Rightarrow\int cosecv\,dv\ = -\int\frac{dx}{x}$$

⇒ log |cosec v – cot v| = -log |x|+ log $|c_{1|}$ ⇒x (cosec v - cot v) =$\pm c_1$

⇒ x $\left(cosec\frac{y}{x} -\ cot\ \frac{y}{x}\right)$ = C where C ==$\pm c_1$

5) Solve the differential equation $(1 - x)\,dy\ - (3 + y)dx = 0$

Solution:

The given differential equation is $(1 - x)\,dy\ - (3 + y)dx = 0$

$\Rightarrow(1 - x)dy = (3 + y)dx$

$$\frac{dy}{3 + y} = \frac{dx}{1 - x}$$

On integrating both sides we get $\int \frac{dy}{3 + y} = \int \frac{dx}{1 - x}$

$\Rightarrow \log|3 + y| = -log|1 - x| + \log C$

$\Rightarrow \log|3 + y| + log|1 - x| = \log C$

$\Rightarrow$(1-x)(3+y) = C which is the required solution of the given differential equation.

6) Find the general solution of the differential equation

$$y^2 dx + (x^2 - xy + y^2)dy = 0$$

$$Solution:$$

The given differential equation is $y^2 dx + (x^2 - xy + y^2)dy = 0$

We can write this as $\frac{dx}{dy} = -\left(\frac{x^2}{y^2} - \frac{x}{y} + 1\right)$-----(1)

Which is a homogeneous differential equation of the form

$\frac{dx}{dy} = F(x,y)$ $Put\ x = vy$

$\Rightarrow \frac{dx}{dy} = v + y.\frac{dv}{dy}$; $v + y.\frac{dv}{dy} = -(v^2 - v + 1)$

$y.\frac{dv}{dy} = -(v^2 - v + 1)$; $\Rightarrow . \frac{dv}{1 + v^2} = -\frac{dy}{y}$

On integrating both sides we get $v = -log|y| + C$

$\Rightarrow \frac{x}{y} + log|y| = C$ which is the required differential equation.

7) Find the particular solution of the differential equation :

$$2xy + y^2 - 2x^2 \frac{dy}{dx} = 0: y = 2\ when\ x = 1$$

Solution :

The given differential equation : $2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$

$\Rightarrow$2xy + $y^2 = 2x^2 \frac{dy}{dx}$; $\Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$-----(1)

Put y = vx ; $\frac{dy}{dx} = v + x.\frac{dv}{dx}$ ---(2)

Substituting eqn (2) in eqn (1) we get

$$v + x.\frac{dv}{dx} = \frac{2x.vx + v^2x^2}{2x^2}$$

$$\Rightarrow v + x.\frac{dv}{dx} = \frac{2vx^2 + v^2x^2}{2x^2} \Rightarrow v + x.\frac{dv}{dx} = \frac{2v + v^2}{2} \Rightarrow x.\frac{dv}{dx} = \frac{2v + v^2 - 2v}{2}$$

$$\Rightarrow \frac{2\,dv}{v^2} = \frac{dx}{x}$$

$2 . -\frac{-1}{v} = log|x| + C$; $2.\frac{-x}{y} = log\,x + C$ ------(3) $Given\ y = 2\ when\ x = 1$

$2.\frac{-1}{2} = log\,1 + C$ $\Rightarrow$ C = -1

Substituting C = -1 in eqn (3) we get $\frac{-2x}{y} = log\,x - 1$ which is the required solution.

8) Find the particular solution of the differential equation

$\frac{dy}{dx} + sec^2x\,.y = tanx\,.sec^2x$ $,given\ that\ y(0) = 0.$

$Solution:$ The given differential equation is

$\frac{dy}{dx} + sec^2x\,.y = tanx\,.sec^2x$ $which\ is\ of\ the\ form\ \frac{dy}{dx} + P\,y = Q$

Here P = sec^2x Q= $tanx\,.sec^2x$

Integrating factor (I.F) = $e^{\int P\,dx} = e^{\int sec^2x\,dx} = e^{tan\,x}$

The solution of the linear differential equation is given by y (I.F) $=\int [Q\,.(I.F)]\,dx\ + C$

$\Rightarrow$ y $e^{\tan x} = \int tanx\,.sec^2x\, e^{\tan x}\ dx\ + C$ ----(1)

Put tanx =t sec^2x dx =dt

y $e^{\tan x} = \int t\, e^t\, dt$ + C $\quad\Rightarrow\ y\, e^{\tan x} = t\,e^t - e^t + C$

$\Rightarrow$ y $e^{\tan x} = e^t(t - 1) + C \quad\Rightarrow$ y $e^{\tan x} = e^{tanx}(tanx - 1) + C$

$Given\ y(0) = 0. i.e., When\ x = 0, y = 0$

0 =(0 – 1)$e^0 + C \quad \Rightarrow\ 0 = -1 + C \quad \Rightarrow\ C = 1$

Hence the particular solution of the differential equation is

y $e^{\tan x} = e^{tanx}(tanx - 1) + 1$

9) Find the general solution of the differential equation
$ydx = (x + 2y^2)dy$

$Solution$: The given differential equation is

$ydx = (x + 2y^2)dy \Rightarrow \frac{dx}{dy} = \frac{x+2y^2}{y} \Rightarrow \frac{dx}{dy} - \frac{1}{y}x = 2y$

This is a linear differential equation of the form

$\frac{dx}{dy} + P\,x = Q. \qquad Here\ P = \frac{-1}{y}\ , Q = 2y$

Integrating factor = $e^{\int P\,dy} = e^{\int \frac{-1}{y} dy} = e^{y} = e^{\log y^{-1}} = \frac{1}{y}$

The solution is given by x.(I.F) = $\int Q.(I.F)\,dy\ + C$

$\Rightarrow$ x.$\frac{1}{y} = \int 2y.\frac{1}{y}\,dy + C$

$\Rightarrow\ x = 2\,y^2 + Cy$ is the general solution of the differential equation.

LONG ANSWER TYPE QUESTIONS

1) Given $x + (\,y + 1)\ \frac{dy}{dx} = 2$

(i) Solve the differential equation and show that the solution represents a family of circles.

(ii) Find the radius of a circle belonging to the above family that passes through the origin.

Solution : Given $x + (\,y + 1)\ \frac{dy}{dx} = 2$

$$(\,y + 1\,)\frac{dy}{dx} = 2 - x \ \Rightarrow (y + 1)dy = (2 - x)dx$$

$$\Rightarrow \int (y + 1)\,dy = \int (2 - x)\,dx$$

$$\Rightarrow \frac{y^2}{2} + y = 2\,x - \frac{x^2}{2} + C_1$$

$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = 2\,x - y + C_1\ ;\Rightarrow x^2 + y^2 = 2(2x - y + C_1)$

$$\Rightarrow x^2 - 4x + y^2 + 2y = 2\,C_1$$

$$\Rightarrow x^2 - 4x + 4 + y^2 + 2y + 1 = 2\,C_1 + 5$$

$$\Rightarrow (x - 2)^2 + (y + 1)^2 = C \quad where\ C = 2\,C_1 + 5$$

$Hence\ it\ represents\ a\ family\ of\ circles\ with\ centre\ at\ (\,2\,, -1)$

(ii) $\quad We\ have\ equation\ representing\ family\ of\ circles$

$$(x - 2)^2 + (y + 1)^2 = C\ -----\ (1)$$

$$Since\ (1)\ passes\ through\ (0{,}0)$$

$$(0 - 2)^2 + (0 + 1)^2 = C$$

$\Rightarrow$ 4 + 1 = C ;$\Rightarrow$ C = 5

Therefore (1) becomes $(x - 2)^2 + (y + 1)^2 = 5$

$$(x-2)^2+(y+1)^2=\left(\sqrt{5}\right)^2$$

$Centre\ is\ (2,-1)\ and\ radius\ =\sqrt{5}$ units

$So\ its\ centre\ is\ (2,-1)\ and\ radius\ =\sqrt{5}\ units$

2) Find the general solution of the following differential equation.

$sinx\frac{dy}{dx}=\left(y+\sqrt{sin^2x-y^2}\right)cosx$

Solution: Put sin x =t $cosx\ \frac{dx}{dt}=1\ \ \frac{dx}{dt}=\frac{1}{cosx}$

$\frac{dy}{dt}=\frac{y+\sqrt{t^2-y^2}}{t}$ where t=sinx

$Substitutes\ y=vt\ and\ gets\ \frac{dy}{dt}=v+t.\frac{dv}{dt}$

$v+t.\frac{dv}{dt}=v+\sqrt{1-v^2}\ where\ v=\frac{y}{t}\ and\ reduces\ it\ to$

$\frac{dv}{\sqrt{1-v^2}}=\frac{dt}{t};\ \ sin^{-1}v=log|t|+C$

$sin^{-1}\frac{y}{t}=log|t|+C;\ \ sin^{-1}\left(\frac{y}{sinx}\right)=log|sinx|+C$

3) Solve the differential equation $(x^2+y^2)\,dx+xy\,dy=0, y(1)=1$

Solution : The given differential equation is

$(x^2+y^2)\,dx+xy\,dy=0, y(1)=1$

$\Rightarrow\frac{dy}{dx}=-\frac{(x^2+y^2)}{xy}\ \ \Rightarrow\frac{dy}{dx}=-\frac{1+\left(\frac{y}{x}\right)^2}{\frac{y}{x}}$-----(1) $\frac{dy}{dx}=v+x.\frac{dv}{dx}$

(1) becomes $v+x.\frac{dv}{dx}=\frac{1+v^2}{v}\ \Rightarrow\ \ x.\frac{dv}{dx}=-\frac{1+v^2}{v}-v$

$\Rightarrow\ \ x.\frac{dv}{dx}=-\left(\frac{1+2v^2}{v}\right)\Rightarrow\frac{v}{1+2v^2}=-\frac{dx}{x}$

$$\int\frac{v}{1+2v^2}dv=-\int\frac{dx}{x}$$

$$\frac{1}{4}\int\frac{4v}{1+2v^2}dv=-\int\frac{dx}{x}$$

$\Rightarrow\frac{1}{4}\ log\,(1+2v^2)$ = -log $|x|+log\ C\Rightarrow log\,(1+2v^2)=log\ \frac{C^4}{x^4}$

$\Rightarrow$ 1 +2 $\left(\frac{y}{x}\right)^2=\frac{D}{x^4}$, D =$C^4$Given y (1)=1; When x =1, y =1 , D=3

The solution of the differential equation is 3 = $(2\,y^2+x^2)x^2$

4) Find the particular solution of the differential equation

$tan\,x\frac{dy}{dx}=2x\ tanx+x^2-y\ (tan\,x\neq 0)\ .Given\ that\ y=0\ when\ x=\frac{\pi}{2}$

Solution :

Given differential equation is $tan\,x\frac{dy}{dx}=2x\ tanx+x^2-y$

$\Rightarrow\frac{dy}{dx}=2x+\frac{x^2}{tan\,x}-\frac{y}{tan\,x}\ \ \Rightarrow\frac{dy}{dx}+\frac{1}{tan\,x}y=2x+x^2cot\,x$

$\Rightarrow\frac{dy}{dx}+cot\,x\,y=2x+x^2cot\,x$

This is a linear differential equation of the form

$\frac{dy}{dx}+Py=Q.\quad Here\ P=cot\,x\ ,Q=2x+x^2cot\,x$

Integrating factor = $e^{\int P\,dx}=e^{\int cot\,x\,dx}=e^{log\,(sinx)}=sinx$

The solution of the linear differential equation is given by

y (I.F) =$\int[Q\,.(I.F)]\,dx+C$

$y.sinx=\int(2x+x^2cot\,x)sinx\ dx+C$

$\Rightarrow y.sinx=\int 2x\ sinx\ dx+\int x^2\ cos\,x\ dx+C$

$\Rightarrow y.sinx=\int 2x\ sinx\ dx+x^2sinx-\int 2x\ sinx\ dx$ (Using integration by parts)

$\Rightarrow y.sinx=x^2sinx$ + C----(1) $Given\ that\ y=0\ when\ x=\frac{\pi}{2}$.

Putting x =$\frac{\pi}{2}andy=0ineqn(1)weget\ \ 0=\frac{\pi^2}{4}+C;\Rightarrow$ C = $\frac{--\pi^2}{4}$

$y. \sin x = x^2 \sin x - \frac{\pi^2}{4}$ $\Rightarrow 4y \sin x = 4x^2 \sin x - \pi^2$

5) Birds are sensitive to microwaves that are emitted by mobile phones, which has resulted in a decline in the bird population, especially sparrows. The population of sparrows in a certain region is decreasing according to the following equation due to the extensive use of mobile phones.

$$\frac{dy}{dt} = k\,y$$

where, y represents the population of sparrows at time t (in years) and k is a constant. The population, which was e^{10} five years ago, has decreased by 25% in that time. Find k. (Note: Take ln 3 ≈ 1.09 and ln 4 ≈ 1.38.)

Solution: We have $\frac{dy}{dt} = k\,y$

Uses the variable separable form to rewrite the given equation as:

$$\frac{dy}{y} = k\,dt$$

Integrates the above equation on both sides to get:

$$\int \frac{dy}{y} = \int k\ dt$$

log y= kt +C, where C is a constant.

Writes the above equation in terms of e as:

$$y = e^{kt+C} \Rightarrow y = e^{kt}.e^{C}$$

Uses the given conditions to write:

At t=0, y $= e^{C} = e^{10}$ where e^{10} is the initial population.

At t=5, y $= \frac{3}{4}e^{10} = e^{5k}.e^{10};\quad e^{5k} = \frac{3}{4}$

Takes the natural logarithm on both sides to find the value of k as-0.058.The

$$\log(e^{5k}) = \log\left(\frac{3}{4}\right) \Rightarrow 5k = \log 3 - \log 4 \quad \Rightarrow 5k=1.09-1.38$$

$$\Rightarrow 5k = -0.29 \quad \Rightarrow k = \frac{-0.29}{5} K = -0.058$$

CASE STUDY BASED QUESTIONS

1) A Veterinary doctor was examining a sick cat brought by a pet lover. When it was brought to the hospital, it was already dead. The pet lover wanted to find its time of death. He took the temperature of the cat at 11.30 pm which was 94.6°F. He took the temperature again after one hour; the temperature was lower than the first observation. It was 93.4°F. The room in which the cat was put is always at 70°F. The normal temperature of the cat is taken as 98.6°F when it was alive. The doctor estimated the time of death using Newton law of cooling which is governed by the differential equation: $dT\ dt \propto (T-70)$, where 70°F is the room temperature and T is the temperature of the object at time t. Substituting the two different observations of T and t made, in the solution of the differential equation $dT = k(T-70)$ *where k is a constant of proportion, time of death is calculated.*

(i) Find the order and degree of the above given differential equation.

(ii) Which method of solving a differential equation helped in calculation of the time of death?

(iii) Find the solution of the differential equation $\frac{dT}{dt} = k\,(T - 70)$

Solution :

(i) Order is one and degree is 1

(ii) Given $\frac{dT}{dt} = k(T-70) \Rightarrow \frac{dT}{T-70} = k\,dt$

variable separable method

(iv)The given differential equation is $\frac{dT}{dt} = k(T - 70)$

Integrating both sides we get

$$\int \frac{dT}{T - 70} = \int k \; dt$$

log | T – 70| = kt + C

2) Polio drops are delivered to 50K children in a district. The rate at which polio drops are given is directly proportional to the number of children who have not been administered the drops. By the end of 2^{nd} week half the children have been given the polio drops. How many will have been given the drops by the end of 3^{rd} week that can be estimated using the solution to the differential equation

$\frac{dy}{dx} = \mathbf{k}(\mathbf{50-y})$ where x denotes the number of weeks and y the number of children (in thousands) who have been given the drops. Based on the above information answer the following questions :

(i) Find the order and degree of the above given differential equation.

(ii) Find the solution of the differential equation $\frac{dy}{dx} = k(50 - y)$

(iii) Find the particular solution of the given differential equation, given that y (0) = 0

Solution :

(i) Given differential equation is $\frac{dy}{dx} = k(50 - y)$.

The highest order derivative present in the differential equation is $\frac{dy}{dx}. So\ its\ order\ is\ one.$

Here degree is the highest exponent of $\frac{dy}{dx}$ which is one.Thus its degree is one.

(ii) Given differential equation is $\frac{dy}{dx} = k(50 - y) \Rightarrow \frac{dy}{50-y} = k\,dx$

Integrating both sides we get $\int \frac{dy}{50-y} = \int k \; dx$

$\Rightarrow$ - log |50 - y|= k x + C

(iii) The general solution of the differential equation $\frac{dy}{dx} = k(50 - y)$ is

- log |50 - y|= k x + C

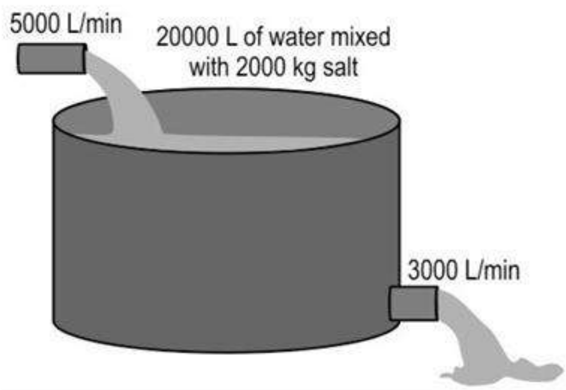
Given y(0) =0 ;i.e., when x=0, y=0; -log|50 - 0|= k.0 + C

$\Rightarrow$ C = - log 50

- log |50 - y|= k x -log 50 is the required particular solution.

3.Answer the questions based on the given information.

The mixing tank shown below generates saline water (a mixture of salt and water) for the cooling of a thermoelectric power plant.



The tank initially holds 20000 L of water in which 2000 kg of salt has been dissolved. Then, pure water is poured into the tank at a rate of 5000 L per minute. The mixture in the tank, which is The tank initially holds 20000 L of water in which 2000 kg of salt has been dissolved. Then, pure water is poured into the tank at a rate of 5000 L per minute. The mixture in the tank, which is stirred continuously, flows out at a rate of 3000 L per minute.

The quantity of salt in the tank at time t is denoted by Q_t where t is in minutes and Q_t is in kilograms.

The rate of flow of salt into the tank is measured as :$\left(\frac{dQ_t}{dt}\right)_{in} = 0\ kg/min$

The rate of flow of salt out of the tank is measured as :

$$\left(\frac{dQ_t}{dt}\right)_{out} = Rate\ of\ flow\ of\ water\ out\ of\ the\ tank \times \frac{Quantity\ of\ salt\ in\ the\ tank\ at\ time\ t}{Amount\ of\ water\ in\ the\ tank\ at\ time\ t}$$

The rate of change of quantity of salt in the tank with respect to time is given by

$$\left[\left(\frac{dQ_t}{dt}\right)_{in} - \left(\frac{dQ_t}{dt}\right)_{out}\right]$$

(i). Find the expression for the rate of change of quantity of salt in the tank with time t.

(ii). Find the general solution of the differential equation corresponding to the rate of change of quantity of salt in the tank with time t.

(iii). Find the general solution of the differential equation corresponding to the rate of change of quantity of salt in the tank with time t.

Solution :

(i) According to the given condition, the rate of change of quantity of salt can be written as

$$\frac{dQ_t}{dt} = 0 - 3000 \times \frac{Q_t}{20000 + 5000t - 3000t} = \frac{-3Q_t}{20 + 2t}$$

Which is the required expression.

(ii) Consider the differential equation ; $\frac{dQ_t}{dt} = \frac{-3Q_t}{20+2t}$

Separates the variables of the differential equation as follows:

$$\frac{dQ_t}{-3Q_t} = \frac{dt}{20 + 2t}$$

Integrates both sides to obtain the following:

$$\int \frac{dQ_t}{-3Q_t} = \int \frac{dt}{20 + 2t}$$

$$-\frac{1}{3}logQ_t = \frac{1}{2}log(10 + t) + C$$

(ii)

Takes initial condition $Q_t(0) = 2000\ to\ obtain\ c\ as\ follows$:

$$-\frac{1}{3}log\ 2000 = \frac{1}{2}log\ 10\ + C$$

$$C = -\frac{1}{3}log\ 2 - \frac{3}{2}log\ 10$$

Frames the equation as : $-\frac{1}{3}logQ_t = \frac{1}{2}log(10 + t) - \frac{1}{3}log\ 2 - \frac{3}{2}log\ 10$

CHAPTER-10-VECTOR ALGEBRA

Gist/Summary of the lesson (Definitions and Formulae)

PRODUCT OF TWO VECTORS

There are two types of products between two vectors.

1. Scalar Product (OR) Dot Product.
2. Vector Product (OR) Cross Product.

Scalar Product:

If $\vec{a}$, $\vec{b}$ are two non-zero vectors and if the angle between them is θ then $\vec{a}.\vec{b}$ is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\,Cos\theta$

Case (1). If $\vec{a}$, $\vec{b}$ are Like vectors then θ = 0° .

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\,Cos\,0^0 = |\vec{a}||\vec{b}|$$

Case (2). If $\vec{a}$, $\vec{b}$ are Unlike vectors then θ = 180° .

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\,Cos180° = -|\vec{a}||\vec{b}|$$

Case(3). If $\vec{a}$, $\vec{b}$ are Perpendicular vectors then θ = 90° .

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\,Cos90° = 0 \Rightarrow \vec{a}.\,\vec{b} = 0$$

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ are two vectors then the Scalar Product $\vec{a}.\vec{b}$ is defined as $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ (Scalar)

Properties of Dot Product:

1. $\vec{a}.\vec{b} = \vec{b}.\,\vec{a}$ 2. $\vec{a}.(-\vec{b}) = -(\vec{a}.\vec{b})$ 3. $\vec{a}.\vec{a} = |\vec{a}||\vec{a}|$ Cos 0° = $|\vec{a}|^2 = a^2$
4. $(m\,\vec{a}).\,\vec{b} = m(\vec{a}.\vec{b}) = \vec{a}.m(\vec{b})$

Angle between vectors:

Let θ be the angle between two non-zero vectors $\vec{a}$, $\vec{b}$. Then

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\,Cos\theta \quad \Rightarrow \quad Cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Composition Table of Dot Product:

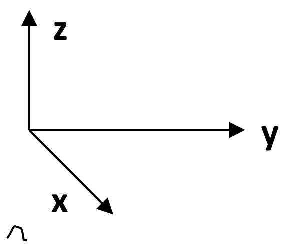
•	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\hat{\imath}$	1	0	0
$\hat{\jmath}$	0	1	0
$\hat{k}$	0	0	1

Projection of a Vector:

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Let θ be the angle between $\vec{a}$ and $\vec{b}$.

∴The Projection of $\vec{b}$ on $\vec{a} = OM = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$, Similarly the Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Vector (or Cross) Product of two vectors:



The vector product of two non – zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\,X\,\vec{b}$ and is defined as

$\vec{a}\,X\,\vec{b} = |\vec{a}||\vec{b}|$ Sinθ $\hat{n}$

Where θ is angle between $\vec{a}$ and $\vec{b}$, where $0 \leq \theta \leq \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

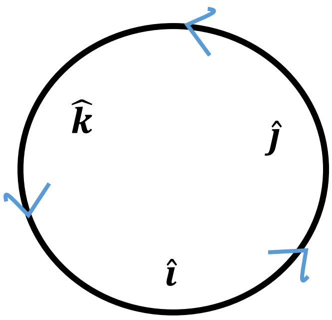
(i). If either $\vec{a} = \vec{\mathbf{0}}$ or $\vec{b} = \vec{\mathbf{0}}$ then $\vec{a} \times \vec{b} = \vec{\mathbf{0}}$

(ii). $\vec{a} \times \vec{b}$ is a vector.

(iii). If $\vec{a}$ *and* $\vec{b}$ are parallel then $\vec{a} \times \vec{b} = \vec{0}$

(iv). If θ = 90° then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$

Composition Table of Cross Product:



X	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	0	$\hat{k}$	$-\hat{j}$
$\hat{j}$	$-\hat{k}$	0	$\hat{i}$
$\hat{k}$	$\hat{j}$	$-\hat{i}$	0

Angle between vectors:

Let θ be the angle between two non-zero vectors $\bar{a}$ and $\bar{b}$. Then

$\vec{a} X \vec{b} = |\vec{a}||\vec{b}| \text{Sin}\theta\, \hat{n} \quad \Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \text{Sin}\theta \quad \Rightarrow \quad \text{Sin}\theta = \frac{|\vec{a} X \vec{b}|}{|\vec{a}||\vec{b}|}$

Unit vector ⊥ both the vectors $\vec{a}$ and $\vec{b}$:

$\vec{a} X \vec{b} = |\vec{a}||\vec{b}| \text{Sin}\theta\, \hat{n}$ …(1) then

$|\vec{a} X \vec{b}| = |\vec{a}||\vec{b}| \text{Sin}\theta$ …..(2) (1) ÷ (2) $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors then the Vector Product $\vec{a} \times \vec{b}$ is defined as $\vec{a} X \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Area of a Parallelogram: $|\vec{a} \times \vec{b}|$

Note: Area of a triangle with $\vec{a}$ and $\vec{b}$ as adjacent sides = Area = $\frac{1}{2}|\vec{a} \times \vec{b}|$

Note: When diagonals $\vec{d_1}$ and $\vec{d_2}$ are given, Area of a Parallelogram = $\frac{1}{2}|\vec{d_1} \times \vec{d_2}|$

Relation between Dot and Cross Product:

$$(\vec{a}.\vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$$

Note: Stress on Position Vector Concept

Convert a position vector to Cartesian form of a point and vice-versa

Position Vector (Vector Geometry)	Point in 3D Geometry(Cartesian Form)
$\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$	P=(x,y,z)
$\vec{OA} = 2\hat{i} + 3\hat{j} - \hat{k}$	A=(2,3,-1)
	C=(1,-2,-3)
$\vec{OX} = 3\hat{i} - 5\hat{j} + 4\hat{k}$	
	Y=(-2,-5,-7)

Activity to remember the concepts:

Requirement	If one vector is given in the Question (Not applicable to position vector) $\vec{r} = a\vec{i} + b\vec{j} + c\vec{k}$	If two points are given in the Question A(x_1,y_1,z_1) and B(x_2,y_2,z_2) **OR** $\vec{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ & $\vec{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$
Direction Ratios of a vector		
Magnitude of a vector		
Direction Cosines of a vector		

Unit Vector in the direction of_____		
A vector with magnitude k units and in the direction of____		

MULTIPLE CHOICE QUESTIONS

1. If the position vectors of the vertices of a triangle be $6\mathbf{i}+4\mathbf{j}+5\mathbf{k}$, $4\mathbf{i}+5\mathbf{j}+6\mathbf{k}$ and $5\mathbf{i}+6\mathbf{j}+4\mathbf{k}$, then the triangle is

(a) Right angled (b) Isosceles (c) Equilateral (d) None of these

Sol: (c) Equilateral, since each side is of length $\sqrt{6}$.

2. The perimeter of the triangle whose vertices have the position vectors $(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $(5\mathbf{i}+3\mathbf{j}-3\mathbf{k})$ and $(2\mathbf{i}+5\mathbf{j}+9\mathbf{k})$, is given by

(a) $15+\sqrt{157}$ (b) $15-\sqrt{157}$ (c) $\sqrt{15}-\sqrt{157}$ (d) $\sqrt{15}+\sqrt{157}$ **Sol: (a)**

$\mathbf{a}=4\mathbf{i}+2\mathbf{j}-4\mathbf{k} \Rightarrow |\mathbf{a}|=\sqrt{16+16+4}=6$

$\mathbf{b}=-3\mathbf{i}+2\mathbf{j}+12\mathbf{k} \Rightarrow |\mathbf{b}|=\sqrt{144+4+9}=\sqrt{157}$

$\mathbf{c}=-\mathbf{i}-4\mathbf{j}-8\mathbf{k} \Rightarrow |\mathbf{c}|=\sqrt{64+16+1}=9$

Hence perimeter is $15+\sqrt{157}$.

3. The position vectors of two points A and B are $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}+\mathbf{k}$ respectively. Then $|\overrightarrow{AB}|=$

(a) 2 (b) 3 (c) 4 (d) 5 **Sol: (b)**

$\overrightarrow{AB}=\mathbf{i}-2\mathbf{j}+2\mathbf{k} \Rightarrow |\overrightarrow{AB}|=3.$

4. The magnitudes of mutually perpendicular forces **a, b** and **c** are 2, 10 and 11 respectively. Then the magnitude of its resultant is

(a) 12 (b) 15 (c) 9 (d) None **Sol: (b)**

$R=\sqrt{4+100+121}=15.$

5. The system of vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ is

(a) Orthogonal (b) Coplanar (c) Collinear (d) None of these **Sol: (a)**

It is a fundamental concept.

6. The direction cosines of the resultant of the vectors $(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $(-\mathbf{i}+\mathbf{j}+\mathbf{k})$, $(\mathbf{i}-\mathbf{j}+\mathbf{k})$ and $(\mathbf{i}+\mathbf{j}-\mathbf{k})$, are

(a) $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{6}}\right)$ (b) $\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$ (c) $\left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$ (d) $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ **Sol: (d)**

Resultant vector $=2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

Direction cosines are $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

7. The position vectors of P and Q are $5\mathbf{i}+4\mathbf{j}+a\mathbf{k}$ and $-\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ respectively. If the distance between them is 7, then the value of a will be

(a) - 5, 1 (b) 5, 1 (c) 0, 5 (d) 1, 0 **Sol: (a)**

$7=\sqrt{(5+1)^2+(4-2)^2+(a+2)^2} \Rightarrow a+2=\pm 3 \qquad a=-5,\ 1.$

8. A zero vector has

(a) Any direction (b) No direction (c) Many directions (d) None of these **Sol: (a)**

Direction is not determined.

9. A unit vector **a** makes an angle $\frac{\pi}{4}$ with z-axis. If $\mathbf{a}+\mathbf{i}+\mathbf{j}$ is a unit vector, then **a** is equal to

(a) $\frac{\mathbf{i}}{2}+\frac{\mathbf{j}}{2}+\frac{\mathbf{k}}{\sqrt{2}}$ (b) $\frac{\mathbf{i}}{2}+\frac{\mathbf{j}}{2}-\frac{\mathbf{k}}{\sqrt{2}}$ (c) $-\frac{\mathbf{i}}{2}-\frac{\mathbf{j}}{2}+\frac{\mathbf{k}}{\sqrt{2}}$ (d) None of these **Sol: (c)**

Let $\mathbf{a}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$, where $l^2+m^2+n^2=1$.

a makes an angle $\frac{\pi}{4}$ with $z-$axis.

$\therefore n=\frac{1}{\sqrt{2}},\ l^2+m^2=\frac{1}{2}$ .....(i)

$\therefore \mathbf{a}=l\,\mathbf{i}+m\,\mathbf{j}+\frac{\mathbf{k}}{\sqrt{2}}$

$\mathbf{a}+\mathbf{i}+\mathbf{j}=(l+1)\mathbf{i}+(m+1)\mathbf{j}+\frac{\mathbf{k}}{\sqrt{2}}$

Its magnitude is 1, hence $(l+1)^2+(m+1)^2=\frac{1}{2}$ .....(ii)

$l^2+1+2l+m^2+1+2m=\frac{1}{2}\Longrightarrow\frac{1}{2}+2+2l+2m=\frac{1}{2}\Rightarrow l+m=-1\Rightarrow l=m=\frac{-1}{2}$

From (i) and (ii),

Hence $\mathbf{a}=-\frac{\mathbf{i}}{2}-\frac{\mathbf{j}}{2}+\frac{\mathbf{k}}{\sqrt{2}}$.

10. A force is a

(a) Unit vector (b) Localized vector (c) Zero vector (d) Free vector **Sol: (b)**

It is a fundamental concept.

ASSERTION AND REASON BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason(R).Pick the correct option:

A) Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).
B) Both Assertion (A) and Reason(R) are true but Reason(R) is NOT the correct explanation of Assertion (A).
C) Assertion (A) is true but Reason(R) is false.
D) Assertion (A) is false but Reason(R) is true.

1. **Assertion :** $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$ then $\vec{b}=\lambda\vec{a}+\vec{c}$

Reason : If $\vec{a}\times\vec{b}=0$, $\vec{a}$ is collinear to $\vec{b}$

Sol. **[A]** $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, $\vec{a}\times(\vec{b}-\vec{c})=0$

i.e. $\vec{b}-\vec{c}=\lambda\vec{a}\ \Rightarrow\vec{b}=\lambda\vec{a}+\vec{c}$

2. **Assertion :** If $\bar{a}\times\bar{b}=\bar{c}\times\bar{d}$, and $\bar{a}\times\bar{c}=\bar{b}\times\bar{d}$ then $\bar{a}-\bar{d}$ is perpendicular to $\bar{b}-\bar{c}$.

Reason : If $\bar{r}$ is perpendicular to $\bar{q}$ then $\bar{r}.\bar{q}=0$

Sol.[D](Assertion false & reason is true)

$\bar{a}\times\bar{b}=\bar{c}\times\bar{d}$ … (1)

$\bar{a}\times\bar{c}=\bar{b}\times\bar{d}$ … (2)

subtract

$\bar{a}\times(\bar{b}-\bar{c})=(\bar{c}-\bar{b})\times\bar{d}$;$(\bar{a}-\bar{d})\times(\bar{b}-\bar{c})=\vec{\mathbf{0}}$; So $\bar{a}-\bar{d}$ is parallel to $\bar{b}-\bar{c}$

3. **Assertion :** If three points P, Q, R have position vectors $\vec{a},\vec{b},\vec{c}$ respectively and $2\vec{a}+3\vec{b}-5\vec{c}=0$, then the points P, Q, R must be collinear.

Reason: If for three points A, B, C, ; $\overrightarrow{AB}=\lambda\,\overrightarrow{AC}$, then the points A, B, C must be collinear.

Sol. [A] $\overrightarrow{2a}+\overrightarrow{3b}-\overrightarrow{5c}=0$, $3(\vec{b}-\vec{a})=5(\vec{c}-\vec{a})$

$$\vec{b}-\vec{a}=5/3(\vec{c}-\vec{a}) \ , \ \overrightarrow{AB}=5/3\ \overrightarrow{AC}$$

$\overrightarrow{AB}$ & $\overrightarrow{AC}$ must be parallel since there is common point A. The points A, B , C must be collinear.

4. **Assertion:** If the difference of two unit vectors is again a unit vector then angle between them is 60^{o}
Reason : If angle between $\vec{a}$ & $\vec{b}$ is acute than $|\vec{a}.\vec{b}| < |\vec{a}|\,|\vec{b}|$
Sol. [B]
5. **Assertion :** $\vec{a}=\hat{i}+p\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+q\hat{k}$ are parallel vector. If p =3/2, q = 4.
Reason: If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are parallel $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$
Sol. [A]
6. **Assertion:** If $\vec{a}$ & $\vec{b}$ are unit vectors & θ is the angle between them, then $\sin\frac{\theta}{2}=\frac{|\vec{a}-\vec{b}|}{2}$
Reason : The number of vectors of unit length perpendicular to the vectors $\vec{a}=\hat{i}+\hat{j}$ & $\vec{b}=\hat{j}+\hat{k}$ is two :
Sol.[B] Assertion: $|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\cos\theta=2(1-\cos\theta)$ $(|\vec{a}|=|\vec{b}|=1)$
$|\vec{a}-\vec{b}|^2=2.2\sin^2\frac{\theta}{2} \Rightarrow \sin\frac{\theta}{2}=\frac{|\vec{a}-\vec{b}|}{2}$
Reason: Number of vectors of unit length perpendicular to the vectors $\vec{a}$ & $\vec{b}$ are two,
i.e. $\pm\frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$
7. **Assertion:** If $\vec{a}\times\vec{b}=\vec{0}$ and $\vec{a}.\vec{b}=\vec{\mathbf{0}}$ then either $\vec{a}=\vec{\mathbf{0}}$ or $\vec{b}=\vec{\mathbf{0}}$
Reason: $\vec{a}\times\vec{b}=\vec{\mathbf{0}}$, $\vec{a}=\vec{\mathbf{0}}$, $\vec{b}=\vec{\mathbf{0}}$ or $\vec{a}$ is parallel to $\vec{b}$.
$\vec{a}.\vec{b}=0$, $\vec{a}=\vec{\mathbf{0}}$, $\vec{b}=\vec{\mathbf{0}}$ or $\vec{a}$ is perpendicular to $\vec{b}$.
Sol: A
Since $\vec{a}$ cannot be both parallel and perpendicular to $\vec{b}$, we have $\vec{a}=0$ or $\vec{b}=0$.
8. **Assertion:** If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then equation $\vec{r}\times(2\hat{i}-\hat{j}+3\hat{k})=3\hat{i}+\hat{k}$ represent a straight line.
Reason: If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then equation $\vec{r}\times(\hat{i}+2\hat{j}-3\hat{k})=2\hat{i}-\hat{j}$ represent a straight line

Sol.[D] Reason:

$$\vec{r}\times(\hat{i}+2\hat{j}-3\hat{k})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\x&y&z\\1&2&-3\end{vmatrix}=\hat{i}(-3y-2z)-\hat{j}(-3x-z)+\hat{k}(2x-y)$$

$-3y-2z=2,\ 3x+z=-1,\ 2x-y=0$
i.e. $-6x+2z=2,\ 3x+z=-1$
Straight line $2x-y=0,\ 3x+z=-1$

Assertion :

$$\vec{r}\times(2\hat{i}-\hat{j}+3\hat{k})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\x&y&z\\2&-1&3\end{vmatrix}=\hat{i}(3y+z)-\hat{j}(3x-2z)+\hat{k}(-x-2y)$$

$3y + z = 3$, $3x - 2z = 0$, $-x - 2y = 1$, $3x - 2(3 - 3y) = 0$, $3x + 6y = 6$, $x + 2y = 2$

Now, $x + 2y = -1$, $x + 2y = 2$ are parallel planes

$\vec{r} \times (2\hat{i} - \hat{j} + 3\hat{k}) = 3\hat{i} + \hat{k}$ is not a straight line

9. **Assertion**: Angle between $\hat{\imath}+\hat{\jmath}$ and $\hat{\imath}$ is 45^0

Reason: $\hat{\imath}+\hat{\jmath}$ is equally inclined to both $\hat{\imath}$ and $\hat{\jmath}$ and the angle between $\hat{\imath}$ and $\hat{\jmath}$ is 90^0

Sol: B $\Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

10. **Assertion:** If $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ and $\vec{b} = -\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$, then $|\vec{a}| = |\vec{b}|$

Reason: If $|\vec{a}| = |\vec{b}|$, then it necessarily implies that $\vec{a} = \pm\vec{b}$. **Sol. [C]**

$|\vec{a}| = |\vec{b}|$ need not imply $\vec{a} = \pm\vec{b}$, eg: $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ and $\vec{b} = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$

11. **Assertion:** For two nonzero vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

Reason: For two non-zero vectors $\vec{a}$ and $\vec{b}$, we have $\vec{a} \times \vec{b} = \vec{b} \times \vec{a}$

Sol: [C] Fundamental concept

VERY SHORT ANSWER TYPE QUESTIONS

1. Given $\overrightarrow{AB} = 3\hat{i} - \hat{j} - 5\hat{k}$ and co-ordinate of the terminal point is (0, 1, 3). Find the coordinate of the initial point.

SOL: Given $\overrightarrow{AB} = 3\hat{i} - \hat{j} - 5\hat{k}$

B (0, 1, 3) $\Rightarrow$ $\overrightarrow{OB} = \hat{j} + 3\hat{k}$ then $\overrightarrow{OA} = ?$

We know that,

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $\overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{AB}$

$\overrightarrow{OA} = \hat{j} + 3\hat{k} - (3\hat{i} - \hat{j} - 5\hat{k}) = \hat{j} + 3\hat{k} - 3\hat{i} + \hat{j} + 5\hat{k} - = -3\hat{i} + 2\hat{j} + 8\hat{k}$

2. If $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, find the value of $\vec{a} \bullet \vec{b}$.

SOL: Given $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, $\vec{a} \bullet \vec{b} = ?$

$|\vec{a}||\vec{b}| \sin\theta = 8 \Rightarrow (2)(5)\sin\theta = 8$

Then, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta = (2)(5)\frac{3}{5} = 6$

3. If $\vec{a}$, $\vec{b}$ are any two unit vectors and θ is the angle between them, then show that $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\vec{a} - \vec{b}|$

SOL: Given $|\vec{a}| = 1$, $|\vec{b}| = 1$, $\vec{a}.\vec{a} = |\vec{a}||\vec{a}|\cos 0° = |\vec{a}|^2 = a^2$

$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a} \bullet \vec{b} = (1)^2 + (1)^2 - 2|\vec{a}||\vec{b}|\cos\theta$

$= 1 + 1 - 2(1)(1)\cos\theta = 2(1 - \cos\theta)$

$= 2\left(2\sin^2\frac{\theta}{2}\right) \Rightarrow |\vec{a} - \vec{b}| = 2\sin\frac{\theta}{2}$, $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}|\vec{a} - \vec{b}|$

4. If $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 5\hat{k}$, find a unit vector in the direction of $\vec{a} - \vec{b}$

SOL: Given: $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 5\hat{k}$

Then, $\vec{a} - \vec{b} = \hat{i} + 2\hat{j} - \hat{k} - (3\hat{i} + \hat{j} - 5\hat{k}) = -2\hat{i} + \hat{j} + 4\hat{k} = \vec{c}$ (say)

A unit vector in the direction of $\vec{c}$ is $\frac{\vec{c}}{|\vec{c}|}$

Unit vector in the direction $of\ \vec{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{-2\hat{i} + \hat{j} + 4\hat{k}}{\sqrt{4+1+16}} = \frac{-2\hat{i}}{\sqrt{21}} + \frac{\hat{j}}{\sqrt{21}} + \frac{4\hat{k}}{\sqrt{21}}$

5. **If the position vectors of the points A and B are $2\hat{i} + 3\hat{j} - \hat{k}$ and $3\hat{i} + 2\hat{j} + \hat{k}$ then find the vector of magnitude 6 units in the direction of $\overrightarrow{AB}$**

SOL: Given $\overrightarrow{OA} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\overrightarrow{OB} = 3\hat{i} + 2\hat{j} + \hat{k}$ then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \hat{i} - \hat{j} + 2\hat{k}$

A Unit vector in the direction of $\overrightarrow{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{1+1+4}} = \frac{\hat{i}}{\sqrt{6}} - \frac{\hat{j}}{\sqrt{6}} + \frac{2\hat{k}}{\sqrt{6}}$

The vector of magnitude 6 units in the direction of $\overrightarrow{AB} = 6\left(\frac{\hat{i}}{\sqrt{6}} - \frac{\hat{j}}{\sqrt{6}} + \frac{2\hat{k}}{\sqrt{6}}\right)$

6. **If P (1, 5, 4) and Q (4, 1, -2), find the direction ratios of $\overrightarrow{PQ}$**

SOL: Let P (1, 5, 4) and Q (4, 1, -2), then $\overrightarrow{OP} = \hat{i} + 5\hat{j} + 4\hat{k}$ $\quad \overrightarrow{OQ} = 4\hat{i} + \hat{j} - 2\hat{k}$

and $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 3\hat{i} - 4\hat{j} - 6\hat{k}$.

The direction ratios of $\overrightarrow{PQ}$ = 3, – 4, – 6

7. **Find λ, if the vectors $3\hat{i} + \hat{j} - 5\hat{k}$ and $3\hat{i} + \hat{j} - \lambda\hat{k}$ are parallel**

SOL: The vectors $3\hat{i} + \hat{j} - 5\hat{k}$ and $3\hat{i} + \hat{j} - \lambda\hat{k}$ are parallel

Then, $\frac{3}{3} = \frac{1}{1} = \frac{-5}{-\lambda}$ $\qquad$ So, $\lambda = 5$

8. **Find λ, if the vectors $3\hat{i} - \hat{j} - 5\hat{k}$ and $2\hat{i} + 3\hat{j} - \lambda\hat{k}$ are perpendicular**

SOL: Let $\vec{a} = 3\hat{i} - \hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - \lambda\hat{k}$

If $\vec{a}$ and $\vec{b}$ are perpendicular then $\vec{a} \bullet \vec{b} = 0$

$\rightarrow 3(2) - 1(3) - 5(-\lambda) = 0 \Rightarrow 5\lambda = -3$ then $\lambda = \frac{-3}{5}$

9. **If $\vec{a} = 4\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 5\hat{i} + 2\hat{j} - 3\hat{k}$ find the angle between $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$**

SOL: Let $\vec{a} = 4\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 5\hat{i} + 2\hat{j} - 3\hat{k}$

Then $\vec{a} + \vec{b} = 9\hat{i} + 4\hat{j} - 4\hat{k} = \vec{c}$, $\qquad \vec{a} - \vec{b} = -\hat{i} + 2\hat{k} = \vec{d}$

So, the angle between $\vec{c}$ and $\vec{d}$ = $\cos\theta = \frac{\vec{c}.\vec{d}}{|\vec{c}||\vec{d}|} = \frac{9(-1) + 4(0) - 4(2)}{\sqrt{81+16+16}\ \sqrt{1+4}} = \frac{(-17)}{\sqrt{113}\sqrt{5}}$

10. **Find the projection of $\vec{a} = \hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$**

SOL: Given the projection of $\vec{a} = \hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$

Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{1(2) + 1(1) + 4(2)}{\sqrt{4+1+4}} = \frac{11}{3}$

SHORT ANSWER TYPE QUESTIONS

1. **Find the position vector of a point R which divided the line segment joining the points P and Q with position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ration 2 : 1 i) internally ii) externally.**

SOL: The position vector of a point R which divided the line segment joining the points P and Q with position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ration 2 : 1

Position Vector of a Point R is $\overrightarrow{OR} = \frac{m\vec{b} + n\vec{a}}{m+n} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2+1} = \frac{-2\hat{i} + 2\hat{j} + 2\hat{k} + \hat{i} + 2\hat{j} - \hat{k}}{3} = \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3}$

2. **Show that the points A (2, 6, 3), B (1, 2, 7) and C (3, 10, -1) are collinear.**

SOL: Given points A (2, 6, 3), B (1, 2, 7) and C (3, 10, –1)

Then, $\overrightarrow{OA} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, $\quad \overrightarrow{OB} = \hat{i} + 2\hat{j} + 7\hat{k}$, $\quad \overrightarrow{OC} = 3\hat{i} + 10\hat{j} - \hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\hat{i} - 4\hat{j} + 4\hat{k}$ $\qquad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \hat{i} + 4\hat{j} - 4\hat{k}$

$\overrightarrow{AB} = (-1)\,\overrightarrow{AC}$, $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$ and A is common point. (Alter D.Rs of two vectors are in proportional) So, the given points A, B, C are collinear

3. **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three - unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.**

SOL: Given $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$ $\quad \vec{a} + \vec{b} = -\vec{c}$

By pre cross multiplication of $\vec{a}$ and b on both the sides respectively, we have

$\vec{a} \times (\vec{a} + \vec{b}) = -\vec{a} \times \vec{c}$ implies $\vec{a} \times \vec{a} + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$ that is $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ….(1)

$\vec{b} \times (\vec{a} + \vec{b}) = -\vec{b} \times \vec{c} \Rightarrow \vec{b} \times \vec{a} + \vec{b} \times \vec{b} = -\vec{b} \times \vec{c} \Rightarrow \vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ….(2)

From (1) and (2), we conclude that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

4. **Find the area of the parallelogram with diagonals $3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$.**

SOL: If $\overrightarrow{d_1}$ and $\overrightarrow{d_2}$ are diagonals of Parallelogram then Area = $\frac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right|$

$$\overrightarrow{d_1} \times \overrightarrow{d_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 2 \\ 1 & -3 & 4 \end{vmatrix}$$

. $= \hat{\imath}(4+6) - \hat{\jmath}(12-2) + \hat{k}(-9-1) = 10\hat{\imath} - 10\hat{\jmath} - 10\hat{k}$

$\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right| = \sqrt{100+100+100} = 10\sqrt{3}$

$$\text{Area of Parallelogram} = \frac{1}{2}\left|\overrightarrow{d_1} \times \overrightarrow{d_2}\right| = 5\sqrt{3} \text{ square units}$$

5. **Find the unit vector perpendicular to vector $\vec{a} = \hat{\imath} - 7\hat{\jmath} + 7\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$**

SOL: Given $\vec{a} = \hat{\imath} - 7\hat{\jmath} + 7\hat{k}$, $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

A Unit vector perpendicular to $\vec{a}$ and $\vec{b} = \hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

. $= \hat{\imath}(-14+14) - \hat{\jmath}(2-21) + \hat{k}(-2+21) = 19\hat{\jmath} - 19\hat{k}$

$\left|\vec{a} \times \vec{b}\right| = \sqrt{(19)^2 + (19)^2} = 19\sqrt{2}$

A Unit vector perpendicular to $\vec{a}$ and $\vec{b} = \hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{19\hat{\jmath} - 19\hat{k}}{19\sqrt{2}} = \frac{\hat{\jmath} - \hat{k}}{\sqrt{2}}$

6. **Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\,(\vec{a} \times \vec{b})$**

SOL: LHS = $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$ Note: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$= (\vec{a} \times \vec{a}) + (\vec{a} \times \vec{b}) - (\vec{b} \times \vec{a}) - (\vec{b} \times \vec{b}) = 0 + (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{b}) - 0$

$= 2\,(\vec{a} \times \vec{b})$

7. **If $\vec{a}$, $\vec{b}$ are any two unit vectors and θ is the angle between them, then show that $Cos\left(\frac{\theta}{2}\right) = \frac{1}{2}\left|\vec{a} + \vec{b}\right|$**

SOL: Given $|\vec{a}| = 1$, $|\vec{b}| = 1$, we have $\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}| \operatorname{Cos} 0° = |\vec{a}|^2 = a^2$

$\left|\vec{a} + \vec{b}\right|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a} \bullet \vec{b}$

$= (1)^2 + (1)^2 + 2\,|\vec{a}||\vec{b}| \operatorname{Cos}\theta; \; = 1 + 1 + 2\,(1)(1) \operatorname{Cos}\theta$

$= 2(1 + \operatorname{Cos}\theta) \; = 2\left(2 \operatorname{Cos}^2 \frac{\theta}{2}\right)$

$\left|\vec{a} + \vec{b}\right| = 2 \operatorname{Cos}\frac{\theta}{2}$ then $\operatorname{Cos}\left(\frac{\theta}{2}\right) = \frac{1}{2}\left|\vec{a} + \vec{b}\right|$

8. **If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are 3 vectors with magnitudes 3,4 and 5 respectively and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ find $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$**

SOL: Given $|\vec{a}|=3$, $|b|=5$, $|\vec{c}|=4$, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$(\vec{a}+\vec{b}+\vec{c})^2=(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$$

$$=|\vec{a}|^2+|b|^2+|\vec{c}|^2+2\vec{a}\bullet\vec{b}+2\vec{b}\bullet\vec{c}+2\vec{c}\bullet\vec{a}$$

$$0=9+25+16+2\left(\vec{a}\bullet\vec{b}+\vec{b}\bullet\vec{c}+\vec{c}\bullet\vec{a}\right)$$

$$\vec{a}\bullet\vec{b}+\vec{b}\bullet\vec{c}+\vec{c}\bullet\vec{a}=\frac{-50}{2}=-25$$

9. **If $\vec{a}=2\hat{\imath}+2\hat{\jmath}+3\hat{k}$ $\vec{b}=-\hat{\imath}+2\hat{\jmath}+\hat{k}$ and $\vec{c}=3\hat{\imath}+\hat{\jmath}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$ then find the value of λ**

SOL: Given $\vec{a}=2\hat{\imath}+2\hat{\jmath}+3\hat{k}$ $\vec{b}=-\hat{\imath}+2\hat{\jmath}+\hat{k}$ $\vec{c}=3\hat{\imath}+\hat{\jmath}$

$$\vec{a}+\lambda\vec{b}=2\hat{\imath}+2\hat{\jmath}+3\hat{k}+\lambda(-\hat{\imath}+2\hat{\jmath}+\hat{k})=(2-\lambda)\hat{\imath}+(2+2\lambda)\hat{\jmath}+(3+\lambda)\hat{k}$$

$\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}\Rightarrow(\vec{a}+\lambda\vec{b}).\vec{c}=0$

$$(3)(2-\lambda)+(1)(2+2\lambda)+(0)(3+\lambda)=0\Rightarrow 6-3\lambda+2+2\lambda=0\Rightarrow\lambda=8$$

10. **If $\vec{a}=\hat{\imath}+\alpha\hat{\jmath}+2\hat{k}$, $\vec{b}=\beta\hat{\imath}+\hat{\jmath}-\hat{k}$ are orthogonal and $|\vec{a}|=|\vec{b}|$ then find the values of α and β.**

SOL: Given $\vec{a}=\hat{\imath}+\alpha\hat{\jmath}+2\hat{k}$, $\vec{b}=\beta\hat{\imath}+\hat{\jmath}-\hat{k}$ are orthogonal and $|\vec{a}|=|\vec{b}|$

$$\vec{a}.\vec{b}=0\Rightarrow 1.\beta+\alpha.1+2.-1=0\Rightarrow\beta+\alpha=2---(1)\ and$$

$$\sqrt{1^2+\alpha^2+2^2}=\sqrt{\beta^2+1^2+(-1)^2}\Rightarrow\alpha^2+5=\beta^2+2\Rightarrow\beta^2-\alpha^2=+3$$

Implies $\beta-\alpha=\frac{3}{2}---(2)$ by solving (1) and (2) $\alpha=\frac{1}{4},\beta=\frac{7}{4}$

CASE BASED TYPE QUESTIONS

1. **Relative to a fixed origin O, the points A, B and C have respective position vectors $\hat{\imath}+10\hat{k}$, $4\hat{\imath}+3\hat{\jmath}+7\hat{k}$ and $8\hat{\imath}+7\hat{\jmath}+3\hat{k}$.**
   (i) Show that A,B and C are collinear, and find the ration AB:BC.
   (ii) Calculate the area of the triangle OAC.
   Sol:

(i) $\overrightarrow{AB}=(4\hat{\imath}+3\hat{\jmath}+7\hat{k})-(\hat{\imath}+10\hat{k})=3\hat{\imath}+3j-3\hat{k}$

$\overrightarrow{BC}=(8\hat{\imath}+7\hat{\jmath}+3\hat{k})-(4\hat{\imath}+3\hat{\jmath}+7\hat{k})=4\hat{\imath}+4j-4\hat{k}$

Clearly, the direction ratios of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, B is a common point, so A,B and C are collinear. Ratio of AB:BC is 3:4

(ii) Area of triangle ABC$=\frac{1}{2}|AC||OB|$

$$=\frac{1}{2}|(OC)-(OA)|\sqrt{4^2+3^2+7^2}=\frac{1}{2}|(7\hat{\imath}+7\hat{\jmath}-7\hat{k})|\sqrt{74}$$

$$=\frac{1}{2}\sqrt{10878}=\frac{7}{2}\sqrt{222}\ sq.units$$

2. **Let $\vec{a}=2\hat{i}+3\hat{j}-6\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$ and $\vec{c}=-2\hat{i}+3\hat{j}+6\hat{k}$. Let $\vec{a}_1$ be projection of $\vec{a}$ on $\vec{b}$ and $\vec{a}_2$ be the projection of $\vec{a}_1$ on $\vec{c}$, then**

   (i) Find the vector $\vec{a}_2$ (ii) Find the value of $\vec{a}_1.\vec{b}$

Sol.

(i) $\vec{a}_1 = \left[(2\hat{i}+3\hat{j}-6\hat{k}).\frac{(2\hat{i}-3\hat{j}+6\hat{k})}{7}\right]\frac{2\hat{i}-3\hat{j}+6\hat{k}}{7} = \frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})$

$\vec{a}_2 = \frac{-41}{49}\left[(2\hat{i}-3\hat{j}+6\hat{k}).\frac{(-2\hat{i}+3\hat{j}+6\hat{k})}{7}\right]\frac{-2\hat{i}+3\hat{j}+6\hat{k}}{7} = \frac{-41}{(49)^2}(-4-9+36)\ (-2\hat{i}+3\hat{j}+6\hat{k})$

$= \frac{943}{49^2}\ (2\hat{i}-3\hat{j}-6\hat{k})$

(ii) $.\vec{a}_1.\vec{b} = \frac{-41}{49}(2\hat{i}-3\hat{j}+6\hat{k})(2\hat{i}-3\hat{j}+6\hat{k}) = -41$.

3. If each of $\vec{a}$, $\vec{b}$, $\vec{c}$ is orthogonal to the sum of the two. Also given $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5$. On the basis of above information, answer the following questions.

(i) If $\vec{a}$ makes angles of equal measures with all three axes, then the tangent of angle becomes.

(ii) If $\vec{a}.\vec{c} = 9$ then the value of $|\vec{a}\times\vec{b}|$

(iii) Find the range of the value $|\vec{a}-\vec{b}|$ (OR)

Find the Value of $|\vec{a}+\vec{b}+\vec{c}|$

Sol: Given that, **$|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5$-------(1)**

And also, $\vec{a}$, $\vec{b}$, $\vec{c}$ is orthogonal to the sum of the two then

$\vec{a}.(\vec{b}+\vec{c}) = 0 \Rightarrow \vec{a}.\vec{b}+\vec{a}.\vec{c} = 0$

$\vec{b}.(\vec{c}+\vec{a}) = 0 \Rightarrow \vec{b}.\vec{c}+\vec{b}.\vec{a} = 0$

$\vec{c}.(\vec{a}+\vec{b}) = 0 \Rightarrow \vec{c}.\vec{a}+\vec{c}.b = 0$

So, by adding above equations then

$2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0----(2)$

(i) If $\vec{a}$ makes angles of equal measures with all three axes then

$3cos^2\theta = 1 \Rightarrow cos\theta = \pm\frac{1}{\sqrt{3}}$ Then $Tan\theta = \pm\sqrt{2}$

(ii) If $\vec{a}.\vec{c} = 9 \Rightarrow 3.4\, cos(\vec{a},\vec{c}) = 9 \Rightarrow cos\alpha = \frac{3}{4}$ $(\vec{a}.\vec{b}+\vec{a}.\vec{c} = 0)$

$\vec{a}.\vec{b} = -\vec{a}.\vec{c} = -9 \Rightarrow |\vec{a}||\vec{b}|cos(\vec{a},\vec{b}) = -9 \Rightarrow cos(\vec{a},\vec{b}) = \frac{-3}{4}$

Then, $\sin\left((\vec{a},\vec{b})\right) = \frac{\sqrt{7}}{4}$ $|\vec{a}\times\vec{b}| = 3.4\, sin\alpha = 12.\frac{\sqrt{7}}{4} = 3\sqrt{7}$

then the value of $|\vec{a}\times\vec{b}| = 3\sqrt{7}$

(iii) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - 2(\vec{a}.\vec{b}) = 9.16 - 2.3.4(cos\theta) = 25 - 24[-1,1] = [1,49]$

$|\vec{a}-\vec{b}|$**=[1,7] (OR)**

$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$

$= 9+16+25+0 = 50$

Then, $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$

CHAPTER-11- THREE DIMENSIONAL GEOMETRY

Gist/Summary of the lesson (Definitions and Formulae)

Condition of perpendicular: If the given lines are perpendicular, then $\theta = 90°$ *i.e.,* $\cos\theta = 0$

$\Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Condition of parallelism: If the given lines are parallel, then $\theta = 0^o$ *i.e.,* $\sin\theta = 0 \Rightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

Similarly, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Intersection of two lines

Determine whether two lines intersect or not. In case they intersect, the following algorithm is used to find their point of intersection.

Algorithm:

Let the two lines be $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ ..(i) and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ …..(ii)

Step I : Write the co-ordinates of general points on (i) and (ii). The co-ordinates of general points on (i) and (ii) are given by $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} = \lambda$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} = \mu$ respectively.

i.e., $(a_1\lambda + x_1, b_1\lambda + y_1 + c_1\lambda + z_1)$ and $(a_2\mu + x_2, b_2\mu + y_2, c_2\mu + z_2)$.

Step II : If the lines (i) and (ii) intersect, then they have a common point.

$a_1\lambda + x_1 = a_2\mu + x_2, b_1\lambda + y_1 = b_2\mu + y_2$

and $c_1\lambda + z_1 = c_2\mu + z_2$.

Step III : Solve any two of the equations in λ and μ obtained in step II. If the values of λ and μ satisfy the third equation, then the lines (i) and (ii) intersect, otherwise they do not intersect.

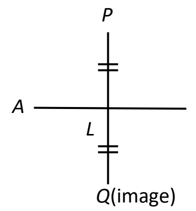
Step IV : To obtain the co-ordinates of the point of intersection, substitute the value of λ (or μ) in the co-ordinates of general point (*s*) obtained in step I.

Foot of perpendicular from a point $A(\alpha, \beta, \gamma)$ to the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$: If *P* be the foot of perpendicular, then *P* is $(lr + x_1, mr + y_1, nr + z_1)$. Find the direction ratios of *AP* and apply the condition of perpendicularity of *AP* and the given line. This will give the value of *r* and hence the point *P,* which is foot of perpendicular.

Length and equation of perpendicular : The length of the perpendicular is the distance *AP* and its equation is the line joining two known points *A* and *P*.

The length of the perpendicular is the perpendicular distance of given point from that line.

Reflection or image of a point in a straight line : If the perpendicular *PL* from point *P* on the given line be produced to *Q* such that *PL* = *QL*, then *Q* is known as the image or reflection of *P* in the given line. Also, *L* is the foot of the perpendicular or the projection of *P* on the line.



- ⇨ The number of lines which are equally inclined to the co-ordinate axes is 4.
- ⇨ If *l, m, n* are the d.c.'s of a line, then the maximum value of $lmn = \frac{1}{3\sqrt{3}}$.

Distance between two skew lines

(i)$\overrightarrow{r_1} = \vec{a}_1 + \lambda\overrightarrow{b_1} \& \overrightarrow{r_2} = \vec{a}_2 + \mu\vec{b}_2$ *is* $\left|\frac{(\vec{b}_1 \times \vec{b}_2)(\overrightarrow{a_2} - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$ units

(ii) $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ &) $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is $\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$

(iii)Parallel lines $\vec{r_1}=\vec{a}_1+\lambda\vec{b}$ & $\vec{r_2}=\vec{a}_2+\mu\vec{b}$ is $|\frac{\vec{b}\times(\vec{a_2}-\vec{a}_1)}{|\vec{b}|}|$ units

Angle Between two lines

If 'θ' is the acute angle between $\vec{r_1}=\vec{a}_1+\lambda\vec{b_1}$ & $\vec{r_2}=\vec{a}_2+\mu\vec{b}_2$ then,

cos θ= $\left|\frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$.

If $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2}=\frac{y-y_1}{m_2}=\frac{z-z_1}{n_2}$ are the equations of two lines,then acute angle between them is cosθ= $|l_1l_2+n_1m_2+n_1n_2|$

Activity to remember the main concepts:

Given form	Standard form of a line in Cartesian form	Standard form of a line in Vector form	Figure	D.r's of a line are	Any point on the line
$\vec{r}=(2+3s)\vec{i}-(s-2)\vec{j}+(1-s)\vec{k}$					
$\frac{2-x}{-3}=\frac{2y-3}{4}=z$					
3x=2y=z					
x=ay+b; z=cy+d					
$\frac{x-2}{1}=\frac{2-y}{-3}; z=2$					

MULTIPLE CHOICE TYPE QUESTIONS

1. **If α, β, γ be the angles which a line makes with the positive direction of co-ordinate axes, then $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=$**

   (a) 2 (b) 1 (c) 3 (d) 0

   Sol: a) Since $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

   $\Rightarrow\Sigma\sin^2\alpha=3-1=2$.

2. **If α,β,γ be the direction angles of a vector and $\cos\alpha=\frac{14}{15}$, $\cos\beta=\frac{1}{3}$ then $\cos\gamma=$**

   (a) $\pm\frac{2}{15}$ (b) $\frac{1}{5}$ (c) $\pm\frac{1}{15}$ (d) None of these

   Solution: (a)$cos^2\alpha+cos^2\beta+cos^2\gamma=1\Rightarrow cos\,\gamma=\sqrt{1-\left(\frac{14}{15}\right)^2-\left(\frac{1}{3}\right)^2}=\sqrt{\frac{8}{9}-\left(\frac{196}{225}\right)}=\pm\frac{2}{15}$.

3. **The direction cosines of the line $x=y=z$ are**

   (a) $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$ (b) $\frac{1}{3},\frac{1}{3},\frac{1}{3}$ (c) 1, 1, 1 (d) None of these

   Sol: (a) Direction cosines $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

4. **If a line makes angles of 30^o and 45^o with x-axis and y-axis, then the angle made by it with $z-$ axis is**

(a) 45^o (b) 60^o (c) 120^o (d) None of these

Solution: (d) $cos\,\gamma = \sqrt{1-\frac{3}{4}-\frac{1}{2}} = \sqrt{\frac{-1}{4}}$, which is not possible.

5. **If the co-ordinates of the points P, Q, R, S be (1, 2, 3), (4, 5, 7), (– 4, 3, – 6) and (2, 0, 2) respectively, then**

(a) $PQ \| RS$ (b) $PQ \perp RS$ (c) $PQ = RS$ (d) None of these

Solution: (d)Find angle between the lines PQ and RS, we get that neither $PQ \| RS$ nor $PQ \perp RS$. Also $PQ \neq RS$.

6. **If the projections of a line on the co-ordinate axes be 2, –1, 2, then the length of the lines is**

(a) 3 (b) 4 (c) 2 (d) $\frac{1}{2}$

Solution: (a) $r = \sqrt{4+1+4} = 3$.

7. **A line makes angles α, β, γ with the co-ordinate axes. If $\alpha + \beta = 90^o$, then $\gamma =$**

(a) 0 (b) $90°$ (c) $180°$ (d) None of these

Solution: (b) Here, $cos^2\,\alpha + cos^2(90-\alpha) + cos^2\,\gamma = 1$

$\Rightarrow \cos^2\alpha + \sin^2\alpha + \cos^2\gamma = 1 \quad \Rightarrow \cos^2\gamma + 1 = 1 \Rightarrow \gamma = 90^o$.

8. **Points (–2, 4, 7), (3, –6, –8) and (1, –2, –2) are**

(a) Collinear (b) Vertices of an equilateral triangle

(c) Vertices of an isosceles triangle (d) None of these

Solution: (a) Here, $\frac{(3-(-2))}{1-3} = \frac{-6-4}{-2-(-6)} = \frac{-8-7}{-2-(-8)}$

$\Rightarrow -\frac{5}{2} = -\frac{5}{2} = -\frac{5}{2}$. Obviously, points are collinear.

9. **The direction ratios of the line joining the points (4, 3, –5) and (–2, 1, –8) are**

(a) $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$ (b) 6, 2, 3 (c) 2, 4, –13 (d) None of these

Solution: (b) Direction ratios are, $l = 4-(-2) = 6$, $m = 3-1 = 2$ and $n = -5+8 = 3$

a=6, b=2, c=3

10. **The co-ordinates of a point which is equidistant from the points $(\mathbf{0}, \mathbf{0}, \mathbf{0}), (\boldsymbol{a}, \mathbf{0}, \mathbf{0}), (\mathbf{0}, \boldsymbol{b}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{0}, \boldsymbol{c})$ are given by**

(a) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (b) $\left(-\frac{a}{2}, -\frac{b}{2}, \frac{c}{2}\right)$ (c) $\left(\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right)$ (d) $\left(-\frac{a}{2}, \frac{b}{2}, -\frac{c}{2}\right)$

Solution: (a) Let point be (x, y, z), then $r^2 = x^2 + y^2 + z^2$

$= (x-a)^2 + y^2 + z^2 = x^2 + (y-b)^2 + z^2 = x^2 + y^2 + (z-c)^2$

Therefore $x = \frac{a}{2}, y = \frac{b}{2}$ and $z = \frac{c}{2}$.

11. **The angle between the lines $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ is**

(a) $cos^{-1}\left(\frac{-1}{5}\right)$ (b) $\cos^{-1}\frac{1}{3}$ (c) $\cos^{-1}\frac{1}{2}$ (d) $\cos^{-1}\frac{1}{4}$

Solution: (a) $\theta = cos^{-1}\left(\frac{3+0-5}{\sqrt{1+1}\sqrt{9+16+25}}\right) = cos^{-1}\left(\frac{-2}{10}\right) = cos^{-1}\left(-\frac{1}{5}\right)$.

12. **If $\frac{x-1}{l} = \frac{y-2}{m} = \frac{z+1}{n}$ is the equation of the line through (1, 2, –1) and (–1, 0, 1), then (l, m, n) is**

(a) (–1, 0, 1) (b) (1, 1, –1) (c) (1, 2, –1) (d) (0, 1, 0)

Solution: (b) $\frac{-2}{l} = \frac{-2}{m} = \frac{2}{n}$; $\therefore$ (l, m, n) are $(1, 1, -1)$.

13. **The point of intersection of lines $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is**

(a) (–1, –1, –1) (b) (–1, –1, 1) (c) (1, –1, –1) (d) (–1, 1, –1)

Solution: (a) **Trick :** Both lines are satisfied by (–1,–1, –1).

14. **The angle between the lines whose direction cosines are proportional to (1, 2, 1) and (2, –3, 6) is**

(a) $\cos^{-1}\left(\frac{2}{7\sqrt{6}}\right)$ (b) $\cos^{-1}\left(\frac{1}{7\sqrt{6}}\right)$ (c) $\cos^{-1}\left(\frac{3}{7\sqrt{6}}\right)$ (d) $\cos^{-1}\left(\frac{5}{7\sqrt{6}}\right)$

Solution: (a) $\theta = cos^{-1}\left[\frac{(1)(2)+(2)(-3)+(1)(6)}{\sqrt{1^2+2^2+1^2}\sqrt{2^2+(-3)^2+6^2}}\right]=cos^{-1}\left[\frac{2-6+6}{\sqrt{6}\sqrt{49}}\right]=cos^{-1}\left[\frac{2}{7\sqrt{6}}\right]$.

15. If P($\vec{p}$), Q($\vec{q}$), R($\vec{r}$) and S($\vec{s}$) be four points such that $3\vec{p}+8\vec{q}=6\vec{r}+5\vec{s}$, then the lines PQ and RS are -

(A) skew (B) intersecting (C) parallel (D) none of these

Sol.[B] Given $3\vec{p}+8\vec{q}=6\vec{r}+5\vec{s}$

$$\frac{3\vec{p}+8\vec{q}}{8+3}=\frac{6\vec{r}+5\vec{s}}{5+6}$$

The point which divides PQ in ratio 8 : 3 is the same as the point which divides RS in the ratio 5 : 6. Hence, the line PQ and RS intersect.

16. The perpendicular distance of the point (2, 4, –1) from the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$ is

(a) 3 (b) 5 (c) 7 (d) 9

Solution: (c) The perpendicular distance of (2, 4, – 1) from the line $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$ is

$$=\left\{(2+5)^2+(4+3)^2+(-1-6)-\left[\frac{1(2+5)+4(4+3)-9(-1-6)}{\sqrt{1+16+81}}\right]^2\right\}^{1/2}$$

$$=\sqrt{147-\left(\frac{98}{\sqrt{98}}\right)^2}=\sqrt{147-98}=\sqrt{49}=7 \text{ (or) } d=\frac{|\overrightarrow{AP}X\vec{b}|}{|\vec{b}|}$$

17. The angle between two lines $\frac{x+1}{2}=\frac{y+3}{2}=\frac{z-4}{-1}$ and $\frac{x-4}{1}=\frac{y+4}{2}=\frac{z+1}{2}$ is (a) $\cos^{-1}\left(\frac{1}{9}\right)$

(b) $\cos^{-1}\left(\frac{2}{9}\right)$ (c) $\cos^{-1}\left(\frac{3}{9}\right)$ (d) $\cos^{-1}\left(\frac{4}{9}\right)$

Solution: (d) $\theta = cos^{-1}\left(\frac{(2)(1)+(2)(2)+(-1)(2)}{\sqrt{2^2+2^2+1^2}\sqrt{1^2+2^2+2^2}}\right)=cos^{-1}\frac{4}{9}$

18. Equation of x-axis is

(a) $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$ (b) $\frac{x}{0}=\frac{y}{1}=\frac{z}{1}$ (c) $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$ (d) $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$

Solution: (c) It is obvious.

19. The angle between the pair of lines with direction ratios (1, 1, 2) and $(\sqrt{3}-1,-\sqrt{3}-1,4)$ is

(a) 30° (b) 45° (c) 60° (d) 90°

Solution: (c) $cos\,\theta=\frac{1(\sqrt{3}-1)-1(\sqrt{3}+1)+2\times4}{\sqrt{6}\sqrt{24}}=\frac{6}{12}\Rightarrow\theta=60^o$.

20. The acute angle between the line joining the points (2,1,–3), (–3,1,7) and a line parallel to $\frac{x-1}{3}=\frac{y}{4}=\frac{z+3}{5}$ through the point (–1, 0, 4) is

(a) $\cos^{-1}\left(\frac{7}{5\sqrt{10}}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$ (c) $\cos^{-1}\left(\frac{3}{5\sqrt{10}}\right)$ (d) $\cos^{-1}\left(\frac{1}{5\sqrt{10}}\right)$

Solution: (a) Direction ratio of the line joining the point $(2, 1, -3)$, $(-3, 1, 7)$ are (a_1, b_1, c_1)

$\Rightarrow (-3-2, 1-1, 7-(-3)) \Rightarrow (-5, 0, 10)$

Direction ratio of the line parallel to line $\frac{x-1}{3} = \frac{y}{4} = \frac{z+3}{5}$ are $(a_2, b_2, c_2) \Rightarrow (3, 4, 5)$

Angle between two lines,

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \qquad \cos\theta = \frac{(-5\times 3)+(0\times 4)+(10\times 5)}{\sqrt{25+0+100}\sqrt{9+16+25}}$$

$$\cos\theta = \frac{35}{25\sqrt{10}} \Rightarrow \theta = \cos^{-1}\left(\frac{7}{5\sqrt{10}}\right).$$

ASSERTION - REASON BASED QUESTIONS

In the following questions, a statement of Assertion (A) is followed by a statement of Reason(R).Pick the correct option:

A) Both Assertion (A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion (A).

B) Both Assertion (A) and Reason(R) are true but Reason(R) is NOT the correct explanation of Assertion (A).

C) Assertion (A) is true but Reason(R) is false.

D) Assertion (A) is false but Reason(R) is true.

1. ASSERTION: Equation of a line passes through the point P (2, – 1, 3) & perpendicular to the lines L_1: $\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + \alpha(2\hat{i} - 2\hat{j} + \hat{k})$ & L_2: $\vec{r} = (\hat{i} + 3\hat{k}) + \beta(\hat{i} + 2\hat{j} + 2\hat{k})$ both is $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + s(-6\hat{i} - 2\hat{j} + 6\hat{k})$

REASON: Let, L_1 : $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ & L_2 : $\vec{r} = \vec{a_2} + \mu\vec{b_2}$. Then equation of a line passing through the point P($\vec{a}$) & perpendicular to the lines L_1 & L_2 both is $\vec{r} = \vec{a} + t(\vec{b_1} \times \vec{b_2})$

Sol.[A] $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ By L_1 : $\vec{b_1} = 2\hat{i} - 2\hat{j} + \hat{k}$ By L_2 : $\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\hat{i} - 2\hat{j} + 6\hat{k}$$

Then, required equation of line is $\vec{r} = \vec{a} + t(\vec{b_1} \times \vec{b_2})$

2. ASSERTION : If the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{d}$ intersects at a point then $(\vec{c} - \vec{a}).\{\vec{b} \times \vec{d}\} = 0$

REASON: Two coplanar lines always intersects.

Sol. **[C]**

3. ASSERTION: The angle between the rays of with d.r's (4, –3, 5) and (3, 4, 5) is $\frac{\pi}{3}$.

REASON: The angle between the rays whose d.c's are l_1, m_1, n_1 and l_2, m_2, n_2 is given by whose $\cos\alpha = l_1l_2 + m_1m_2 + n_1n_2$

Sol. [B] $\cos\theta = \left(\frac{12-12+25}{\sqrt{50}\sqrt{50}}\right) = \frac{25}{50} = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

4. ASSERTION: A line makes 60° with x-axis and 30° with y-axis then it makes 90° with z-axis.

REASON:: If a ray makes angles α, β , γ with x-axis, y-axis and z-axis respectively then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1$

Sol. [C] $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

5. **ASSERTION:** If lines x = ay + b, z = 3y + 4 and x = 2y + 6, z = ay + d are perpendicular to each other then a = -1/5

   REASON: If two lines with d.rs a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular then $a_1a_2 + b_1b_2 + c_1c_2 = 0$ **Sol. [D]**

   $\frac{x-b}{a} = \frac{y}{1} = \frac{z-4}{3}$ and $\frac{x-6}{2} = \frac{y}{1} = \frac{z-d}{a} \Rightarrow 2a + 1 + 3a = 0 \Rightarrow a = \frac{-1}{5}$

6. **ASSERTION:** Equation of a line passing through the points (1,2,3) and (3,-1,3) is $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z-3}{0}$.

   REASON: Equation of a line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$.***Sol:*** A

7. **ASSERTION:** A line through the points (4,7,8) and (2,3,4) is parallel to a line through the points (-1,-2,1) and (1,2,5).

   REASON: Lines $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\overrightarrow{b_2}$ are parallel if $\overrightarrow{b_1}.\overrightarrow{b_2} = 0$. **Sol:** C

   D.rs are in proportion, hence the lines are parallel but $\overrightarrow{b_1}.\overrightarrow{b_2} = 0$ gives that lines are perpendicular to each other.

8. **ASSERTION:** Quadrilateral formed by the points A(0,0,0), B(3,4,5), C(8,8,8) and D(5,4,3) is a Rhombus.

   REASON: ABCD is a Rhombus if AB=BC=CD=DA and AC≠BD. **Sol:** A

9. **ASSERTION:** A line in space cannot be drawn perpendicular to x,y and z axes simultaneously.

   REASON: For any line making angles α, β, γ with the positive directions of x,y and z axes respectively, $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$.**Sol:** D

10. **Assertion:** The vector equation of the line passing through the points (6, –4, 5) and (3, 4, 1) is $\vec{r} = (6\hat{\imath} - 4\hat{\jmath} + 5\hat{k}) + \mu(-3\hat{\imath} + 8\hat{\jmath} - 4\hat{k})$

    Reason: The vector equation of the line passing through the points $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \mu(\vec{b} - \vec{a})$. **Sol:** A, It is fundamental concept

VERY SHORT ANSWER TYPE QUESTIONS

1. **Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of 5 units from the point P(1,3,3).**

Sol: Given line is $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \mu$ then,

General point on the line is R$(3\mu - 2, 2\mu - 1, 2\mu + 3)$

Distance from R to P(1, 3, 3) is 5 units

$$\sqrt{(3\mu-2-1)^2 + (2\mu-1-3)^2 + (2\mu+3-3)^2} = 5$$

$\therefore \mu = 0, 2.$

∴ R(-2, -1, 3) or R(4 , 3 , 7)

2. **The equation of a line is $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line.**

Sol: Given equation of a line is $5x - 3 = 15y + 7 = 3 - 10z$ … (1)

Here coefficients of x, y and z are 5, 15 and 10. (without sign)

LCM (5, 15, 10) = 30. Thus, dividing by 30 we have eq. (1) becomes

$$\frac{5x-3}{30}=\frac{15y+7}{30}=\frac{3-10z}{30}$$

$$\frac{5(x-\frac{3}{5})}{30}=\frac{15(y+\frac{7}{15})}{30}=\frac{-10(z-\frac{3}{10})}{30}$$

$$\frac{x-\frac{3}{5}}{6}=\frac{y+\frac{7}{15}}{2}=\frac{z-\frac{3}{10}}{-3} \qquad \text{…. (2)}$$

The standard form of equation is given as

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \qquad \text{…. (3)}$$

Comparing the above standard equation with Eq. (2), we get 6, 2, -3 are the direction ratios of the given line.

Now $\sqrt{6^2+2^2+(-3)^2}=\sqrt{49}=7$

Now, the direction cosines of given line are $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$.

3. **If α, β, γ are the angles made by the lines with x,y and z axes then show that $\cos 2\alpha+\cos 2\beta+\cos 2\gamma= -1$**

Sol: If α, β, γ are the angles made by the lines with x,y and z axes then we have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma= 1$

By writing $\cos 2x = 2\cos^2 x-1$

$\frac{1+\cos 2\alpha}{2}+\frac{1+\cos 2\beta}{2}+\frac{1+\cos 2\gamma}{2}=1$; $\cos 2\beta+\cos 2\beta+\cos 2\gamma+3=2$

So, $\cos 2\alpha+\cos 2\beta+\cos 2\gamma= -1$

4. **If α, β and γ are the angles which a line makes with positive direction of ,x , y and z axes respectively , then prove that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.**

Sol **:** We have $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$$\Rightarrow 1-\sin^2\alpha+1-\sin^2\beta+1-\sin^2\gamma=1$$
$$\Rightarrow \sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$$

5. **Find the angle between the pair of lines given by $\frac{x-3}{1}=\frac{y+2}{-2}=\frac{z+4}{2}$; $\frac{x-5}{-3}=\frac{y+2}{2}=\frac{z}{6}$**

Sol: By observing the two lines, the direction ratios of two lines are <1,-2, 2> and <-3,2,6>

Let θ be the angle between the two given lines

Hence, $\cos\theta=\frac{-3-4+12}{3\times 7}=\frac{5}{21}\Rightarrow\theta=\cos^{-1}\left(\frac{5}{21}\right)$

6. **If the lines $\frac{x-1}{-3}=\frac{2y-2}{4k}=\frac{3-z}{-2}$ and $\frac{x-1}{3k}=\frac{3y-1}{6}=\frac{z-6}{-5}$ are perpendicular to each other, find the value of k.**

Sol**:** The equations of the lines in standard form are $\frac{x-1}{-3}=\frac{y-1}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-\frac{1}{3}}{2}=\frac{z-6}{-5}$

So, the direction ratios of two lines are <-3,2k,2> and <3k,2,-5>

Since the lines are perpendicular $\vec{a}.\vec{b}=0$

That is $a_1a_2+b_1b_2+c_1c_2=0$; $-9k+4k-10=0\Rightarrow k=-2$

7. **If the angle between the lines $\frac{x-5}{\alpha}=\frac{y+2}{-5}=\frac{z+\frac{24}{5}}{\beta}$ and $\frac{x}{1}=\frac{y}{0}=\frac{z}{1}$ is $\frac{\pi}{4}$. Find the relation between α and β.**

Sol**:** The direction ratios of two given lines are < α,-5, β> and <1,0,1>

and $\theta = \frac{\pi}{4}$ be the angle between the two given lines. Then

$$\cos\frac{\pi}{4} = \frac{\alpha+\beta}{(\sqrt{\alpha^2+\beta^2+25}).\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{\alpha+\beta}{\sqrt{\alpha^2+\beta^2+25}\ .\sqrt{2}} \Rightarrow \alpha\beta = \frac{25}{2}$$

8. Write the vector equation of a line passing through point $(1,-1,2)$ and parallel to the line whose equation is $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{-2}$.

Sol: The Vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $r^{\rightarrow} = a^{\rightarrow} + \lambda b^{\rightarrow}$, $where\ \lambda \in R$

Given point on the line is $(1,-1,2)$ then $a^{\rightarrow} = \hat{i} - \hat{j} + 2\hat{k}$

and a vector parallel to the line $\frac{(x-3)}{1} = \frac{(y-1)}{2} = \frac{(z+1)}{-2}$ is $b^{\rightarrow} = \hat{i} + 2\hat{j} - 2\hat{k}$

Hence, Direction Ratios of a required line are $1,2$ and -2

So, required vector equation of line is

$$r^{\rightarrow} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 2\hat{k}), where\ \lambda \in R$$

9. Find the vector and Cartesian equations of a line that passes through the point $A(1,2,-1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$.

Sol: The given line is $\frac{x-5}{\frac{1}{5}} = \frac{y-2}{-\frac{1}{7}} = \frac{z}{\frac{1}{35}}$

hence, Cartesian equation of a line passing through $A(1,2,-1)$ and parallel to given line is

$$\frac{x-1}{\frac{1}{5}} = \frac{y-2}{-\frac{1}{7}} = \frac{z+1}{\frac{1}{35}}\ or\ \frac{x-1}{7} = \frac{y-2}{-5} = \frac{z+1}{1}$$

and the corresponding vector equation is $\vec{r} = \hat{\imath} + 2\hat{\jmath} - \hat{k} + \lambda(7\hat{\imath} - 5\hat{\jmath} + \hat{k})$.

10. If the equation of a line is $= ay + b$, $z = cy + d$, then find the direction ratios of the line and a point on the line.

Sol: The given equation of a line is not in standard form

$x - b = ay\ and\ z - d = cy$ $\quad \frac{x-b}{a} = y\ and\ \frac{z-d}{c} = y$ then $\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$

Hence, the direction ratios are a ,$1, c$ and a point on the line is $(b, 0, d)$

SHORT ANSWER TYPE QUESTIONS

1. Find the coordinates of the point where the line through the points (–1, 1, –8) and (5, –2, 10) crosses the ZX-plane.

Sol: The equation of line passing through two given points (–1, 1, –8) and (5, –2, 10) is

$$\frac{x+1}{6} = \frac{y-1}{-3} = \frac{z+8}{18} = \lambda$$

Any point on this line is $(6\lambda - 1, -3\lambda + 1, 18\lambda - 8)$

If a line crosses ZX-plane i.e. co-ordinate of y is zero

$-3\lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{3}$

Substitute value of λ in the point, then Required point on ZX-Plane is (1, 0, –2)

2. Find the shortest distance between the Lines: $\vec{r} = (3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$ and $\vec{r} = (5\hat{\imath} - \hat{\jmath}) + \mu(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$. Also, find whether the lines are intersecting or not.

Sol: The given two lines are

$\vec{r} = (3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + \lambda(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$ and $\vec{r} = (5\hat{\imath} - \hat{\jmath}) + \mu(3\hat{\imath} + 2\hat{\jmath} + 6\hat{k})$

By observing the given two lines, we have

$\overrightarrow{a_1} = 3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}, \overrightarrow{b_1} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$ $;\overrightarrow{a_2} = 5\hat{\imath} - \hat{\jmath}, \overrightarrow{b_2} = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$

Hence, Shortest distance between the above lines = $\left|\frac{(\overrightarrow{a_2}-\overrightarrow{a_1}).(\overrightarrow{b_1}\times\overrightarrow{b_2})}{|\overrightarrow{b_1}\times\overrightarrow{b_2}|}\right|$

$$(\overrightarrow{a_2}-\overrightarrow{a_1}) = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$$

and $\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 2 \\ 3 & 2 & 6 \end{vmatrix} = 8\hat{\imath} + 0\hat{\jmath} - 4\hat{k}$

$\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{80}$ Now, S.D = $\left|\frac{(2\hat{\imath}-3\hat{\jmath}+4\hat{k}).(8\hat{\imath}+0\hat{\jmath}-4\hat{k})}{\sqrt{80}}\right|$ $= \frac{0}{\sqrt{80}} = 0$

If shortest distance is zero then the given two lines are intersecting.

3. Equations of sides of a parallelogram ABCD are as follows

$\boldsymbol{AB}: \frac{x+1}{1} = \frac{y-2}{-2} = \frac{z-1}{2}$ $\boldsymbol{BC}: \frac{x-1}{3} = \frac{y+2}{-5} = \frac{z-5}{3}$ $\boldsymbol{CD}: \frac{x-4}{1} = \frac{y+7}{-2} = \frac{z-8}{2}$

$\boldsymbol{DA}: \frac{x-2}{3} = \frac{y+3}{-5} = \frac{z-4}{3}$ **.**

Find the equation of diagonal BD.

Sol: Let P be any point on line AB is (λ – 1, –2λ + 2, 2λ +1) and Q be any point on line BC is (3μ+ 1, –5μ −2, 3μ+5)

For some value of λ and μ, both the lines are intersecting at a point B.

P = Q

$$\Rightarrow (\lambda - 1, -2\lambda + 2, 2\lambda + 1) = (3\mu + 1, -5\mu - 2, 3\mu + 5)$$

B is the point of intersection of AB and BC , so coordinates of B are $(1, -2, 5)$

Similarly, any point on line CD is (λ +4, –2λ −7, 2λ+8) and any point on line DA is (3μ+2, –5μ −3, 3μ+4)

D is the point of intersection of CD and DA , so coordinates of B are $(2, -3, 4)$.

The equation of a line passing through the points $B(1, -2, 5)$ *and* $D(2, -3, 4)$ *is*

$\frac{x-1}{2-1} = \frac{y-(-2)}{-3-(-2)} = \frac{z-5}{4-5}$; $\frac{x-1}{1} = \frac{y+2}{-1} = \frac{z-5}{-1}$

4. Check whether the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are skew or not.

Sol: From the given two lines, we have

Let $\overrightarrow{a_1} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $\overrightarrow{a_2} = 4\hat{\imath} + \hat{\jmath}$ $\overrightarrow{b_1} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$, $\overrightarrow{b_2} = 5\hat{\imath} + 2\hat{\jmath} + \hat{k}$

and $\overrightarrow{a_2} - \overrightarrow{a_1} = 3\hat{\imath} - \hat{\jmath} - 3\hat{k}$ & $\overrightarrow{b_1} \times \overrightarrow{b_2} = -5\hat{\imath} + 18\hat{\jmath} - 11\hat{k}$

Now, $(\overrightarrow{a_2} - \overrightarrow{a_1}).\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right) = (3\ X - 5) + (-1\ X\ 18) + (-3\ X - 11)$ = -15-18+33=0

Hence, given lines are not skew lines.

5. Find the equation of the line which bisects the line segment joining points $A(2,3,4)$ and $B(4,5,8)$ and is perpendicular to the lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$$

Sol: The midpoint of line joining points $A(2,3,4)$ and $B(4,5,8)$ is (3,4,6)

So, the equation of line passing through (3,4,6) with direction ratios a,b,c is

$\frac{x-3}{a} = \frac{y-4}{b} = \frac{z-6}{c}$

Since this line is perpendicular to $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

$\Rightarrow 3a - 16b + 7c = 0 \ \Rightarrow 3a + 8b - 5c = 0$

by using cross multiplication method, we have a=2,b=3,c=6

Hence, the required equation of a line passing through (3,4,6) with d.r's <2,3,6> is

$$\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-6}{6}$$

6. **Find the vector and Cartesian equations of the line which is perpendicular to the lines with equations $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ *and* $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and passes through the point (1, 1, 1).**

   Sol: Let equation of line through (1, 1, 1) with direction ratios a,b, c be
   $\frac{x-1}{a}=\frac{y-1}{b}=\frac{z-1}{c}$.................(i)
   If Line(i) perpendicular to the lines $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ *and* $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$
   Therefore, a+2b+4c=0 and 2a+3b+4c=0 ∴ dr's are -4, 4, -1 or 4, -4, 1
   ∴ Hence, the Cartesian equation of a line is $\frac{x-1}{4}=\frac{y-1}{-4}=\frac{z-1}{1}$ and vector equation of a line is r=(i+j+k)+$\mu(4i-4j+k)$

7. **Find the value of p for which the lines $\vec{r}=\lambda\hat{\imath}+(2\lambda+1)\hat{\jmath}+(3\lambda+2)\hat{k}$ and $\vec{r}=\hat{\imath}-3\mu\hat{\jmath}+(p\mu+7)\hat{k}$ are perpendicular to each other and also intersect. Also find the point of intersection of the given lines.**

   Sol: By writing the given two equations in standard form
   The direction ratios of a given two lines are (1,2,3) and (0,-3,p) respectively.
   If the lines are perpendicular to each other then $a_1a_2+b_1b_2+c_1c_2=0$
   $$\Rightarrow \quad 1\times 0+2\times(-3)+3\times p=0 \Rightarrow p=2$$
   Any point on the line $\vec{r}=\lambda\hat{\imath}+(2\lambda+1)\hat{\jmath}+(3\lambda+2)\hat{k}$ is $(\lambda, 2\lambda+1, 3\lambda+2)$
   Any point on the line $\vec{r}=\hat{\imath}-3\mu\hat{\jmath}+(p\mu+7)\hat{k}$ is $(1,-3\mu,2\mu+7)$
   For point of intersection, we can equate the coordinates
   $$(\lambda, 2\lambda+1, 3\lambda+2)=(1,-3\mu,2\mu+7)$$
   and solving we get $\lambda=1$ and $\mu=-1$
   The point of intersection is (1,3,5).

8. **Find the values of p, so that the lines**
   $$l_1: \frac{1-x}{3}=\frac{7y-14}{p}=\frac{z-3}{2} \text{ and } l_2: \frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$$
   are perpendicular to each other. Also, find the equation of a line passing through a point $(3,2,-4)$ and parallel to linel_1.

   Sol: Writing the given line in standard form as
   $$\frac{(x-1)}{-3}=\frac{y-2}{\frac{p}{7}}=\frac{z-3}{2}=r_1(let) \quad .... (1)$$
   $$\text{and} \quad \frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}=r_2(let) \quad .... (2)$$
   Two lines with DR's a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular if
   $$a_1a_2+b_1b_2+c_1c_2=0$$
   Thus line (1) and (2) will intersect at right angle, if
   $$-3\left(\frac{-3p}{7}\right)+\frac{p}{7}(1)+2(-5)=0 \;, \frac{9p}{7}+\frac{p}{7}=10 \;, \frac{10p}{7}=10 \Rightarrow p=7$$
   This is the required value of p.
   Also, we know that the equation of a line which passes through the point (x_1,y_1,z_1) with direction ratios a, b, c is given by

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Since, required line is parallel to line l_1.

So, $\quad a = -3, b = \frac{7}{7} = 1 \; and \; c = 2$

Now, equation of line passing through the point (3,2,-4) and having direction ratios (-3,1,2) is $\quad \frac{x-3}{-3} = \frac{y-2}{1} = \frac{z+4}{2} \Rightarrow \frac{3-x}{3} = \frac{y-2}{1} = \frac{z+4}{2}$

9. **Find the vector and the Cartesian equations of a line passing through the point $(1, 2, -4)$ and parallel to the line joining the points $A(3, 3, -5)$ and $B(1, 0, -11)$. Hence find the distance between the two lines.**

Sol: The direction ratios of a line joining the points $A(3,3,-5)$and $B(1,0,-11)$are (3-1,3-0, -5-(-11))=(2,3,6)

So, the Vector equation of required line through (1, 2, – 4) with direction ratios (2,3,6) is

$$\vec{r} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k} + \lambda(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$$

and the cartesian equation is $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$

Equation of line through A(3, 3, – 5) and B(1, 0, – 11) is

$$\vec{r} = 3\hat{\imath} + 3\hat{\jmath} - 5\hat{k} + \lambda(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$$

Distance between parallel lines is given by $d = \frac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|}$

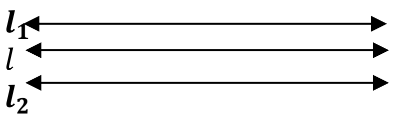
Here $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$, $\vec{a_1} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k}$, $\vec{a_2} = 3\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$

$(\vec{a_2} - \vec{a_1}) = 2\hat{\imath} + \hat{\jmath} - \hat{k}$, $(\vec{a_2} - \vec{a_1}) \times \vec{b} = 9\hat{\imath} - 14\hat{\jmath} + 4\hat{k}$

$$|(\vec{a_2} - \vec{a_1}) X \vec{b}| = \sqrt{81 + 196 + 16} = \sqrt{293} \; and \; |\vec{b}| = \sqrt{4 + 9 + 36} = 7$$

$$so, d = \frac{\sqrt{293}}{7}.$$

10. **Find the equation of a line l_2 which is the mirror image of the line l_1 with respect to line(l) : $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$, given that line l_1 passes through the point P($1, 6, 3$) and parallel to line l.**



Sol : Since l_1 passes through the point P(1,6,3) and parallel to line l equation of l_1 is

$$\frac{x - 1}{1} = \frac{y - 6}{2} = \frac{z - 3}{3} = \mu$$

Since line l_2 is the mirror image of the line l_1 with respect to line l , l_2 is parallel to l.

Foot of perpendicular of P(1,6,3)to line l is (1,3,5)

So point on l_2 is (1,0,7) , image of P(1,6,3) with respect to line l

So, equation of l_2 which passes through (1,0,7) and parallel to l is

$$\frac{x-1}{1} = \frac{y}{2} = \frac{z-7}{3} = \lambda$$

LONG ANSWER TYPE QUESTIONS

1. **Find the co-ordinates of the foot of the perpendicular drawn from the point $A(-1, 8, 4)$ to the line joining points $B(0, -1, 3)$ and $C(2, -3, -1)$. Hence find the image of the point A in the line BC.**

Sol: The equation of a line joining the points B (0, – 1, 3) and C (2, – 3, – 1) is

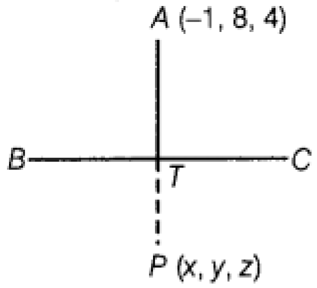
$$\vec{r} = (0\hat{\imath} - \hat{\jmath} + 3\hat{k}) + \lambda[(2\hat{\imath} - 3\hat{\jmath} - 1\hat{k}) - (0\hat{\imath} - \hat{\jmath} + 3\hat{k})]$$

$\Rightarrow \vec{r} = (-\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 2\hat{j} - 4\hat{k})$

$\Rightarrow \vec{r} = (2\lambda)\hat{i} + (-2\lambda - 1)\hat{j} + (-4\lambda + 3)\hat{k}, \lambda \in \mathbb{R}$

So any point on line BC is to the form (2λ, – 2λ – 1, – 4λ + 3)

Let foot of the perpendicular drawn from point A to the line BC be T(2λ, – 2λ – 1, – 4λ + 3).



Now, DR's of line AT is (2λ + 1, – 2λ – 1 – 8, – 4λ + 3 – 4) or (2λ + 1, 2λ – 9, – 4λ – 1).

Since, AT is perpendicular to BC.

∴ 2 × (2λ + 1) + (- 2) × (- 2λ– 9) + (- 4) (- 4λ– 1) = 0

[∵ $a_1a_2 + b_1b_2 + c_1c_2 = 0$]

⇒ 4λ + 2 + 4λ + 18 + 16λ + 4 = 0 ⇒ 24λ + 24 = 0 , λ = -1

∴ Coordinates of foot of perpendicular is

T (2 × (- 1)), – 2 × (- 1) – 1, – 4 × (- 1) + 3) or T(- 2, 1, 7)

Let P(x, y, z) be the image of a point A with respect to the line BC. So, point T is the mid-point of AP.

∴ Coordinates of T = Coordinates of mid-point of AP

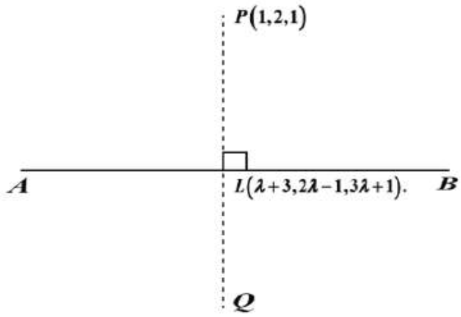
⇒ (- 2, 1, 7) = [(x−1)/2,(y+8)/2,(z+4)/2]

On equating the corresponding coordinates, we get

– 2 = (x−1)/2, 1 = (y+8)/2 and 7 = (z+4)/2 .⇒ x = -3, y = – 6 and z = 10

Hence, Image=(-3,-6,10)

2. Find the image of the point (1,2, 1) with respect to the line $\frac{x-3}{1} = \frac{y+1}{2} = \frac{z-1}{3}$. Also find the equation of the line joining the given point and its image.



Sol: Let P(1,2,1) be the given point and L be the foot of the perpendicular from P to the given line AB (as shown in the figure above).

Let's put $\frac{x-3}{1} = \frac{y+1}{2} = \frac{z-1}{3} = \lambda$. Then, $x = \lambda + 3, y = 2\lambda - 1, z = 3\lambda + 1$

Let the coordinates of the point L be $(\lambda + 3, 2\lambda - 1, 3\lambda + 1)$.

So, direction ratios of PL are $(\lambda + 3 - 1, 2\lambda - 1 - 2, 3\lambda + 1 - 1)$ *i.e.*, $(\lambda + 2, 2\lambda - 3, 3\lambda)$

Direction ratios of the given line are 1,2 and 3, which is perpendicular to PL. Therefore, we have,

$$(\lambda + 2) \cdot 1 + (2\lambda - 3) \cdot 2 + 3\lambda \cdot 3 = 0 \Rightarrow 14\lambda = 4 \Rightarrow \lambda = 2/7$$

Then, $\lambda + 3 = \frac{2}{7} + 3 = \frac{23}{7};\ 2\lambda - 1 = 2\left(\frac{2}{7}\right) - 1 = -\frac{3}{7}; 3\lambda + 1 = 3\left(\frac{2}{7}\right) + 1 = \frac{13}{7}$

Therefore, coordinates of the point L are $\left(\frac{23}{7}, -\frac{3}{7}, \frac{13}{7}\right)$.

Let $Q(x_1, y_1, z_1)$ be the image of P(1,2,1) with respect to the given line. Then, L is the mid-point of PQ.

Therefore, $\frac{1+x_1}{2} = \frac{23}{7}, \frac{2+y_1}{2} = -\frac{3}{7}, \frac{1+z_1}{2} = \frac{13}{7} \Rightarrow x_1 = \frac{39}{7}, y_1 = -\frac{20}{7}, z_1 = \frac{19}{7}$

Hence, the image of the point P(1,2,1) with respect to the given line Q $\left(\frac{39}{7}, -\frac{20}{7}, \frac{19}{7}\right)$.

The equation of the line joining P(1,2,1) and Q $\left(\frac{39}{7}, -\frac{20}{7}, \frac{19}{7}\right)$ is

$$\frac{x-1}{32/7} = \frac{y-2}{-34/7} = \frac{z-1}{12/7} \Rightarrow \frac{x-1}{16} = \frac{y-2}{-17} = \frac{z-1}{6}$$

3. Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\vec{r} = (-\hat{i} - \hat{j} - \hat{k}) + \lambda(7\hat{i} - 6\hat{j} + \hat{k})$ and $\vec{r} = (3\hat{i} + 5\hat{j} + 7\hat{k}) + \mu(\hat{i} - 2\hat{j} + \hat{k})$. where λ and μ are parameters.

Sol: Given that equation of lines are

$$\vec{r} = (-\hat{i} - \hat{j} - \hat{k}) + \lambda(7\hat{i} - 6\hat{j} + \hat{k}) \ldots (i) \text{ and}$$
$$\vec{r} = (3\hat{i} + 5\hat{j} + 7\hat{k}) + \mu(\hat{i} - 2\hat{j} + \hat{k}) \ldots\ldots (ii)$$

The given lines are non-parallel lines as vectors $7\hat{i} - 6\hat{j} + \hat{k}$ and $\hat{i} - 2\hat{j} + \hat{k}$ are not parallel. There is a unique line segment PQ (lying on line (i) and Q on the other line (ii) , which is at right angles to both the lines PQ&is the shortest distance between the lines.

Hence, the shortest possible distance between the lines $= PQ$. Let the position vector of the point P lying on the line

$\vec{r} = (-\hat{i} - \hat{j} - \hat{k}) + \lambda(7\hat{i} - 6\hat{j} + \hat{k})$ where' λ' is a scalar,

is $(7\lambda - 1)\hat{i} - (6\lambda + 1)\hat{j} + (\lambda - 1)\hat{k}$, for some λ and the position vector of the point Q lying on the line $\vec{r} = (3\hat{i} + 5\hat{j} + 7\hat{k}) + \mu(\hat{i} - 2\hat{j} + \hat{k})$ where ' μ ' is a scalar, is $(\mu + 3)\hat{i} + (-2\mu + 5)\hat{j} + (\mu + 7)\hat{k}$, for some μ. Now, the vector

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$\overrightarrow{PQ} = (\mu + 3 - 7\lambda + 1)\hat{i} + (-2\mu + 5 + 6\lambda + 1)\hat{j} + (\mu + 7 - \lambda + 1)k$$

i.e., $\overrightarrow{PQ} = (\mu - 7\lambda + 4)\hat{i} + (-2\mu + 6\lambda + 6)\hat{j} + (\mu - \lambda + 8)\hat{k}$;

(where ' O ' is the origin), is perpendicular to both the lines, so the vector $\overrightarrow{PQ}$ is perpendicular to both the vectors $7\hat{i} - 6\hat{j} + \hat{k}$ and $\hat{i} - 2\hat{j} + \hat{k}$.

$$\Rightarrow (\mu - 7\lambda + 4) \cdot 7 + (-2\mu + 6\lambda + 6) \cdot (-6) + (\mu - \lambda + 8) \cdot 1 = 0$$
$$\&(\mu - 7\lambda + 4) \cdot 1 + (-2\mu + 6\lambda + 6) \cdot (-2) + (\mu - \lambda + 8) \cdot 1 = 0$$
$$\Rightarrow 20\mu - 86\lambda = 0 \Rightarrow 10\mu - 43\lambda = 0 \& 6\mu - 20\lambda = 0 \Rightarrow 3\mu - 10\lambda = 0$$

On solving the above equations, we get $\mu = \lambda = 0$

So, the position vector of the points P and Q are $-\hat{i} - \hat{j} - \hat{k}$ and $3\hat{i} + 5\hat{j} + 7\hat{k}$ respectively.

$\overrightarrow{PQ} = 4\hat{i} + 6\hat{j} + 8\hat{k}$ and $|\overrightarrow{PQ}| = \sqrt{(4^2 + 6^2 + 8^2)} = \sqrt{116} = 2\sqrt{29}$ units.

(OR) Find the shortest distance between the lines
$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

SOL: Given lines: $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Shortest Distance between the lines = $\frac{|(\overrightarrow{a_2} - \overrightarrow{a_1}) \cdot (\overrightarrow{b_1} \times \overrightarrow{b_2})|}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|}$

$\overrightarrow{a_1} = -\hat{\imath} - \hat{\jmath} - \hat{k} \qquad \overrightarrow{a_2} = 3\hat{\imath} + 5\hat{\jmath} + 7\hat{k} \qquad \overrightarrow{a_2} - \overrightarrow{a_1} = 4\hat{\imath} + 6\hat{\jmath} + 8\hat{k}$

$\overrightarrow{b_1} = 7\hat{\imath} - 6\hat{\jmath} + \hat{k} \qquad \overrightarrow{b_2} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \hat{\imath}(-6+2) - \hat{\jmath}(7-1) + \hat{k}(-14+6) = -4\hat{\imath} - 6\hat{\jmath} - 8\hat{k}$$

$|\overrightarrow{b_1} \times \overrightarrow{b_2}| = \sqrt{16+36+64} = \sqrt{116}$

SD = $\frac{|(\overrightarrow{a_2} - \overrightarrow{a_1}) \cdot (\overrightarrow{b_1} \times \overrightarrow{b_2})|}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|} = \left|\frac{(4\hat{\imath} + 6\hat{\jmath} + 8\hat{k}).(-4\hat{\imath} - 6\hat{\jmath} - 8\hat{k})}{\sqrt{116}}\right|$

SD = $\left|\frac{(4)(-4)+(6)(-6)+(8)(-8)}{\sqrt{116}}\right| = \left|\frac{-16-36-64}{\sqrt{116}}\right| = \frac{116}{\sqrt{116}} = \sqrt{116} = 2\sqrt{29}$

4. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{\imath} + (2-t)\hat{\jmath} + (3-2t)\hat{k}$ **&** $\vec{r} = (s+1)\hat{\imath} + (2s-1)\hat{\jmath} - (2s+1)\hat{k}$

SOL: The equations of the given lines are

$\vec{r} = (1-t)\hat{\imath} + (2-t)\hat{\jmath} + (3-2t)\hat{k}$ & $\vec{r} = (s+1)\hat{\imath} + (2s-1)\hat{\jmath} - (2s+1)\hat{k}$

After writing standard equation of a line, then we have

$\vec{r} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k} + t(-\hat{\imath} + \hat{\jmath} - 2\hat{k})$ & $\vec{r} = \hat{\imath} - \hat{\jmath} - \hat{k} + s(\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$

Shortest Distance between the lines = $\frac{|(\overrightarrow{a_2} - \overrightarrow{a_1}) \cdot (\overrightarrow{b_1} \times \overrightarrow{b_2})|}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|}$

$\overrightarrow{a_1} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k} \qquad \overrightarrow{a_2} = \hat{\imath} - \hat{\jmath} - \hat{k} \qquad \overrightarrow{a_2} - \overrightarrow{a_1} = 0\hat{\imath} + \hat{\jmath} - 4\hat{k}$

$\overrightarrow{b_1} = -\hat{\imath} + \hat{\jmath} - 2\hat{k} \qquad \overrightarrow{b_2} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

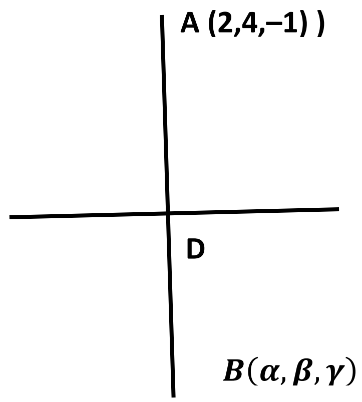
$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = \hat{\imath}(-2+4) - \hat{\jmath}(2+2) + \hat{k}(-2-1) = 2\hat{\imath} - 4\hat{\jmath} - 3\hat{k}$$

$|\overrightarrow{b_1} \times \overrightarrow{b_2}| = \sqrt{4+16+9} = \sqrt{29}$

SD = $\frac{|(\overrightarrow{a_2} - \overrightarrow{a_1}) \cdot (\overrightarrow{b_1} \times \overrightarrow{b_2})|}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|} = \left|\frac{(0\hat{\imath} + \hat{\jmath} - 4\hat{k}).(2\hat{\imath} - 4\hat{\jmath} - 3\hat{k})}{\sqrt{29}}\right|$

SD = $\left|\frac{(0)(2)+(1)(-4)+(-4)(-3)}{\sqrt{29}}\right| = \left|\frac{-4+12}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}} = \frac{8\sqrt{29}}{29}$

5). Find the image of the point (2, 4, –1) in the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$



SOL: Given point (2, 4, –1)

Given line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$

Let $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9} = k$

Any Point on given line D (k–5, 4k–3, –9k+6)

DRs of AD: $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$

k–5 –2, 4k–3 –4, –9k+6 +1

k–7, 4k–7, –9k+7

DRs of the given line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$ are 1, 4, –9

AD is perpendicular to given line: $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

1(k – 7) + 4(4k – 7) +(– 9) (–9k+7)

k – 7 + 16k – 28 + 81k – 63 = 0

98k – 98 = 0

k = 1

Substitute k=1 in D then
Foot of the perpendicular = D (– 4, 1, – 3)
Let B(α, β, γ) be the image of A
Then mid-point of AB = D
$\left(\frac{\alpha+2}{2}, \frac{\beta+4}{2}, \frac{\gamma-1}{2}\right) = (-4, 1, -3)$
$\frac{\alpha+2}{2} = -4$, $\alpha + 2 = -8$, $\alpha = -10$
$\frac{\beta+4}{2} = 1$ $\beta + 4 = 2$ $\beta = -2$
$\frac{\gamma-1}{2} = -3$ $\gamma - 1 = -6$ $\gamma = -5$
Image = B $(-10, -2, -5)$

CASE STUDY BASED QUESTIONS

1) **Imagine you are at a point A, a café you visit often. Your friend is at the point B, a bookstall a few blocks away on a straight road. You want to meet your friend at a point on the line joining the café and the bookstall. Another friend, who is at home on the other side of the same road represented by point P, also wants to join. You decide to determine the exact meeting point by finding the foot of the perpendicular from P on the line joining the café and the bookstall. Given that co-ordinates of the café (A) are (1,2,4), of the bookstall (B) are (3,4,5) and the home are (2,1,3).**
   (i) Find the location of the meeting point.
   (ii) Find the distance from Home to meeting point? Also, find the equation of the path connected by café and bookstall.

Sol: Let Q be the foot of the perpendicular from P(2,1,3) to AB and let Q divide AB in the ratio k:1.

Then, co-ordinates of Q are $\left(\frac{3k+1}{k+1}, \frac{4k+2}{k+1}, \frac{5k+4}{k+1}\right)$.

Direction ratios of PQ are $\frac{3k+1}{k+1} - 2, \frac{4k+2}{k+1} - 1, \frac{5k+4}{k+1} - 3$

$= \frac{k-1}{k+1}, \frac{3k+1}{k+1}, \frac{2k+1}{k+1}$

Direction ratios of AB are <3-1, 4-2, 5-4>; =<2, 2, 1>

As, $PQ \perp AB$,

$$\mathbf{2}\left(\frac{k-1}{k+1}\right) + 2\left(\frac{3k+1}{k+1}\right) + 1\left(\frac{2k+1}{k+1}\right) = 0$$

$k = \frac{-1}{10}$ and substitute value of k in point Q.

Hence, the co-ordinates of Q are $\left(\frac{7}{9}, \frac{16}{9}, \frac{35}{9}\right)$.

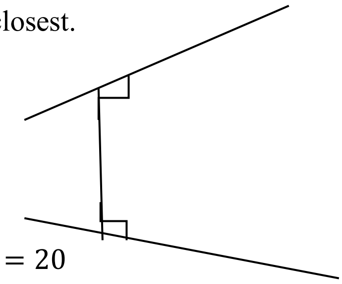
(ii) Distance PQ=$\sqrt{\left(\frac{7}{9} - 2\right)^2 + \left(\frac{16}{9} - 1\right)^2 + \left(\frac{35}{9} - 3\right)^2} = \sqrt{\frac{121}{81} + \frac{49}{81} + \frac{64}{81}} = \frac{\sqrt{234}}{9}\,units$

Equation of line AB is $\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-4}{1}$

(2) **Two drones are being used for radar centre analysis over an area of enemy land. Drone A has been programmed to fly on the path given by $\vec{r} = 6\hat{\imath} + 2\hat{\jmath} + 2\hat{k} + \mu(\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$ and drone B has been programmed to fly on the path $\vec{r} = -4\hat{\imath} - \hat{k} + \beta(\widehat{3\imath} - 2\hat{\jmath} - 2\hat{k})$.**
   (i) At what points on their respective paths should they reach, so that they will be closest to each other?

(ii) Find the shortest distance between these paths.

Sol: Let P and Q be the points on their respective paths when they are closest.



$\therefore PQ\ is\ \perp\ to\ both\ line\ 1\ and\ line\ 2$

So, general point on line-1 is P=$(\lambda+6,-2\lambda+2,\lambda+2)$

General point on line-2 is Q=$(3\mu-4,-2\mu,-2\mu-1)$

Direction ratios of PQ are $<3\mu-\lambda-10,-2\mu+2\lambda-2,-2\mu-2\lambda-3>$

Line -1 is perpendicular to PQ, then

$\Rightarrow -3\lambda+\mu=4\ \ and\ \ line-2\ is\ perpendicular\ to\ PQ\ then -3\lambda+17\mu=20$

By solving the above equations, we have $\mu=1\ and\ \lambda=-1$

Hence, the points are P=(5,4,0) and Q=(-1,-2,-3).

(iii) PQ=$\sqrt{(5+1)^2+(4+2)^2+(0+3)^2}=\sqrt{36+36+9}=\sqrt{81}=9\ units$

3) Two tunnels are planned to be dug through

4) Bisle ghat to improve traffic infrastructure.

5) (Shown in the figure-not to Scale).

Digging at one end of the tunnel is to begin at the point (−9,15,7) at Hassan and continue in the direction 7i-5j-k. The digging at the other end of the tunnel will start at the coordinate (33, 5, −1) near Sampaje and continue in the direction −14i +3k. Both sections are to be straight lines. The coordinates are measured relative to a fixed origin O, where one unit is 500 meters.

(i) Show that the two sections of the tunnel will eventually meet at a point near Bisle Ghat and find the coordinates of this point.

(ii) Find the total length of the two tunnels to the nearest kilometre. (OR)

Find the coordinates of the point which is nearest to the origin on the line

$\vec{r}=(\vec{i}+2\vec{j}+2\vec{k})+\mu(-\vec{i}-3\vec{j}+0\vec{k})$

Sol:

i) Let the equation the lines (paths) are

Hassan to Bisle:

$l_1: \frac{x+9}{7}=\frac{y-15}{-5}=\frac{z-7}{-1}=\lambda$

Bisle to Sampaje $l_2: \frac{x-33}{-14}=\frac{y-5}{0}=\frac{z1}{3}=\mu$

Any points on the line 1 and 2 are respectively

P=$(7\lambda-9,-5\lambda+15,-\lambda+7)$ Q=$(-14\mu+33,5,3\mu-1)$

For some values of λ and , lines are intersecting, so P=Q

$7\lambda-9=-14\mu+33,-5\lambda+15=5,-\lambda+7=3\mu-1$ Hence, $\mu=2,\lambda=2$

So, 5=5 hence the lines are intersecting and point of intersection is (5,5,5)

ii) (Let $\overrightarrow{OA}=-9\hat{\imath}+15\hat{\jmath}+7\hat{k}\ and\ \overrightarrow{OB}=33\hat{\imath}+5\hat{\jmath}-\hat{k}$

Then, $\overrightarrow{AB}=42\hat{\imath}-10\hat{\jmath}-8\hat{k}$

$|\overrightarrow{AB}|=\sqrt{1764+100+64}=\sqrt{1928}\cong 44\ units=44X500\ mtrs=22000\ mtrs$

(OR)

The coordinates of any point on the line are $P=(1-\mu,2-3\mu,2))$

Distance from (0,0,0) to the point P is

OP=$\sqrt{(1-\mu)^2+(2-3\mu)^2+(2)^2}=\sqrt{1+\mu^2-2\mu+4+9\mu^2-12\mu+4}$

D= $\sqrt{10\mu^2-14\mu+9}$, Let f=$10\mu^2-14\mu+9$

Hence, minimum value occurs at $\mu=\frac{-(-14)}{2(10)}=\frac{7}{10}$

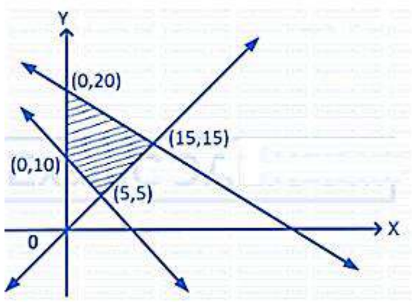
Substitute value of $\mu\ in\ P\ then\ P=(\frac{3}{10},\frac{-1}{10},2)$

CHAPTER-12: LINEAR PROGRAMMING PROBLEMS

- **Linear Programming** is the process used to obtain minimum or maximum value of the linear function under known linear constraints.
- **Objective Function**: Linear function z= ax + by where a and b are constants, which has to be maximized or minimized is called a linear objective function.
- **Constraints:** The linear inequalities or in equations or restrictions on the variables of a linear programming problem. The condition $x \geq 0, y \geq 0$ are known as non-negative restrictions.
- **Feasible Region:** It is defined as a set of points which satisfy all the constraints.
- To find feasible Region: Draw the graph of all the linear in equations and shade common region determined by all the constraints.
- **Feasible solutions:** Points within and on the boundary of the feasible region represents feasible solutions of the constraints.
- **Optimal feasible solution:** Optimal feasible solution which optimizes the objective function is called optimal feasible solution.

MULTIPLE CHOICE QUESTIONS

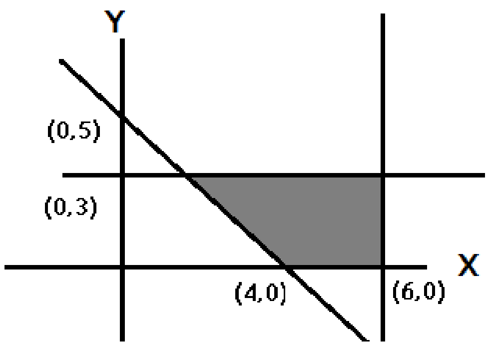
1. The feasible region of an LPP is shown in the figure. If Z=3x+9y, then the minimum value of z occurs at



(a) (0,20) (b) (0,10) (c) (5,5) (d) (15,15) Sol: Option: (c)

Corner Points	Z=3x+9y
(0,20)	180
(15,15)	180
(5,5)	60-MIN
(0,10)	90

2. The constraints of the LPP represented by the following figure is



$(a) 5x + 4y \geq 20, x \leq 6, y \geq 3, x \geq 0, y \geq 0$
$(b) 5x + 4y \leq 20, x \leq 6, y \geq 3, x \geq 0, y \geq 0$
$(c) 5x + 4y \geq 20, x \leq 6, y \leq 3, x \geq 0, y \geq 0$
$(d) 5x + 4y \geq 20, x \geq 6, y \leq 3, x \geq 0, y \geq 0$ Sol : Option: (c)

3. If an LPP admits optimal solution at two consecutive vertices of a feasible region, then
   (a) the LPP under consideration is not solvable
   (b) the LPP under consideration must be reconstructed
   (c) the required optimal solution is at the mid-point of the line joining two points
   (d) the optimal solution occurs at every point on the line joining these two points

   Option : (d)

4. Which of the following statement is correct
   (a) every LPP admits an optimal solution
   (b) If a L.P.P admits two optimal solutions it has an infinite number of optimal solutions
   (c) A L.P.P admits unique optimal solution
   (d) (0,0) is the only optimal solution. Option : (b)

5. Minimum value of $z = x\text{-}5y\text{+}20$ subject to the constraints:
$x - y \geq 0, -x + 2y \geq 2, x \geq 3, y \leq 4$ is given by
(a) (-6) at (4,6) (b) 4 at (4,4) (c) (-9) at (4,3) (d) 6 at (4,5)

Corner Points	Z=x-5y+20
(4,6)	-6=minimum
(4,4)	4
(4,3)	9
(4,5)	-1

So, Option: (a)

6. The corner points of the bounded feasible region determined by a system of linear constraints are (0,3), (1,1) and (3,0). Let $Z = px + qy$, where $p, q > 0$. The condition on p and q so that the minimum of Z occurs at (3,0) and (1,1) is:
(a) $p = 2q$ (b) $p = q/2$ (c) $p = 3q$ (d) $p = q$
Sol:3p=p+q implies 2p=q So Option : (b)

7. Which of the points A(80,10), B(20,20), C(60,60) D(20,25) lie in the feasible region of the constraints given below. $x + 5y \leq 200,\ 2x + 3y \leq 134,\ x \geq 0,\ y \geq 0$
   (a) Points A,B,and C only (b) Points B,C and D only (c) only points A and D (d) only points B and D

Sol: Only B and D satisfies all constraints, so Option : (d)

8. The objective function $Z = ax + by$ of an LPP has maximum value 42 at (4,6) and minimum value 19 at (3,2). Which of the following is true?
   (a) $a = 9, b = 1$ (b) $a = 5, b = 2$ (c) $a = 3, b = 5$ (d) $a = 5, b = 3$

Sol: 4a+6b=42 & 3a+2b=19 on solving a=5, b=2 So, Option (b)

ASSERTION-REASONING QUESTIONS

Select the correct answer from the codes (a), (b), (c) and (d) as given below.

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A)
(b) Both Assertion (A) and Reason (R) are true but Reason (R) is **not** the correct explanation of the Assertion (A)
(c) Assertion (A) is true and Reason (R) is false.

(d) Assertion (A) is false and Reason (R) is true.

1. ASSERTION: Maximum value of $z=11x+5y$ subject to
$3x+2y \le 25, x+y \le 10$ & $x,y \ge 0$ with corner points
$(0,10),(5,5)$ *amd* $(25/3,0)$, occurs at $\left(\frac{25}{3},0\right)$

REASON: If the feasible region of the LPP is bounded then the maximum & minimum value of the objecive function occurs at corner points.

Ans: (a) as both A and R are true and R is the correct explanation for A.

2. ASSERTION: Ifthe open half plane represented by $ax + by > M$ has no point common with the unbounded feasible region then M is the maximum value of z otherwise z has no maximum value.

REASON: A solution that also satisfies the non-negativity restriction of a LPP is called the feasible region.

Ans: (b) Both A and R are true but R is not correct explanation for A

3. ASSERTION: Feasible region of all LPP are convex polygon.

REASON: A polygon is said to be a convex polygon, if the line segment joining any two points of the polygon is completely contained in the polygon.

Ans: (d) as A is false and R is true.

4. ASSERTION: Value of the objective function if it exists is always positive.

REASON: Feasible region of any LPP lies only in the first quadrant as it involves non negative restrictions.

Ans(d) as value of objective function is any real number, so A is false but R is true.

5. ASSERTION: *F*or the bounded feasible region of a LPP if minimum value of the objective function exists at two consecutive corner points, then the LPP has infinitely many optimaol solution.

REASON: If minimum value exists at two consecutive vertices of a bounded feasible region, then every point of the segment joining those two vertices will give same value for the objective function.

Ans: (a) Both A and R are true and R is the correct explanation for A.

6. ASSERTION: The half plane $2x+3y \le 6$, contains (1,1) and (3,0)

REASON: Any point if it satisfies the inequality $ax+by \le c$, should lie in the half plane described by it.

Ans: (a), as both A and R are true and R is correct explanation for A.

7. ASSERTION: Every feasible solution of a LPP is an optimal solution of the LPP

REASON: Feasible solution is a point of feasible region that satisfy all constraints including non negative constraints of LPP

Ans: (d) A is false but R is true.

8. ASSERTION: The point (2,-3) does not lie on the half plane $3x+2y \ge 10$

REASON: Every point of a half plane $3x+2y \ge 10$ should satisfy the inequation.

Ans: (a) both A and R are correct and R is correct explanation for A.

9. ASSERTION: Corner points of a feasible region are (0,6), (3,2) and (3,0) for a LPP with objective function Z = 11x+7y, and maximum value of Z is 47.

REASON: Optimal value of an objective function occurs only at corner points if they exist.

Ans: (a) Both A and R are correct and R is proper reason for the A.

10.ASSERTION: For a LPP, the feasible region is unbounded and minimum value of the objective function Z=px+qy occurs at (a,b) with value "m". Then optimal solution is x=a,y=b with optimal value "m".

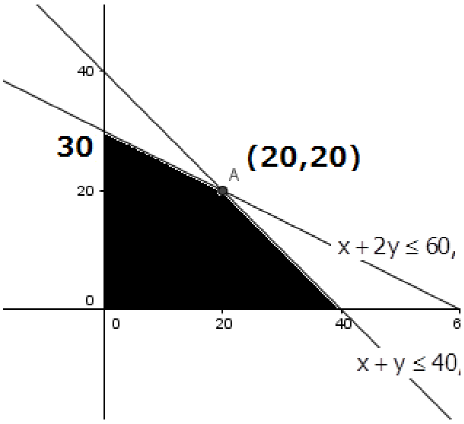
REASON:If feasible region is unbounded, it is to decide minimum exists or not after considering px+qy<m and whether this half plane has common points with feasible region of the problem.

Ans: (c) as A is wrong R is correct.

VERY SHORT ANSWER QUESTIONS

1. Find the maximum value of z=3x+4y subject to the constraints

$x + y \le 40,\ x + 2y \le 60,\ x, y \ge 0$

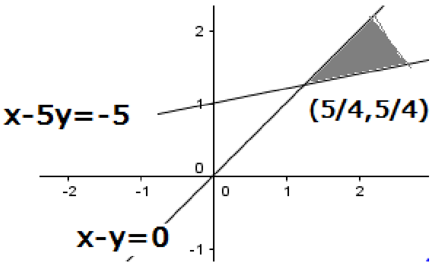


CORNER POINTS	Z=3X+4Y
(0,30)	120
(20,20)	140
(40,0)	120
(0,0)	0

Max z is 140 at (20,200

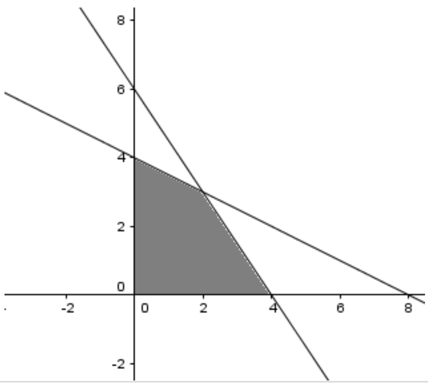
2. If the minimum value of the objective function z=2x+10y exists subject to constraints

$x - y \ge 0,\ x - 5y \le -5,\ x, y \ge 0$ find it.



Minimum value is 15 at (5/4,5/4).

3. Shown below is the feasible region of maximization problem whose objective function is given by z=5x+3y.

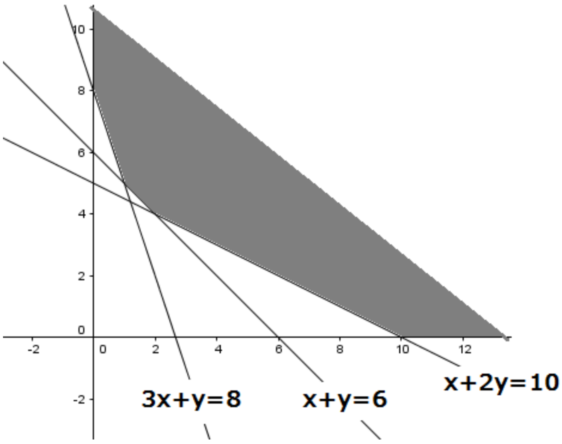


(i) List all the constraints the problem is subjected to.
(ii) Find the optimal solution of the problem.

Constraints are $x+2y \le 8,\ 3x+2y \le 12\ \ \&\ \ x, y \ge 0$

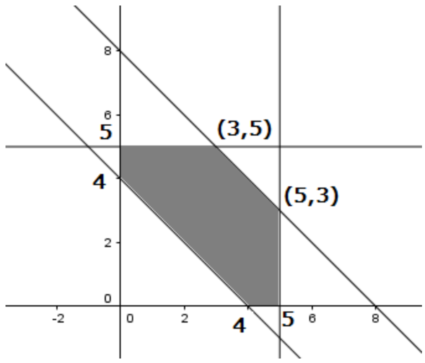
Corner Points	Z=5x+3y
(0,0)	0
(2,3)	19
(0,4)	12
(4,0)	20=Maximum

4. For the feasible region of a LPP given below, write all constraints and corner points of the feasible region.



Constraints are $3x+y \ge 8,\ x+y \ge 6,$
$x+2y \ge 10\ \&\ \ x, y \ge 0$
Corner Points are
(0,8), (1,5), (2,4) (10,0)

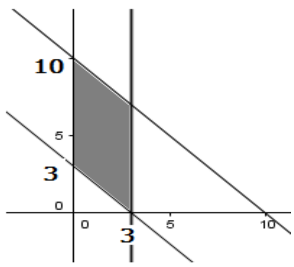
5. If the objective function of the LPP is z=x-7y+200, feasible region is given below.



Find the difference between the optimal values of Z.

Sol: Max=205 , Min=165, Difference=205-165=40

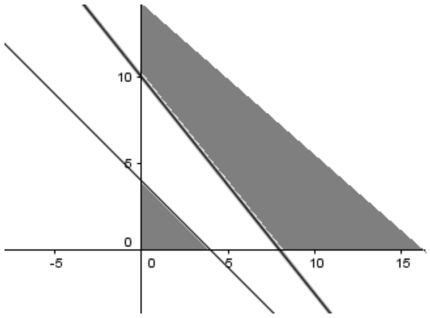
6. What are the constraints of the LPP, whose feasible region is given below, also find the area of the feasible region



Constraints are $x + y \geq 3, x + y \leq 10\ \&\ x\ , y \geq 0$

Area of Parallelogram=base X height =7x3=21sq.units

7. The feasible region of a LPP is as given below. Find the number of optimal solutions for the LPP and justify your answer.



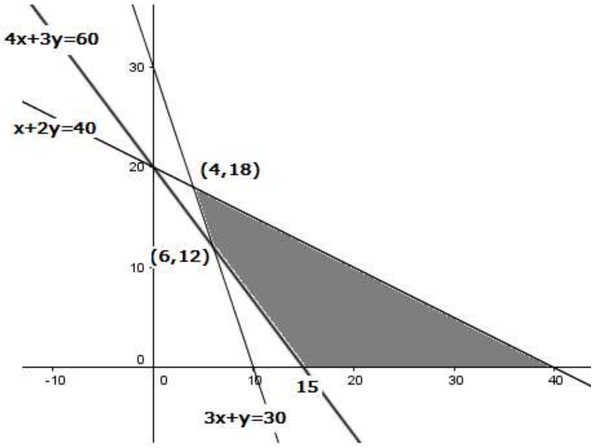
Sol: Zero, as there is no region satisfy all constraints of the problem.

SHORT ANSWER QUESTIONS-SOLVED

1. $Minimize Z = 20x + 10y$

$$Subject\ to\ x + 2y \leq 40, 3x + y \geq 30, 4x + 3y \geq 60, x, y \geq 0$$

Sol:



Corner Pts	Z=20x+10y
(15,0)	300
(40,0)	800
(4,18)	260
(6,12)	240

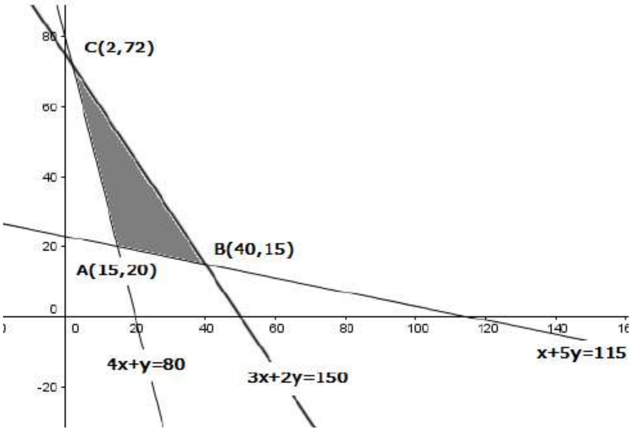
Minimum Value of Z is 240 at (6,12)

Hence x=6,y=12 is the optimal solution

and the optimal value of Z is 240.

2. $Minimize Z = 6x + 3y$

$Subject to 4x + y \geq 80, x + 5y \geq 115, 3x + 2y \leq 150, x, y \geq 0$

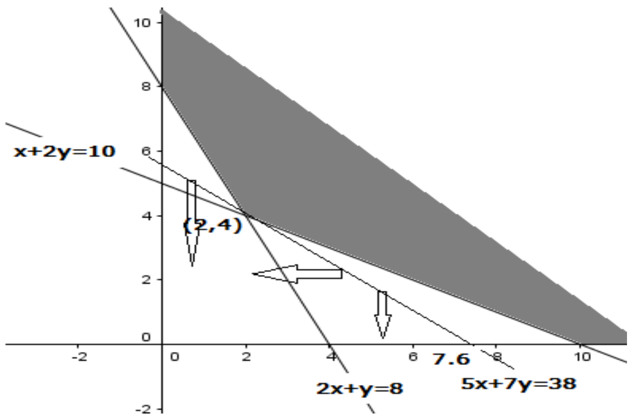


Corner pts	Z=6x+3y
A(15,20)	150
B(40,15)	285
C(2,72)	228

Minimum value of Z is 150 at x=15,y=20

3. $Minimize Z = 5x + 7y$

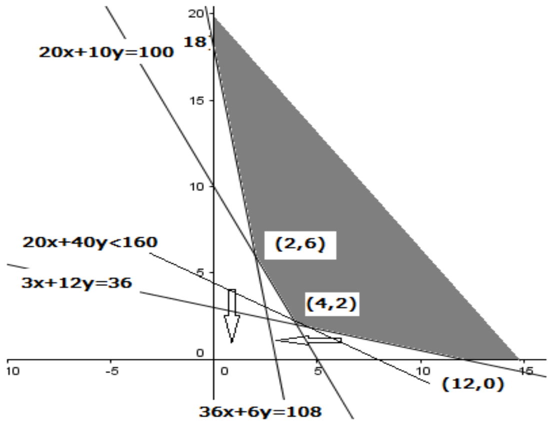
$Subject to 2x + y \geq 8, x + 2y \geq 10 \& x, y \geq 0$



CORNER PTS	Z=5x+7y
(0,8)	56
(2,4)	38 -MINIMUM
(10,0)	50

Feasible Region and halfplane of 5x+7y<38 has no common points , so Minimum value of Z is 38 at (2,4).

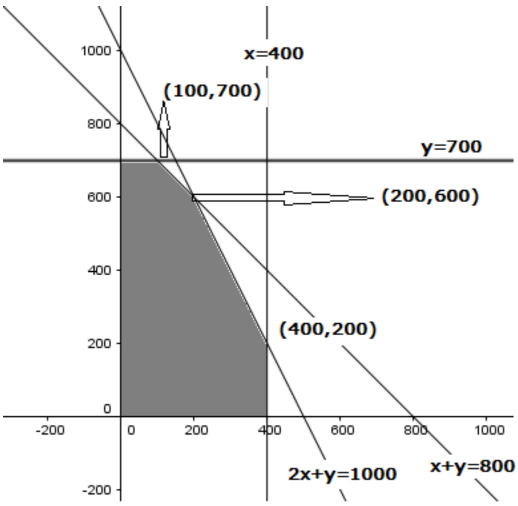
6. $Minimize Z = 20x + 40y$
$Subject to 36x + 6y \geq 108, 3x + 12y \geq 36, 20x + 10y \geq 100 \& x, y \geq 0$



Corner Points	Z=20x+40y
(0,18)	720
(2,6)	280
(4,2)	160=Min
(12,0)	240

Here, Half plane 20x+40y<160 has no common with feasible region, so, Min Z is 160 at x=4, y=2.

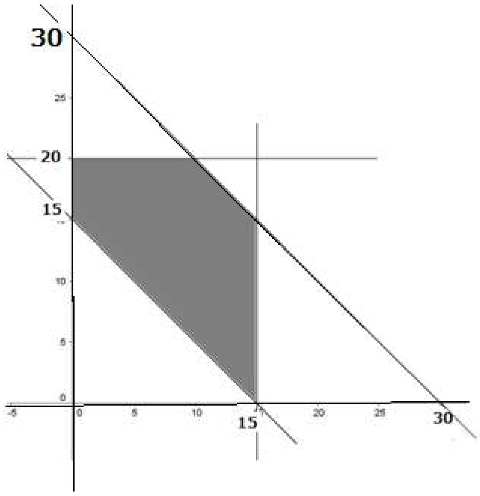
7. $Maximize Z = 2x + 1.5y$
$Subject to 2x + y \leq 1000, x + y \leq 800, 0 \leq x \leq 400, 0 \leq y \leq 700$



Corner Points	Z=2x+1.5y
(0,700)	1050
(100,700)	1250
(200,600)	1300=MAXIMUM
(400,200)	1100
(400,0)	800

Optimal Sol: x=200, y=600
Optimal Value:1300

8. For a LPP, the feasible region is as given below. Objective function is Minimize Z=30x-30y+1800. Write all constraints of the LPP and find the optimal value of the objective function and optimal solution also.



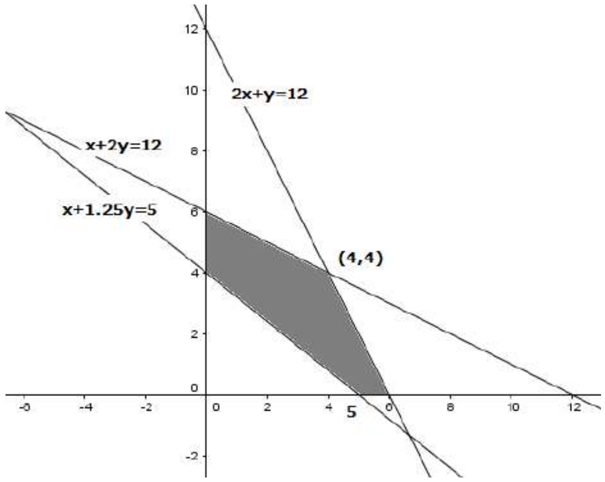
Corner Points	Z=30x-30y+1800
(0,15)	1350
(0,20)	1200=Minimum at x=0, y=20
(15,0)	2250
(10,20)	1500
(15,15)	1800

Sol : Constraints are $x + y \geq 15,\ x + y \leq 30,\ 0 \leq x \leq 15, 0 \leq y \leq 20$

9. Maximize $Z = 600x + 400y$

Subject to $x + 2y \leq 12, 2x + y \leq 12, x + 1.25y \geq 5\ \&\ 0 \leq x, 0 \leq y$

Sol:

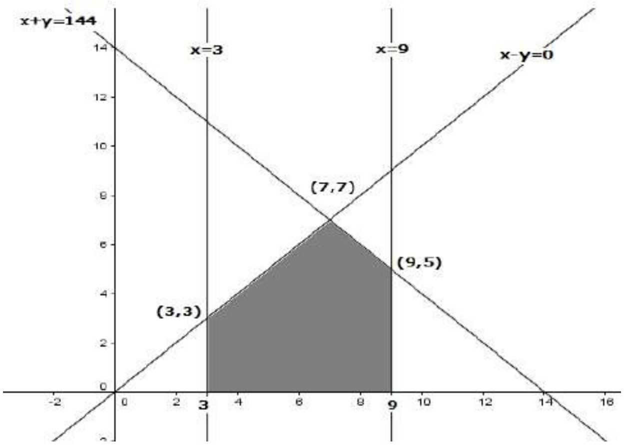


Corner Points	Z=600x+400y
(5,0)	3000
(6,0)	3600
(4,4)	4000=Maximum
(0,6)	2400
(0,4)	1600

Optimal Sol: x=4, y=4

10. Minimize & Maximize $Z = 2x + y$; Subject to $x - y \geq 0, x + y \leq 14, 3 \leq x \leq 9, y \geq 0$

Sol:



Corner Points	Z=2x+y
(3,0)	6=Minimum
(9,0)	18
(9,5)	23=Maximum
(7,7)	21
(3,3)	9

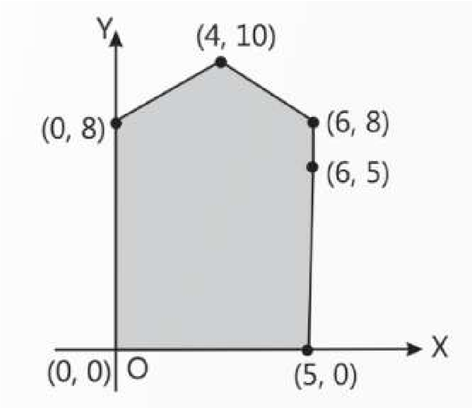
Optimal solutions;

For Minimum ; x=3, y=0

for maximum ; x=9, y=5

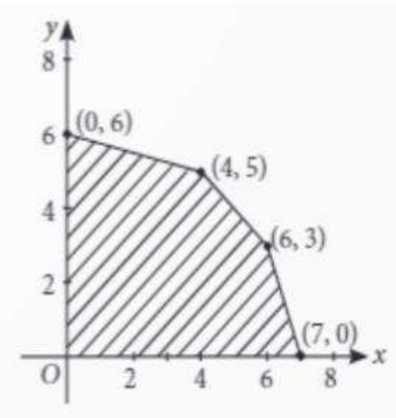
CASE STUDY BASED QUESTIONS

1. The feasible region for a LPP is as shown below, Z=3x-4y
   (a) Find the points at which maximum and minimum of Z occurs.
   (b) Find the difference between the optimum values of Z.



Ans: (a) Minimum Value is -32 at (0,8) Maximum Value is 15 at (5,0)
(b) Difference is 47

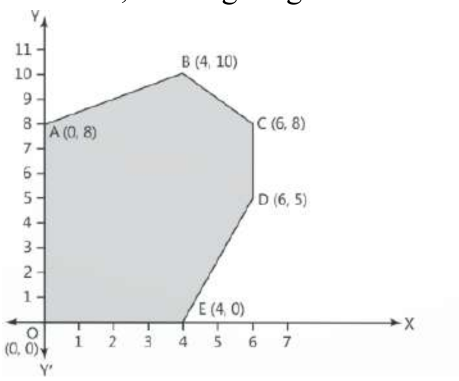
2. On the basis of the feasible region of a LPP as given below,
(a) Find the optimal value and solution of the objective function Z=2x+5y



(b) In general, if the corner points of the feasible region determined by the system of linear constraints are (0,10), (5,5),(15,15),(0,20) and Z= px+qy, where p, q >0. Find the relation between p and q so that the maximum occurs at (15,15) and at (0,20)

Sol: (a) Minimum value is 0 at (0,0) Maximum Value is 33 at (4,5)
(b) 15p+15q=20q implies q=3p

3. Let Z=4x-7y be the objective function, The figure given below is the feasible region.



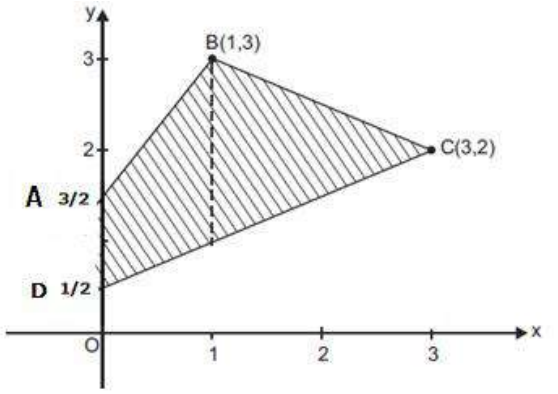
(a) Find the maximum and minimum value of Z and also the optimal solutions.

(b) Let Z=px+qy, where p,q>0 be the objective function. Find the condition on p and q so that the maximum value at B and C are equal. Also mention the number of optimal solutions in this case.

Sol: (a) Minimum Value is (-56) at (0,8) Maximum Value is 16 at (4,0)

(b) 4p+10q=6p+8q implies p=q, in this case, there are infinitely many optimal solution; all points of the segment BC are optimal solutions.

4. Given below is the feasible region of LPP with constraints



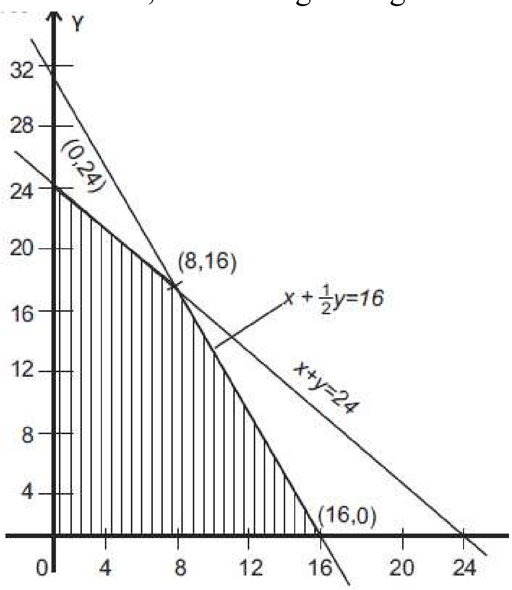
(a) Write all constraints of LPP

(b) Optimal value Maximize Z=5x-2y and corresponding Optimal point.

Sol: Constraints are $2y-3x \le 3,\ x+2y \le 7,\ 2y-x \ge 1\ \&\ x, y \ge 0$

Optimal Value is 11 at C(3,2)

5. Flot a LPP, feasible region is given below.



(a) write all constraints of the LPP

(b) find the maximum value of the objective function Z =3x+8y and also find the optimal solution.

Sol:

Constraints are $x+y \le 24, 2x+y \le 32\ \&\ x, y \ge 0$ Maximum Z=192 at (0,24)

CHAPTER-13-PROBABILITY

Definitions and Formulae:

Conditional Probability: If A and B are two events associated with any random experiment, then P(A/B) represents the probability of occurrence of event A knowing that the event B has already occurred.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0.$$

$P(B) \neq 0$, , means that the event B should not be impossible.

Multiplication Theorem on Probability: If the event A and B are associated with any random experiment and the occurrence of one depends on the other, then

$P(A \cap B) = P(A).P(B/A)$, where $P(A) \neq 0$

Independent Events:

When the probability of occurrence of one event does not depend on the occurrence /non-occurrence of the other event then those events are said to be independent events.

Then P(A/B)=P(A) and P(B/A)=P(B)

So, for any two independent events A and B , $P(A \cap B) = P(A).P(B)$.

Theorem on total probability:

$If E_i (i = 1,2,3, \ldots n) be a partition of sample space and all E_i have non-zero$
$proabability. A be any event associated with the sample space,$
$which occurs with E_1 or E_2 or E_3 or \ldots or E_n then$

$$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3) + \ldots + P(E_n)P(A/E_n)$$

Bayes' Theorem:

Let S be the sample space and $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or,.. or E_n then

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)}$$

MULTIPLE CHOICE QUESTIONS

1. For any two events A and B, P(A')=1/2, P(B')=2/3 and $P(A \cap B)$ =1/4, then $P\left(A'/B'\right)$ equals:

   (a) 8/9 (b) 5/8 (c) 1/8 (d) ¼

   Ans: (b)

   $P(A) = 1/2, P(B) = 1/3$

   $P(AUB) = 1/2 + 1/3 - 1/4 = 7/12$

   $$P(A'/B') = \frac{P(A' \cap B')}{P(B')} = \frac{1 - P(AUB)}{P(B')} = \frac{1 - 7/12}{2/3} = \frac{5/12}{2/3} = 5/8$$

2. For any two events A and B, $P(A) = 4/5$ *and* $P(A \cap B) = 7/10$, then $P(B/A)$ is

   (a) 1/10 (b) 1/8 (c) 17/20 (d) 7/8

   Ans: (d) $P(B/A) = \frac{7/10}{4/5} = 7/8$.

3. If A and B are two independent events such that P(A)=1/2 and P(B)=1/4, then P(B'/A) is

   (a) 1/4 (b) 3/4 (c) 1/8 (d) 1

   Ans: (b) $P(B'/A) = \frac{P(B' \cap A)}{P(A)} = \frac{P(B')P(A)}{P(A)} = P(B') = 3/4$

4. If two events A and B, P(A-B)=1/5 and P(A)= 3/5, then P(B/A) is equal to

   (a) ½ (b) 3/5 (c) 2/5 (d) 2/3

Ans: (d) $P(A-B)=1/5, P(A)=3/5$

$$P(A\cap B)=3/5-1/5=2/5$$

$$P(B/A)=\frac{P(A\cap B)}{P(A)}=\frac{2/5}{3/5}=2/3.$$

5. If P(A/B)=0.3, P(A)= 0.4 and P(B)=0.8, then P(B/A) is equal to

   (a) 0.6 (b) 0.3 (c) 0.06 (d) 0.4

Ans: (a) $P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A\cap B)}{0.8}=0.3$

$$P(A\cap B)=0.24$$

$$P(B/A)=0.24/0.4=0.6$$

6. If A and B are two events such that P(A/B)=2.P(B/A) and P(A)+P(B)=2/3, then P(B) is

   (a) 2/9 (b) 7/9 (c) 4/9 (d0 5/9.

Ans: (a) $\frac{P(A\cap B)}{P(B)}=2\frac{P(A\cap B)}{P(A)}$

$$P(A)=2P(B)$$

$$3P(B)=2/3$$

$$P(B)=2/9$$

7. If the sum of numbers obtained on throwing a pair of dice is 9, then the probability that number obtained on one of the dice is 4, is

   (a) 1/9 (b) 4/9 (c) 1/18 (d) 1/2

Ans: (d) $A=Sum9=\{(3,6),(4,5),(5,4),(6,3)\}$

$B=onedieshows4=\{(1,4),(2,4),(3,4),(5,4),(6,4),(4,1),(4,2),(4,3),(4,5),(4,6)\}$

$P(A)=4/36P(B)=10/36$

$$P(B/A)=\frac{P(B\cap A)}{P(A)}=\frac{2/36}{4/36}=1/2$$

8. Five fair coins are tossed simultaneously. The probability of the events that at least one head comes up is

   (a) 27/32 (b) 5/32 (c) 31/32 (d) 1/32

Ans: (c) $\text{P(atleast one H)}=1-\text{P(none is H)}$

$=1-1/32=31/32.$

9. Ramesh can hit a target 2 out of 3 times. He tried to hit the target twice. The probability that the he missed the target exactly once is

   (a) 2/3 (b) 1/3 (c) 4/9 (d) 1/9

Ans: (c) $P(A)=2/3\ P(A')=1/3\ (A-\text{hit}, A'-\text{not hit})$

$\text{P(only once hit)}=\frac{2}{3}.\frac{1}{3}+\frac{1}{3}.\frac{2}{3}=4/9.$

10. Three dice are thrown simultaneously. The probability of the events that at least one six comes up is

   (b) 27/216 (b) 55/216 (c) 91/216 (d) 1/216

Ans: (c) $\text{P(atleast one 6)}=1-\text{P(none is 6)}$

$=1-125/216=91/216.$

ASSERTION AND REASON QUESTIONS

Select the correct answer from the codes (a), (b), (c) and (d) as given below.

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A)

(b) Both Assertion (A) and Reason (R) are true but Reason (R) is **not** the correct explanation of the Assertion (A)

(c) Assertion (A) is true and Reason (R) is false.

(d) Assertion (A) is false and Reason (R) is true.

1 **Assertion (A):** If A and B are two independent events with P(A)=1/5 and P(B)=1/5, then P(A'/B) is 1/5.

Reason (R) : $\text{P}(A'/B) = \frac{P(A'\cap B)}{P(B)}$ **Ans: (d)**

$$\text{P}(A'/B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(A').P(B)}{P(B)} = P(A')$$

$$= 1 - P(A) = 4/5$$

$$So\ A\ is\ false\ and\ R\ isT\ true.$$

2. **Assertion (A) :** Let A and B be two events such that P (A) =1/5 and P (A or B) =1/2 then P(B)=3/8 for A and B are independent events.

Reason (R) : For independent events P (A or B) = P (A) + P(B) - P(A).P(B).

Ans: (a) as A is true and R is the correct explanation for A.

$ForAssertion$:

$P(AUB) = 1/2$

$P(A) + P(B) - P(A).P(B) = 1/2$

$P(B) = 3/8(\because P(A) = 1 - 1/5 = 4/5)$

$Riscorrectexxplanation.$

3. **Assertion (A):** Two coins are tossed simultaneously. The probability of getting two heads, if it is know that at least one head comes up, is 1/3.

Reason (R) : Let E and F be two events with a random experiment, then

$$P(F/E) = \frac{P(E\cap F)}{P(E)}.$$

Ans: (a) $ForAssertion\ \ F = \{HH\}E = \{HH, HT, TH\}, P(F/E) = \frac{1/4}{3/4} = 1/3$

4. **Assertion (A) :** If A and B are mutually exclusive events with P (A') =5/6 and P (B)=1/3. Then P (A/B')=1/4.

Reason (R) : If A and B are two events such that P (A)=0.2, P(B)=0.6 and P(A/B)=0.2 then the value of P(A/B') is 0.2.

Ans: (b) as A is true and R is not the correct explanation for A.

$$ForAssertion: P(A) = 1/6$$

$$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{1/6 - 0}{1 - 1/3} = 1/4$$

$$Re\,a\,son: P(A \cap B) = 0.12$$

$P(A/B') = \frac{P(A\cap B')}{P(B')} = \frac{P(A)-P(A\cap B)}{1-P(B)} = \frac{0.08}{0.4} = 0.2$

5. Let A and B be two events associated with an experiment such that

$P(A \cap B) = P(A).P(B)$

Assertion(A): P(A/B)=P$(A)$and$P$(B/A)=P(B)

Reason(R): $P(A \cup B)$=P(A)+P(B)

Ans: (c) as A is correct but R is false.

6. Let A and B be two independent events.

Assertion (A): If P(A)=0.3 and P$(A \cup \bar{B})$=0.8 then P(B) is $2/7$

Reason(R) : P$(\bar{E})$=1-P(E), for any event E.

Ans: (a) as $P(AUB') = P(A) + P(B') - P(A).P(B')$

$P(B') = 5/7; P(B) = 2/7$

Hence A is true and R is the correct explanation for A.

7. **Assertion (A):** Consider the experiment of drawing a card from a deck of 52 playing cards, in which the elementary events are assumed to be equally likely.
If E and F denote the events the card drawn is a spade and the card drawn is an ace respectively, then P(E/F)=1/4 and P(F/E)=1/13.
Reason (R): E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other. Such events are called independent events.
Ans: (b) as A is correct but R is not the correct explanation of A.

8. Consider that following statements:
Assertion (A): Let A and B be two independent events. Then P(A and B)= P(A)+P(B)
Reason (R): Three events A,B and C are said to be independent if
$$P(A \cap B \cap C) = P(A).P(B).P(C)$$
Ans: (d) as $P(A \text{ and } B) = P(A) + P(B) - P(A).P(B)$, hence A is false and R is True.

9. **Assertion (A):** In rolling a die, event A= {1, 3, 5} and event B = {2, 4} are mutually exclusive events.
Reason (R): In a sample space two events are mutually exclusive if they do not occur at the same time.
Ans: (a) A is true as $P(A \cap B) = \phi$ and R is the correct explanation of A.

10. For any two independent events A and B. P(A)=p and P(B)=q
Assertion (A): The probability that exactly one of the events A and B occurs is p+q-2pq
Reason (R): $P(A \cup B)=P(A)+P(B)-P(A)P(B)$ **Ans:** (b)
A is correct but R is not the correct explanation of A.

VERY SHORT ANSWER QUESTIONS

1. A pair of dice is thrown. If the two numbers appearing on them are different, find the probability that the sum of the numbers is 6.
Ans: A: Number appearing are different n(A)=30 (except (1,1),(2,2),(3,3),(4,4),(5,5) and (6,6))
B: Sum of the numbers is 6. P(A)=30/36
A and B={(1,5),(2,4),(4,2),(5,1)}
P(A and B)=4/36 $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{4/36}{30/36} = 4/30$

2. In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII given that the chosen student is girl?
Ans: A: Student of Class XII B: The student is a girl.
n(A & B)=10% of 430=43. $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{43}{430} = \frac{1}{10}$

3. Two balls are drawn from a bag containing 2white, 3 red and 4 black balls one by one without replacement. What is the probability that at least one ball is red?
Ans: P(at least one red ball)=1-P(none of the ball is red)
(that is 1st ball is non red and 2nd ball is non red.) $= 1 - \frac{6}{9}.\frac{5}{8} = \frac{42}{72} = \frac{7}{12}$

4. If P(A)=3/8, P(B)=1/2 and P(A and B)=1/4, find $P(A'/B')$
Ans:

$$P(A'/B') = \frac{P(A' \cap B')}{P(B')} = \frac{P(AUB)'}{1 - P(B)} = \frac{1 - P(AUB)}{1 - P(B)}$$
$$P(AUB) = P(A) + P(B) - P(A \cap B)$$
$$= 3/8 + 1/2 - 1/4 = \frac{5}{8}, P(A'/B') = \frac{1 - P(AUB)}{1 - P(B)} = \frac{1 - \frac{5}{8}}{1 - \frac{1}{2}} = \frac{3/8}{1/2} = 3/4$$

5. A committee of 4 students is selected at random from a group of 8 boys and 4 girls. Given that there is at least one girl in the committee, calculate the probability that there are exactly 2 girls in the committee.

Ans:

A: at least one girl in the committee. B: exactly 02 girls in the committee.

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$
$$P(A) = 1 - P(none\ is\ girl) = 1 - \frac{^{8}C_4}{^{12}C_4} = 1 - \frac{70}{495} = \frac{85}{99}$$
$$P(A \cap B) = P(2G\ and\ 2B) = \frac{^{8}C_2.\ ^{4}C_2}{^{12}C_4} = \frac{28.6}{495} = \frac{56}{165}$$

6. Events E and F are independent. Find P(F), if P(E) =0.35 and P(EUF)=0.6.

Ans: $P(EUF) = P(E) + P(F) - P(E \cap F)$

$= P(E) + P(F) - P(E).P(F)$

$0.6 = 0.35 + x - 0.35x; 0.25 = 0.65x; x = \frac{25}{65} = \frac{5}{13}$

7. A and B are two candidates seeking admission in a college. The probability that A is selected is 0.7 and the probability that exactly one of them selected is 0.6. Find the probability that B is selected.

Ans: E: Selecting A F: Selecting B E and F are independent events.

$P(E) = 0.7, P[(E \cap F')U(E' \cap F)] = 0.6$

$P(E).P(F') + P(E').P(F) - P[(E \cap F') \cap (E' \cap F)] = 0.6$

$P(E)(1 - P(F)) + (1 - P(E))P(F) = 0.6$

$P(E) + P(F) - 2.P(E).P(F) = 0.6; 0.7 + x - 2(0.7).x = 0.6$

$0.1 = 0.4xx = \frac{1}{4} = P(F).$

8. A bag contains 3 white, 4 red and 5 black balls. Two balls are drawn at random. Find the probability that both balls are of different colours.

Ans:

$$\text{P(both balls are of different colours)} = 1 - \text{P(both balls of same colour)}$$
$$\text{P(both balls of same colour)} = \frac{^{3}C_2}{^{12}C_2} + \frac{^{4}C_2}{^{12}C_2} + \frac{^{5}C_2}{^{12}C_2} = \frac{3}{66} + \frac{6}{66} + \frac{10}{66} = \frac{19}{66}$$
$$\text{P(both balls are of different colours)} = 1 - \frac{19}{66} = \frac{47}{66}$$

9. An unbiased die is thrown thrice. Find the probability of getting at least 2 sixes.

Ans:

$$\text{P}\left(\text{at least 2 sixes}\right) = \text{P(2 sixes)} + \text{P(3 Sixes)}$$
$$= 3.\frac{1}{6}.\frac{1}{6}.\frac{5}{6} + \frac{1}{6}.\frac{1}{6}.\frac{1}{6} = \frac{16}{216} = \frac{2}{27}$$

10. A problem is given to A, B and C. The probabilities that they solve the problem

correctly are 1/3, 2/7 and 3/8 respectively. If they try to solve the problem simultaneously, find the probability that exactly one of them solve the problem.

Ans:P(exactly one solve)=$P(A\cap B'\cap C')+P(A'\cap B\cap C')+P(A'\cap B'\cap C)$

$$=\frac{1}{3}.\frac{5}{7}.\frac{5}{8}+\frac{2}{3}.\frac{2}{7}.\frac{5}{8}+\frac{2}{3}.\frac{5}{7}.\frac{3}{8}=\frac{75}{168}$$

SHORT ANSWER TYPE QUESTIONS

1. A and B throw a die alternately till one of them get a '6' and wins the game. Find their respective probabilities of winning, if A starts the game first.

Ans: S-Getting 6 F- not getting 6

P(S)=p=1/6 P(F)=q=5/6

$P(A-Wins)=p+qqp+qqqqp+\cdots\ldots$

$=1/6+(5/6)^2 1/6+(5/6)^4 1/6+\ldots.$

$$=\frac{1/6}{1-\frac{25}{36}}=\frac{6}{11},\ P(B\ Wins)=1-\frac{6}{11}=\frac{5}{11}$$

2. A fair coin and an unbiased die are tossed. Let A be the event, "Head appears on the coin" and B is the event, "3 comes on the die". Find whether A and B are independent events or not.

Ans: S=((H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)}

A: H appears B: 3 on die

P(A)=6/12=1/2 P(B)=2/12=1/6 P(A and B)=1/12 P(A).P(B)=1/2.1/6=1/12=P(A and B)

Hence A and B are independent events.

3. There are two coins. One of them is a biased coin such that P(head): P(tail) is 1:3 and the other coin is a fair coin. A coin is selected at random and tossed once. If the coin shows head, then find the probability that it is a biased coin

Ans:

A: Selecting Biased Coin B; Selecting fair coin C: Getting H

P(A)=P(B)=1/2, P(C/A)=1/4 (since ratio for head and tail is 1:3) P(C/B)=1/2

P(C) =P(A).P(C/A)+P(B).P(C/B)=1/2.1/4+1/2.1/2=1/8+1/4=3/8

$$P(B/C)=\frac{P(B).P(C/B)}{P(A).P(C/A)+P(B).P(C/B)}=\frac{1/4}{3/8}=2/3$$

4. A letter is known to have come either from TATANAGAR or from CALCUTTA. What is the probability that on the envelope just two consecutive letters TA are visible?

Ans:

$P(A)=1/2,\ P(B)=1/2$,C-TA visible on the envelope.

P(C/A)=2/8=1/4, P(C/B)=1/7,

$$P(C)=P(A).P(C/A)+P(B).P(C/B)=\frac{1}{2}.\frac{1}{4}+\frac{1}{2}.\frac{1}{7}=11/56$$

5. A and B are independent events such that $P(A\cap\bar{B})=\frac{1}{4}$ and $P(\bar{A}\cap B)=\frac{1}{6}$ Find P(A) and P(B).

Ans:

$P(A).P(B')=1/4\ \&\ P(A').P(B)=1/6\ \ P(A)=x,\ P(B)=y$

$x(1-y)=1/4$ and $(1-x)y=1/6$

on solving we get, $x-y=1/12\quad x=y+1/12$

On substituting, we get $12y^2-11y+2=0$

$y=1/4$ or $y=2/3$, Corresponding $x=1/3$ or $3/4$

$P(A)=1/3\ P(B)=1/4$ (OR) $P(A)=3/4\ P(B)=2/3$

6. There are 2000 scooter drivers, 4000 car drivers and 6000 truck drivers all insured. The probabilities of an accident involving a scooter, a car, a truck are 0.01, 0.03, and 0.15 respectively. What is the probability that one of the insured person meets with an accident.

Ans:

$P(A) = 1/6,\ P(B) = 1/3\ P(C) = 1/2$

D-the insured person meets with an accident.

P(D/A)=1/100, P(D/B)=3/100, P(D/C)=15/100,

$P(D) = P(A).P(D/A) + P(B).P(D/B) + P(C).P(D/C) = 13/150$

7. A bag contains 4 balls. Two balls are drawn at random. What is the probability that both drawn balls are white.

Ans: $A: 2W + 2nonwhite B: 3W + 1nonwhite C: 4W$

$P(A) = 1/3, P(B) = 1/3 P(C) = 1/3$D-both drawn balls are White

$P(\text{D/A}) = {}^2C_2/{}^4C_2=1/6, P(\text{D/B}) = {}^3C_2/{}^4C_2=1/2, P(\text{D/C}) = {}^4C_2/{}^4C_2=1$

$P(D) = P(A).P(D/A) + P(B).P(D/B) + P(C).P(\text{D/C}) = \frac{1}{3}.\frac{1}{6} + \frac{1}{3}.\frac{1}{2} + \frac{1}{3}.1 = \frac{5}{9}$

8. Bag A contains 6 red and 5 blue balls and another bag B contains 5 red and 8 blue balls. A ball is drawn from bag A without seeing its colour and is put into the bag B. Then a ball is drawn from bag B at random. What is the probability that the ball drawn is blue in colour.

Ans:

A:Red$from A(6R + 5B) B: Blue from A(6R + 5B)$

C:getting Blue ball from bag B after transferring.

$P(A) = 6/11, P(B) = 5/11$

$Case: I A \cap C$

$$P(A \cap C) = P(A).P(C/A) = \frac{6}{11}.\frac{8}{14} = \frac{48}{154}$$

$Case: II B \cap C$

$$P(B \cap C) = P(B).P(C/B) = \frac{5}{11}.\frac{9}{14} = \frac{45}{154}$$

Reqd.Probability$= \frac{48}{154} + \frac{45}{154} = \frac{93}{154}$

CASE STUDY BASED QUESTIONS

1 A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by cab, metro, bike or by other means of transport are respectively 3/10, 1/5, 1/10 and 2/5. The probabilities that he will be late are 1/4, 1/3, 1/12 and 1/10 if he comes by cab, metro, bike and other means of transport respectively.

(a) What is the probability that the doctor arrived late?

(b) When the doctor arrives late, what is the probability that he comes by metro?

Ans: $(a) A: Cab B: Metro C: Bike D: other$

$E: Late arrival$

$P(E) = P(A).P(E/A) + P(B).P(E/B) + P(C).P(E/C) + P(D).P(E/D)$

$$= \frac{3}{10}.\frac{1}{4} + \frac{1}{5}.\frac{1}{3} + \frac{1}{10}.\frac{1}{12} + \frac{2}{5}.\frac{1}{10} = \frac{114}{600}$$

$$(b) P(B/E) = \frac{P(B).P(E/B)}{P(E)} = \frac{1/15}{114/600} = 40/114 = 20/57$$

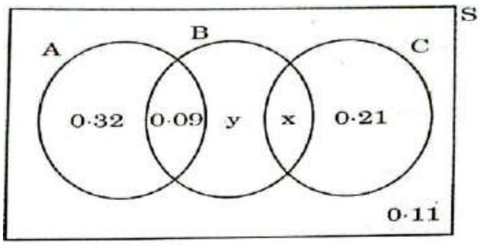
The Venn diagram below represents the probabilities of three different types of Yoga A, B and C performed by the people of a society. Further, it is given that the probability of a member performing type C yoga is 0.44.

2.

(i) Find the value of x. (ii) Find the value of y

(iii) (a) Find P(C/B) (OR)

(b) Find the probability that a randomly selected person of the society does Yoga type A or B but not C.



Ans: (i) x=0.44-0.21=0.23 (ii) y=1-0.96=0.04

(iii) P$(C/B) = \frac{P(C\cap B)}{P(B)} = \frac{0.23}{0.36} = \frac{23}{36} (OR)$

$P(A\ OR\ B\ not\ C) = 0.32 + 0.09 + y = 0.41 + 0.04 = 0.45$

3. Two cards are drawn from a well shuffled pack of 52 cards without replacement.

(a) What is the probability that one is a red queen and the other is a king of black colour

(b) one shows a prime number (from 2 to 10) and other is a face card(other than Ace)

Ans:

$(a) A = Re\ d\ queenin$ 1st attempt B=Black King in 2nd Attempt

C=Black King in 1st Attempt

D= $Re\ d\ queenin$ 2nd attempt

Reqd Prob=P$(A \cap B)$+P$(C \cap D)$ $= \frac{2}{52}.\frac{2}{51} + \frac{2}{52}.\frac{2}{51} = \frac{2}{663}$

$(b) A = Pr\ i\ meNumberin$ 1st attempt

B=face card in 2nd Attempt C=Face card in 1st Attempt D= $Pr\ i\ meNumberin$ 2nd attem

ReqdProb=P$(A \cap B)$+P$(C \cap D)$ $= \frac{16}{52}.\frac{12}{51} + \frac{12}{52}.\frac{16}{51} = \frac{32}{221}$

LONG ANSWER TYPE QUESTIONS

1. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let 3/5 is the probability that he knows the answer and 2/5 be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability 1/3. What is the probability that the student knows the answer, given that he answered it correctly?

Ans: A: Knows the answer B: Guesses the answer E: Answered Correctly

P(A)=3/5, P(B)=2/5 P(E/A)=1, P(E/B)=1/3

By Bayes theorem,

$$P(A/E) = \frac{P(A).P(E/A)}{P(A).P(E/A) + P(B).P(E/B)} = \frac{\frac{3}{5}\cdot 1}{\frac{3}{5}\cdot 1 + \frac{2}{5}\cdot\frac{1}{3}} = \frac{3}{11}$$

2. Three bags contains a number of red and white balls as follow:

Bag-1 contains 3 red balls, bag-2 contains 2 red and 1 white ball, bag-3 contains 3 white balls. The probability that bag -I will be chosen and a ball is selected from it is i/6, i=1,2,3.

(a) What is the probability that a red ball will be selected

(b) What is the probability that a white ball will be selected.

Ans:

(a)A,B,C are events selecting bag-1,bag-2,bag-3respectively.D-selecting red colour ball

$P(D/A)=1, P(D/B)=2/3, P(D/C)=0$

$P(D)=P(A).P(D/A)+P(B).P(D/B)+P(C).P(D/C)$

$=\frac{1}{6}.1+\frac{2}{6}.\frac{2}{3}+0=\frac{7}{18}$ $(b)P(Whiteball)=1-\frac{7}{18}=\frac{11}{18}$

3. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin?

Ans:$A: twoheaded B: Biased(75\%H) coin C: biased(40\%T)$

$P(A)=P(B)=P(C)=1/3 H: getting H$

$P(H/A)=1, P(H/B)=75/100, P(H/C)=60/100$

$$P(A/H)=\frac{P(A).P(H/A)}{P(A).P(H/A)+P(B).P(H/B)+P(C).P(H/C)}=\frac{100}{235}$$

4 A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Ans: $A=6appears, B=nonsixappears$

$P(A)=1/6, P(B)=5/6 C: Reporting 6$

$P(C/A)=3/4, P(C/B)=1/4; P(C)=P(A).P(C/A)+P(B).P(C/B)$

$=\frac{1}{6}.\frac{3}{4}+\frac{5}{6}.\frac{1}{4}=\frac{8}{24}$

$$P(A/C)=\frac{P(A).P(C/A)}{P(A).P(C/A)+P(B).P(C/B)}=\frac{\frac{1}{6}.\frac{3}{4}}{\frac{8}{24}}=\frac{3}{8}$$

5 Assume that the chances of a patient having a heart attack are 40%. It is also assumed that meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options and patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga.

Ans;

$A-Meditation\&Yoga B-takesdrug C-SuffersheartAttack$

$P(A)=P(B)=\frac{1}{2} P(C/A)=\frac{70}{100}.\frac{40}{100}, P(C/B)=\frac{75}{100}.\frac{40}{100}$

$$P(A/C)=\frac{\frac{1}{2}.\frac{70}{100}.\frac{40}{100}}{\frac{1}{2}.\frac{70}{100}.\frac{40}{100}+\frac{1}{2}.\frac{75}{100}.\frac{40}{100}}=\frac{14}{29}$$

6. Bhavani is going to pay a game of chess against one of four opponents in an inter-college sports competition. Each opponent is equally likely to be paired against her. The table below shows the chances of Bhavani losing, where paired against each opponent.

OPPONENT	BHAVANI'S CHANCE OF LOSSING
OPP-1	12%
OPP-2	60%
OPP-3	X%
OPP-4	84%

If the probability that Bhavani losses the game that day is ½. find the probability for Bhavani to be losing when paired against opponent 3.

Sol A,B,C,D be events, respectively Bhavani game with Opp-1,2,3,4 . E be the vent Bhavani Loses the game.

Given P(E)=1/2

P(A)=P(B)=P(C)=P(D)=1/4.

P(E/A)=12/100, P(E/B)=60/100, P(E/C)=x/100,P(E/D)=84/100

P(A)=1/2

$$1/2 = P(A).P(E/A) + P(B).P(E/B) + +P(C).P(E/C) + P(D).P(E/D)$$

$$1/2 = \frac{12+60+X+84}{400}; X = 44p(E/C) = 44\%$$

7. Three persons A , B, C apply for a job a manager in a company. Chances of their selection are in the ratio 1:2:3. The probability that A,B and C can introduce chances to increase the profits of a company are 0.8,0.5 and 0.3 respectively. If increase in the profit does not take place, find the probability that it is due to the appointment of A.

P(A)=1/7, P(B)=2/7, P(C)=4/7

E-not increase the profits

P(E/A)=0.2, P(E/B)=0.5, P(E/C)=0.7

$$P(E/A) = \frac{P(A).P(E/A)}{= P(A).P(E/A) + P(B).P(E/B) + +P(C).P(E/C)} = \frac{2/70}{40/70} = 1/20$$

8. There are two bags I and II, Bag I contains 3 red and 5 black balls and Bag II contains 4 red and 3 black balls. One ball is transferred randomly from Bag I to bag II and then a ball is drawn randomly from bag II. If the ball so drawn is found to be black in colour then find the probability that the transferred ball is also black.

A-transferred ball is Red B-transferred ball is black

C: ball from bag II after transferred , is black.

P(A)=3/8, P(B)= 5/8

P(C/A)=3/8 PC/B)=4/8

$$P(C/B) = \frac{P(A).P(C/A)}{P(A).P(C/A) + P(B).P(C/B)} = \frac{\frac{3}{8}\cdot\frac{3}{8}}{\frac{3}{8}\cdot\frac{3}{8} + \frac{5}{8}\cdot\frac{4}{8}} = \frac{9}{29}$$

EXERCISES AND PAPERS WITH ANSWES

PRACTICE QUESTION PAPERS WITH ANSWERS

USEFUL LINKS

CLASS 12 MATHEMATICS CURRICULUM 2025 - 26	https://cbseacademic.nic.in/curriculum_2026.html
NCERT TEXT BOOK CLASS XII MATHEMATICS PART - I	https://ncert.nic.in/textbook.php?lemh1=0-6
NCERT TEXT BOOK CLASS XII MATHEMATICS PART - II	https://ncert.nic.in/textbook.php?lemh2=0-7
LABORATORY MANUAL – CLASS XII MATHEMATICS	https://ncert.nic.in/science-laboratory-manual.php
CBSE QUESTION PAPER 2025 - CLASS XII MATHEMATICS	https://www.cbse.gov.in/cbsenew/question-paper.html
CBSE MARKING SCHEME 2025 - CLASS XII MATHEMATICS	https://www.cbse.gov.in/cbsenew/marking-scheme.html
CBSE SAMPLE PAPER AND MARKING SCHEME 2024	https://cbseacademic.nic.in/SQP_CLASSXii_2024-25.html
GeoGebra Software Demonstration	https://www.youtube.com/watch?v=o_NN6TvPHp8
Virtual Labs for Mathematics	https://www.youtube.com/watch?v=bQEgiOJeCVU
Listening to Learn: Snippets from Ancient Indian Mathematics	https://www.youtube.com/watch?v=4q2llxXV4-I
Class XII Mathematics – Video Lessons by NCERT	https://www.youtube.com/@NCERTOFFICIAL/search?query=MATHEMATICS%20class%2012
Mathematics Lab Activities	https://www.youtube.com/@NCERTOFFICIAL/search?query=class%2012%20maths%20lab%20activities